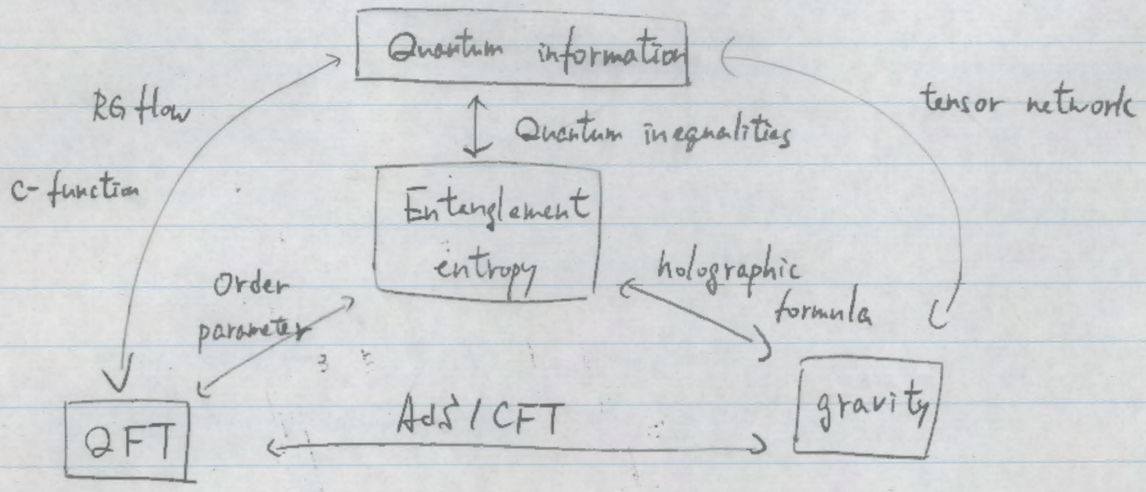
"Entanglement, RG flow and supersymmetry" Taiwan String workshop
Nov. 24 ~ 28, 2014

1. Introduction



Entanglement entropy

- can be defined for any QFT in any dim.
- capture global entanglement, an order parameter for phase transitions
- a measure of effective degrees of freedom under RG flow

Outline

2. Basics of EE
3. EE in QFT

} Part I

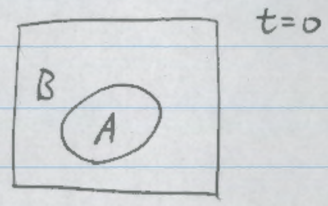
4. EE in RG flow
5. Supersymmetric Rényi entropy

} Part II

2. Basics of EE

Divide a space into two regions A and B

Assume $\underline{\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B}$



Given orthonormal bases

$$\mathcal{H}_{\substack{A\\B}} = \{ |\phi^1\rangle_{\substack{A\\B}}, |\phi^2\rangle_{\substack{A\\B}}, \cdots, |\phi^i\rangle_{\substack{A\\B}} \cdots \}$$

- Reduced density matrix

dimension of $\mathcal{H}_B$

$$\rho_A \equiv \mathrm{tr}_B\, \rho_{tot} = \sum_{i=1}^{d_B} {}_B\langle \phi^i | \rho_{tot} | \phi^i \rangle_B$$

- Entanglement entropy (EE) is the von Neumann entropy of ρ_A

$$\boxed{S_A \equiv -\mathrm{tr}_A\, \rho_A \log \rho_A}$$

We will consider a ground state

$$\rho_{tot} = \frac{|\Psi\rangle\langle\Psi|}{\langle\Psi|\Psi\rangle} \qquad |\Psi\rangle : \text{g.s. wave function}$$

In general

$$|\Psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} C_{ij} \, |\phi^i\rangle_A \, |\phi^j\rangle_B$$

① $\underline{C_{ij} = C_i^A \; C_j^B}$

$$|\Psi\rangle = |\Psi_A\rangle \cdot |\Psi_B\rangle \quad , \quad |\Psi_{A \atop B}\rangle = \sum_i C_i^{A \atop B} |\phi^i\rangle_{A \atop B}$$

$\Rightarrow \quad \rho_A = |\Psi_A\rangle\langle\Psi_A| \quad , \quad \boxed{S_A = 0}$ no entanglement

② $\underline{C_{ij} \neq C_i^A \; C_j^B}$

Singular value decomposition :

$$C_{ij} = U_{ik} \lambda_k V_{kj} \qquad U, V : \text{unitary matrices}$$

$$\lambda_k \geq 0 \qquad k = 1, 2, \cdots, \min(d_A, d_B)$$

- Schmidt decomposition

$$|\Psi\rangle = \sum_k \lambda_k \, |\tilde{\phi}^k\rangle_A \, |\tilde{\phi}^k\rangle_B \qquad \lambda_k \geq 0, \; \sum_k \lambda_k^2 = 1$$

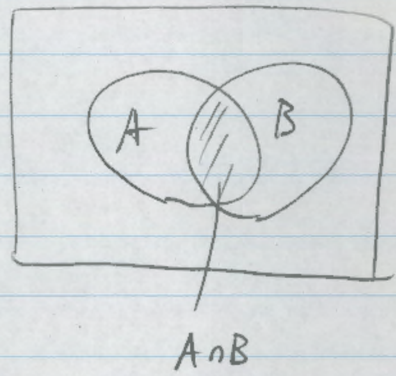
new bases

$$\Rightarrow \quad \boxed{S_A = -\sum_k \lambda_k^2 \log \lambda_k^2 = S_B}$$

- S_A is maximized for $\lambda_1 = \lambda_2 = \cdots = \lambda_k = \cdots = \sqrt{1/\min(d_A, d_B)}$

$\boxed{S_A|_{max} = \log \min(d_A, d_B)}$ maximally entangled

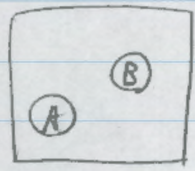
Strong subadditivity



$$S_{A\cup B} + S_{A\cap B} \le S_A + S_B$$

follows from unitarity

Mutual information



$$I(A,B) = S_A + S_B - S_{A\cup B}$$

$$\ge 0$$

$$(\le S_A \text{ for } A\cap B = \phi)$$

always finite

Rényi entropy

For an integer $n > 0$

$$\boxed{S_n(A) = \frac{1}{1-n} \log \mathrm{tr}_A \rho_A^n}$$

$$S_A = \lim_{n\to 1} S_n(A)$$

- Inequalities

$$\partial_n S_n \le 0$$

$$\partial_n \left(\frac{n-1}{n} S_n \right) \ge 0$$

$$\partial_n \left((n-1) S_n \right) \ge 0$$

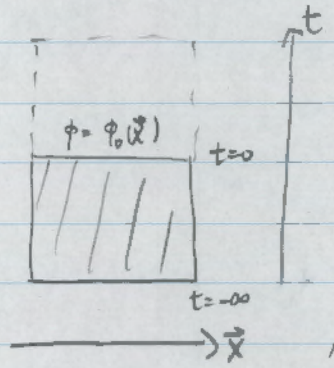
$$\partial_n^2 \left((n-1) S_n \right) \le 0$$

3. EE in QFT

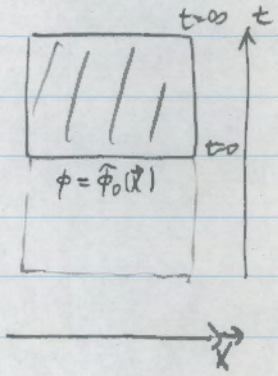
QFT, $\dim \mathcal{H} = \infty$

Replica trick

$\langle \phi_0(\vec{x}) | \Psi \rangle =$



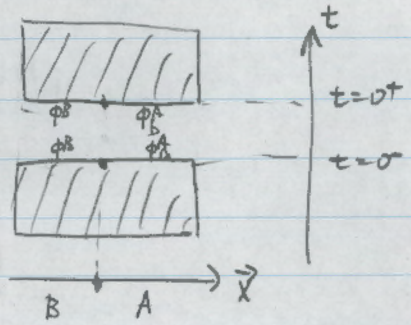
, $\langle \Psi | \hat{\phi}_0(\vec{x}) \rangle =$



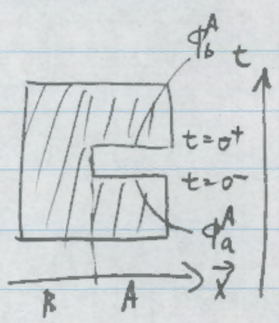
$$[\rho_A]_{ab} = \langle \overset{\mathcal{H}_A}{\phi^A_a} | \rho_A | \phi^A_b \rangle$$

$$= \langle \phi^A_a | \left(\frac{1}{\underset{=\langle\Psi|\Psi\rangle}{Z_1}} \int [D\phi^B(t=0, \vec{x} \in B)] \, \langle \overset{\mathcal{H}^B}{\phi^B} | \Psi \rangle \langle \Psi | \phi^B \rangle \right) | \phi^A_b \rangle$$

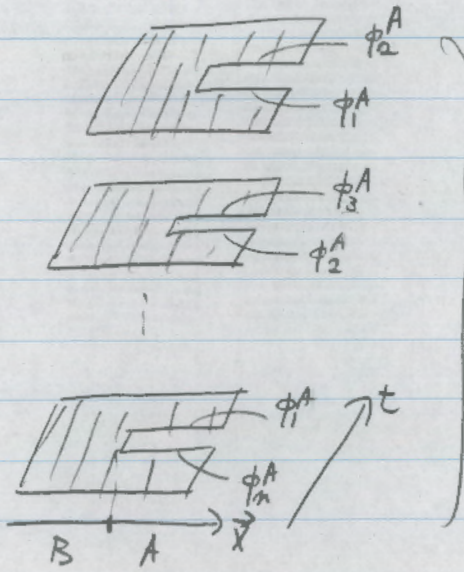
$$= \frac{1}{Z_1} \int D\phi^B(t=0, \vec{x} \in B)$$



$$= \frac{1}{Z_1}$$



$$\mathrm{tr}_A \, \rho_A^n = \frac{1}{(Z_1)^n}$$



n copies $= \dfrac{Z_n}{(Z_1)^n}$

Z_n : partition function on a space $\mathcal{M}_n$ with singular surface at the entangling surface $\Sigma = \partial A$

– Rényi entropy

$$\boxed{S_n(A) = \frac{\log Z_n - n \log Z_1}{1-n}}$$

– EE

$$\boxed{S_A = -(\partial_n - 1) \log Z_n \big|_{n \to 1}}$$

UV structure

In QFTd

$$S_A = \frac{C_{d-2}}{\epsilon^{d-2}} + \frac{C_{d-4}}{\epsilon^{d-4}} + \cdots + \begin{cases} \frac{C_2}{\epsilon^2} + C_0 \log(1/\epsilon) & (d = \text{even}) \\ \frac{C_1}{\epsilon} - \gamma & (d = \text{odd}) \end{cases}$$

$\uparrow$ topological EE

$\boxed{C_{d-2} \propto \text{Area}(\Sigma)}$ Area law

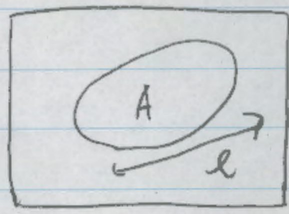
ϵ : UV cutoff

- d = even
  - conformal anomaly
  - $C_0 \sim$ central charge

- d = odd
  - no conformal anomaly
  - γ : topological EE $\gamma \geq 0$

    distinguish topological phases

CFT



$$\ell \frac{d}{d\ell} \log Z_n = \int_{\mathcal{M}_n} d^dx \sqrt{g}\ \underbrace{2 g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \ln Z_n}_{\equiv\ -\langle T_\mu{}^\mu \rangle}$$

$$\Rightarrow \quad \ell \frac{d}{d\ell} S_A = \lim_{n\to 1} (\partial_n - 1) \int_{\mathcal{M}_n} d^dx \sqrt{g}\ \langle T_\mu{}^\mu \rangle \ \equiv\ C_0(\Sigma)$$

(• $\underline{d = \text{odd}}$ $\quad \langle T_\mu{}^\mu \rangle = 0$ $\quad$ no conformal anomaly, $C_0 = 0$)

• $\underline{d : \text{even}}$ $\qquad$ Euler density $\int_{S^d} E_d = 2$

$$\langle T_\mu{}^\mu \rangle = 2(-1)^{\frac{d}{2}} A\, E_d - \sum_i B_i\, I_i$$

central charge $\qquad$ Weyl invariants

$$\therefore \quad \boxed{S_A \supset C_0(\Sigma) \log \frac{\ell}{\epsilon} \quad \epsilon:\ \text{UV cutoff}}$$

• $\underline{d=2}$ $\qquad \langle T_\mu{}^\mu \rangle = -\frac{c}{24\pi} R, \qquad C_0 = \frac{c}{3}$

• $\underline{d=4}$ $\qquad \langle T_\mu{}^\mu \rangle = 2a\, E_4 - c\, I_4,$

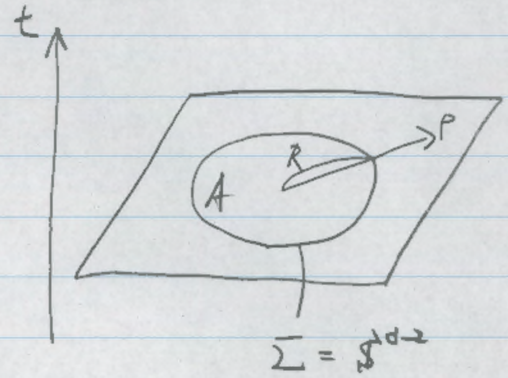
$$C_0(\Sigma) = -2a\, \chi(\Sigma) + \frac{c}{6\pi} \int_\Sigma K_\Sigma$$

Euler of Σ

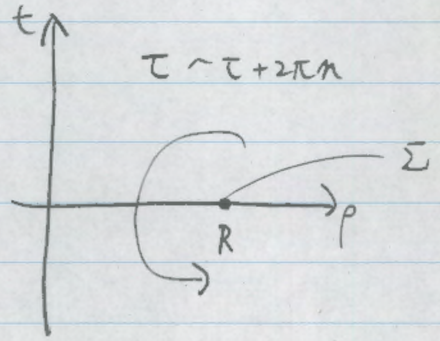
· d = odd

$$\langle T_\mu{}^\mu \rangle = 0 ,$$

Consider $\Sigma = S^{d-2}$



$\mathbb{R}^d$



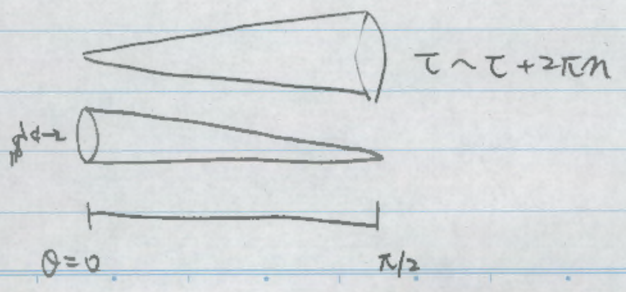
$\mathcal{M}_n$

Conformal map to S^d

$$ds^2_{\mathbb{R}^d} = dt^2 + d\rho^2 + \rho^2 d\Omega^2_{d-2}$$

$$t = \frac{R \sin\tau}{\csc\theta + \cos\tau} , \quad \rho = \frac{R \cot\theta}{\csc\theta + \cos\tau}$$

$$= \Omega^2 \underbrace{\left(d\theta^2 + \sin^2\theta \, d\tau^2 + \cos^2\theta \, d\Omega^2_{d-2}\right)}_{ds^2_{S^d}} , \quad \Omega = \frac{R}{1 + \sin\theta \cos\tau}$$



S^d_n = n branched cover of S^d

In CFT

$$Z_n \equiv Z[\mathcal{M}_n] = Z[S^d_n]$$

– Rényi entropy across $\Sigma = S^{d-2}$

$$\boxed{S_n(A) = \frac{1}{1-n} \log \frac{Z[S^d_n]}{(Z[S^d])^n}}$$

– EE of $\Sigma = S^{d-2}$

$$\boxed{S_1 = \log Z[S^d]}$$

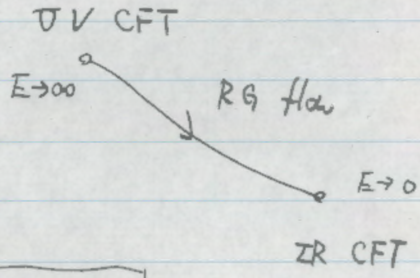
[Casini - Huerta - Myers 11]

$$\log Z[S^d_n] = \log Z[S^d] + \frac{1}{2}\int \underbrace{\delta g_{\mu\nu}}_{(n-1)} \underset{\substack{\| \\ 0}}{\langle T^{\mu\nu}\rangle} + \cdots$$

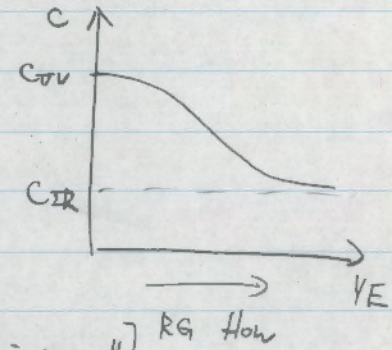
for CFT

4. EE in RG flow

- Is there a "c-function" which monotonically decreases under any RG flow in QFT?



$$\boxed{\text{c-function} = \text{a measure of degrees of freedom}}$$

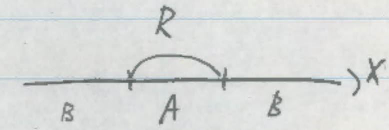


Example

- $d=2$ Zamolodchikov's c-theorem
  $c_{UV} \geq c_{IR}$
- $d=4$ the a-theorem
  [Cardy 88, Komargodski-Schwimmer 11]
  $a_{UV} \geq a_{IR}$

Entropic c-theorem in 2d

$S(R)$: EE of an interval of width R



Def (Entropic c-function)

$$c_E(R) \equiv 3R \frac{\partial S(R)}{\partial R}$$

For CFT_2

central charge

$$S(R) = \frac{c}{3} \log\left(\frac{R}{\epsilon}\right)$$

C - theorem in 3d?

- Thermal c - theorem

$$F_{Therm} \sim C_{Therm} T^3$$

Counter example by [Sachdev 93] O(N) vector model

- C_T - theorem

$$C_T|_{UV} \geq C_T|_{IR} \quad , \quad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle_{CFT} = C_T \frac{I_{\mu\nu,\rho\sigma}(x)}{x^6}$$

Counter example by [TN - Yonekura 13] $\mathcal{N}=2$ WZ model

- F - theorem [Jafferis - Klebanov - Pufu - Safdi 11, Myers - Sinha 10]

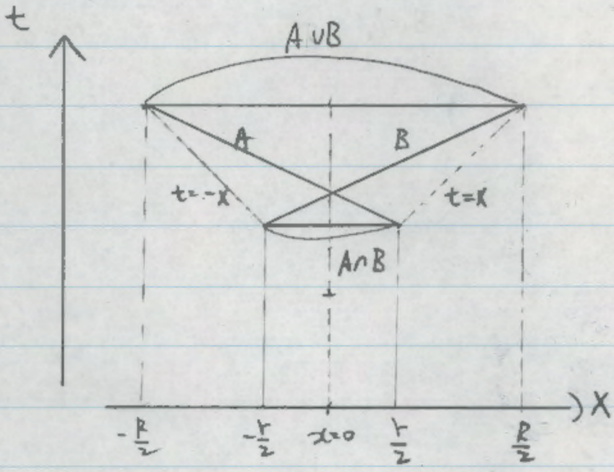
$$F_{UV}(S^3) \geq F_{IR}(S^3) \quad , \quad F \equiv -\log Z[S^3]$$

In 2d, C_{Therm}, $C_T \propto C$

- Entropic c-function satisfies (entropic c-theorem)

① $C_E(R)|_{CFT} = c$ for CFT_2

② $\frac{\partial C_E(R)}{\partial R} \le 0$



Proof of ② [Casini - Huerta 04]

SSA + Lorentz inv

$$\underset{S(R)}{\underset{\shortparallel}{S_{A\cup B}}} + \underset{S(r)}{\underset{\shortparallel}{S_{A\cap B}}} \le \underbrace{S_A + S_B}_{\underset{2S(\sqrt{rR})}{\shortparallel}}$$

$r = R - \epsilon$, $\epsilon \ll 1$

$$\mathcal{O}(\epsilon^2): \quad \frac{C_E'(R)}{3} = S'(R) + R\,S''(R) \le 0 \quad //$$

F - theorem in 3d → ⑫'

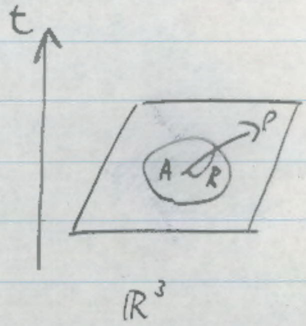
Conjecture (F - theorem) [Jafferis - Klebanov - Pufu - Safdi 11]

In QFT_3,

$F_{UV} \ge F_{IR}$, $F \equiv -\log Z^{(ren)}(S^3)$ ← indep. of UV cutoff

For CFT_3, EE of a disk of radius R

$$\boxed{S(R) = \underbrace{\alpha \frac{2\pi R}{\epsilon}}_{\text{Area law}} - F} \quad — (★)$$



- To prove the F - theorem, we need a function $\mathcal{F}(R)$ satisfying

① $\mathcal{F}(R)\big|_{CFT} = F$

② $\dfrac{\partial \mathcal{F}(R)}{\partial R} \leq 0$

<u>Def</u> (Renormalized EE) [Liu - Mezei 12]

$$\mathcal{F}(R) \equiv (R\partial_R - 1)\, S(R) \,, \quad S(R) : \text{EE of a disk of radius } R \text{ in } QFT_3$$

<u>Outline of the proof</u>

- 定 → ①

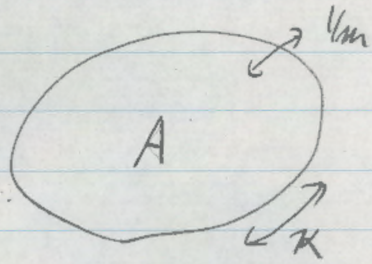
- SSA + Lorentz inv → ② [Casini - Huerta 12]

<u>EE in a gapped phase</u> d = 3

$$S_A = \alpha \frac{l_\Sigma}{\epsilon} + \beta m l_\Sigma - \gamma + \sum_{l=0}^{\infty} \frac{C^{\Sigma}_{-2l-1}}{m^{2l+1}}$$

[Grover - Turner - Vishwanath 11]

$$\begin{cases} m : \text{mass gap} \\ l_\Sigma : \text{Area of } A \end{cases}$$



$$\boxed{C^{\Sigma}_{-2l-1} = \int_\Sigma F(K)}$$

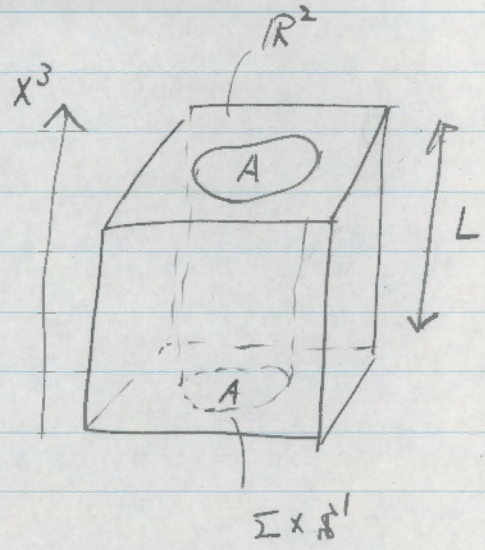
$$F(K) = F(-K) \qquad (S_A = S_{\bar{A}})$$

- Fix C^{Σ}_{-1} for free fields

[Klebanov-TN-Pufu-Safdi 12]

Dimensional reduction

$$\mathbb{R}^{2,1} \times S^1 \to \mathbb{R}^{2,1}$$



$$S^{(3+1)}_{\Sigma \times S^1} = \sum_{n \in \mathbb{Z}} S^{(2+1)}_{\Sigma} \left(m = \left| \frac{2\pi n}{L} \right| \right)$$

$\uparrow$ KK mass

$\cup$

$$\left(\# a\, \chi(\Sigma \times S^1) + \# c \int_{\Sigma \times S^1} K \right) \times \log \epsilon \supset \# \sum_{n \in \mathbb{Z}} \frac{C^{\Sigma}_{-1}}{|m|} \xrightarrow[L \to \infty]{\#} C^{\Sigma}_{-1} \int^{1/\epsilon} dp \, \frac{1}{p} = \# C^{\Sigma}_{-1} \log \epsilon$$

$$\Rightarrow \boxed{C^{\Sigma}_{-1} = \frac{1}{480} (n_0 + 3 n_{1/2}) \oint_{\Sigma} ds \, K^2}$$

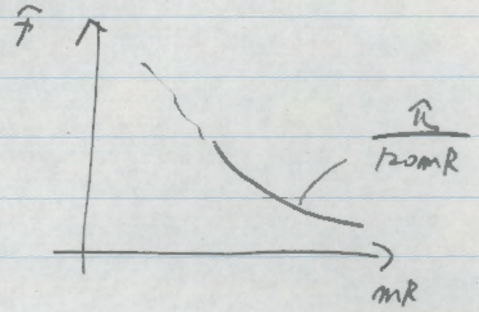
n_0: scalar, $n_{1/2}$: fermion

K: extrinsic curvature of Σ

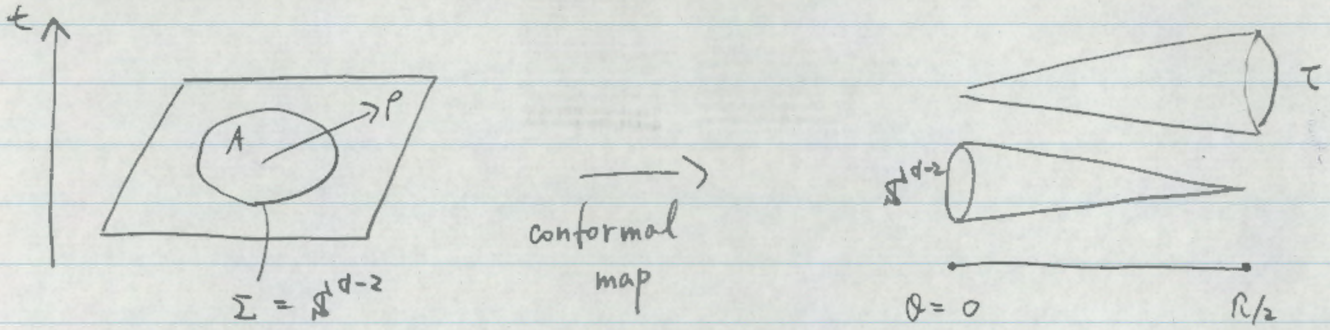
- Applying to $\Sigma = S^1$ of radius R $\left(K = \frac{1}{R} \right)$

$$\boxed{\tilde{\mathcal{F}}(R) = \frac{\pi}{120\, mR} + O\left(1/(mR)^3 \right)}$$

monotonically decreases as $mR \to \infty$



5. Supersymmetric Rényi entropy



$\mathbb{R}^d$

S^d

$$ds^2 = dt^2 + d\rho^2 + \rho^2 d\Omega_{d-2}^2 \qquad\qquad ds^2 = d\theta^2 + \sin^2\theta\, d\tau^2 + \cos^2\theta\, d\Omega_{d-2}^2$$

For CFT_d

$$Z_n[\mathbb{R}^d] = Z[S^d_n]$$

$\leftarrow$ n-branched covering of S^d

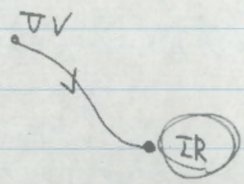
Rényi entropy across $\Sigma = S^{d-2}$ in CFT

$$S_n = \frac{1}{1-n} \log \frac{Z[S^d_n]}{(Z[S^d])^n}$$

When can we compute $Z[S^d_n]$?

- Free fields : $Z[S^d_n]$ is a one-loop determinant
  [Klebanov - Pufu - Sachdev - Safdi 11]

- SUSY gauge theories :



exact $Z[S^d]$ $(n=1)$ by localization

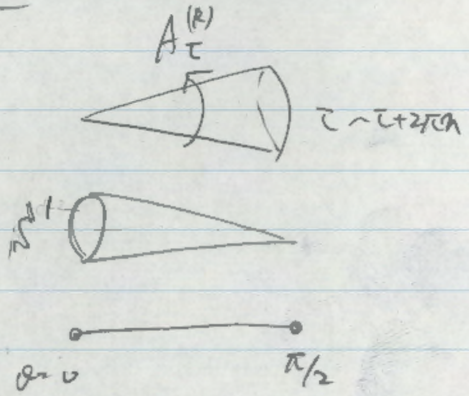
3d $\mathcal{N}=2$ [Kapustin - Willett - Yaakov 09, Jafferis, Hama - Hosomichi - Lee 10]

4d $\mathcal{N}=2$ [Pestun 07]

5d $\mathcal{N}=1$ [Källen - Zabzine 12]

- SUSY is broken on S_n^d!

- To recover SUSY, turn on background fields



· d = 3 $U(1)_R$ gauge field

$$A^{(R)} = \frac{n-1}{2} d\tau$$

- Two choices
  - no b.g. fields (usual RE)
    $\to$ ~~SUSY~~ $\to$ no localization
  - b.g. gauge field
    $\to$ Supersymmetric Rényi entropy $\to$ localization

<u>Def</u> (Super RE) [TN-Yaakov 13] $\leftarrow$ SUSY partition function on S_n^d

$$S_n^{susy} = \frac{1}{1-n} \log \left| \frac{Z^{susy}[S_n^d]}{(Z^{susy}[S^d])^n} \right|$$

<u>$Z^{susy}[S_n^d]$</u>

- d = 3, $\mathcal{N}=2$: $Z^{susy}[S_b^3]$

  S_b^3: squashed sphere w/ $b = \sqrt{n}$

- d = 4, $\mathcal{N}=2$: $Z^{susy}[S^4_{\epsilon_1, \epsilon_2}]$ [Huang-Zhou, Crossley-Dyer-Sonner 14]

  $\uparrow$ ellipsoid w/ $\epsilon_1/\epsilon_2 = n$

- d = 5, $\mathcal{N}=1$: $Z^{susy}[S^5_{\omega_1, \omega_2, \omega_3}]$

  $\uparrow$ squashed S^5 w/ $\omega_1 = 1/n$, $\omega_2 = \omega_3 = 1$

[Alday-Richmond-Sparks, Hama-TN-Ogaiz 14]

Properties

- $S_1^{susy} = S_1$ (EE across S^{d-2})

- $d=3$, $n \sim 1$

$$S_n^{susy} = S_1 + \frac{\pi^2}{16} \tau_{rr} (n-1) + \cdots$$

$\langle j_\mu^{(R)}(x) \, j_\nu^{(R)}(0) \rangle \sim \tau_{rr} I_{\mu\nu}(x)$

$U(1)_R$ current

- Large-N limits

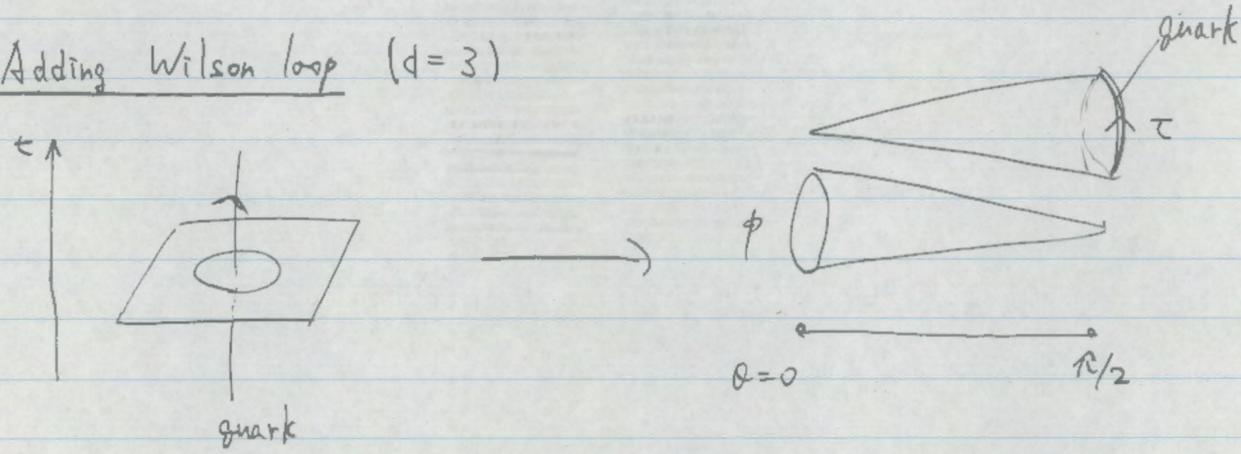
• $d=3$ $\mathcal{N}=2$, $d=4$ $\mathcal{N}=4$

$$S_n^{susy} = \frac{3n+1}{4n} S_1$$

∘ $d=4$ $\mathcal{N}=2$, $d=5$ $\mathcal{N}=1$

$$S_n^{susy} = \frac{19n^2+7n+1}{27n^2} S_1$$

Adding Wilson loop (d = 3)



Quark insertion := Wilson loop [Lewkowycz - Maldacena 13]

$$S^{susy}_{W,n} = \frac{1}{1-n}\left(n \log |\langle W\rangle_1| - \log|\langle W\rangle_n| \right)$$

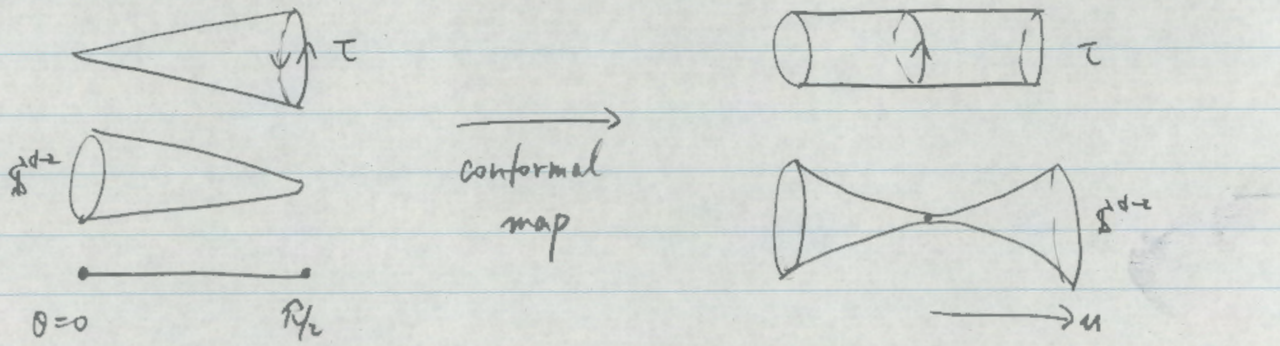
- 1/6 - BPS Wilson loop in ABJM on S^3_n [TN 14]

$$\log|\langle W\rangle_n| \underset{N\to\infty}{=} \frac{\pi(1+n)}{2}\sqrt{2\lambda} + O(\log N)$$

$$\lambda \equiv N/k$$

$$\boxed{S^{susy}_{W,n} = \frac{\pi}{2}\sqrt{2\lambda}}$$ indep of n

Holographic SRE



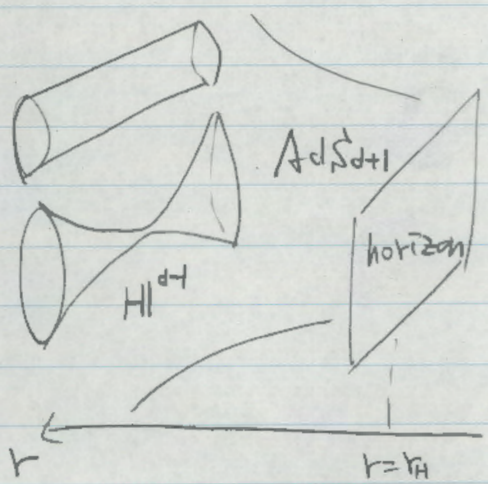
S^d $\qquad\qquad$ $S^1 \times H^{d-1}$

$$ds^2 = d\tau^2 + du^2 + \sinh^2 u \, d\Omega^2_{d-2}$$

- The gravity dual

  BPS charged topological BH in AdS_{d+1}

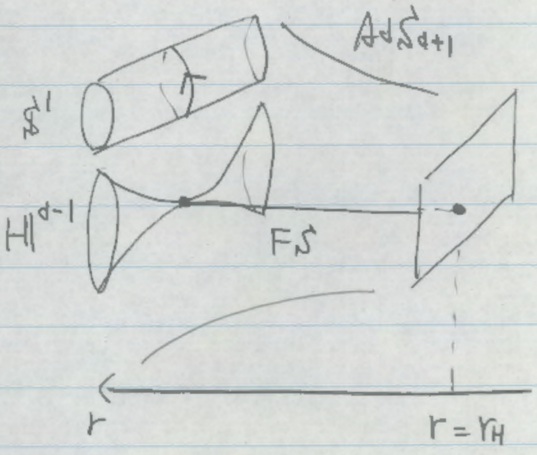
  at $T_{BH} = \dfrac{1}{2\pi n}$



- Reproduces the large-N results!

$Q = M$

Holographic Wilson loop



- Wilson loop is dual to the fundamental string

- reproduces the large-N result in 3d and 5d

6. Summary

- EE as a measure of DOF in QFT

- A susy extention of the Rényi entropy is implemented by introducing b.g. fields

- SRE provides an exact value for interacting theories at IR