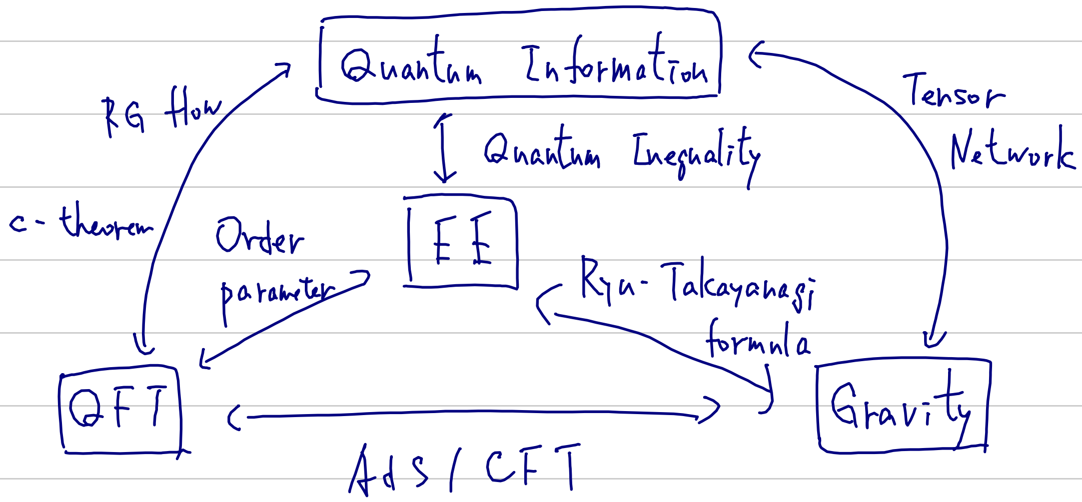
2017 . 2 . 12

"Entanglement in Quantum Field Theories"

Advanced String School
@ Puri

1. Introduction



Entanglement in QFT

- can be defined for any QFT in any dim
- capture global correlation = entanglement
  (non-local like Wilson loop)
- can be an order parameter for phase transitions
  (conf / deconf, quantum PT.)
- measures effective degrees of freedom
  under RG flow
  (Entropic c-theorem)

Outline

2. Basics of EE in quantum mechanics

3. EE in QFT

4. Holographic EE

5. Application to RG flow

Three approaches to EE

- Hamiltonian approach (§2)
  suits for numerical computation

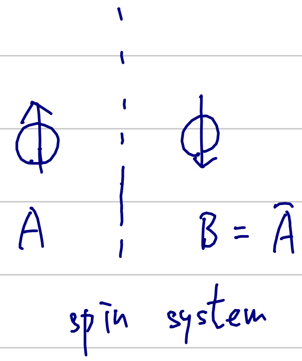
- Path integral ~ (§3)
  most convenient for QFTs

- Holographic ~ (§4)
  simple, efficient when applicable

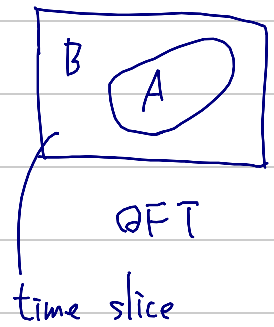
2. Basics of EE

2.1 Definition

For a subsystem A



assume $\boxed{\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}}$



For given orthonormal bases

$$\mathcal{H}_{A/B} = \{ |\phi^1_{A/B}\rangle , |\phi^2_{A/B}\rangle , \cdots , |\phi^i_{A/B}\rangle \cdots \}$$

Def (Reduced density matrix)

$$\rho_A \equiv \mathrm{tr}_B \, \rho_{tot} \equiv \sum_i^{d_B} \langle \phi^i_B | \rho_{tot} | \phi^i_B \rangle$$

$$(d_B \equiv \dim \mathcal{H}_B)$$

Remark

- ρ_A can reproduce all observations localized in A

$$\mathrm{tr}_A [\mathcal{O}_A \rho_A] = \mathrm{tr}_{tot} [(\mathcal{O}_A \otimes 1_B) \rho_{tot}]$$

- ρ_A can be a mixed state even if ρ_{tot} is pure

Example (Thermofield double)

$$|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_A \otimes |n\rangle_B$$

$$\left(Z = \sum_n e^{-\beta E_n} \right)$$

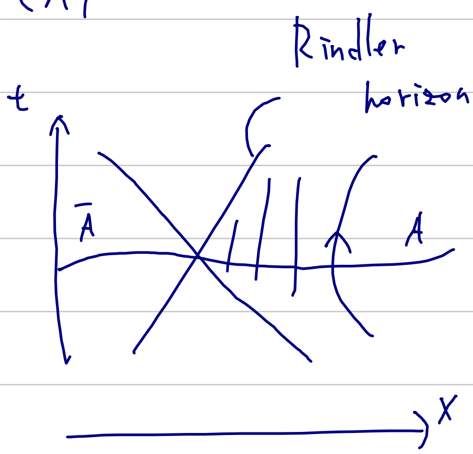
- Suppose a pure total system

$$\rho_{tot} = |\Psi_{TFD}\rangle\langle\Psi_{TFD}|$$

- Thermal state in subsystem A

$$\left(1_A \otimes {}_B\langle n|\right)|\Psi_{TFD}\rangle = \frac{1}{\sqrt{Z}} e^{-\frac{\beta}{2}E_n} |n\rangle_A$$

$$\Rightarrow \rho_A = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle_A {}_A\langle n|$$



cf.) Rindler horizon

An accelerated observer on flat space

can observe only the outside of the Rindler horizon

$\Rightarrow$ thermal bath

To measure how much the state is mixed

<u>Def</u> (Entanglement Entropy)

$$S_A \equiv -\mathrm{tr}_A\, \rho_A \ln \rho_A$$

<u>Example (TFD)</u>

$$\rho_A = \frac{1}{Z}\begin{pmatrix} e^{-\beta E_0} & & 0 \\ & e^{-\beta E_1} & \\ 0 & & \ddots \end{pmatrix}$$

$$S_A = -\sum_n \frac{e^{-\beta E_n}}{Z} \log \frac{e^{-\beta E_n}}{Z}$$

$$= \sum_n \beta E_n \frac{e^{-\beta E_n}}{Z} + \underset{\displaystyle -F}{\underset{\shortparallel}{\log Z}}$$

$$= \beta \langle H \rangle - F \quad \text{(thermal entropy)}$$

$$= -(\beta \partial_\beta - 1) \log Z$$

In general, a state can be written as

$$|\Psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} C_{ij} |\phi_A^i\rangle |\phi_B^j\rangle$$

① $\underline{C_{ij} = C_i^A C_j^B}$ (pure product state)

$\Rightarrow |\Psi\rangle = |\Psi_A\rangle |\Psi_B\rangle$, $|\Psi_{\substack{A\\B}}\rangle = \sum_i C_i^{A,B} |\phi^i_{A,B}\rangle$

$\Rightarrow \rho_A = |\Psi_A\rangle\langle\Psi_A|$, $\boxed{S_A = 0}$

② $\underline{C_{ij} \neq C_i^A C_j^B}$

$$C_{ij} = \underbrace{\left(\begin{array}{c} \cdots \\ \ddots \end{array}\right)}_{d_B} \Big\} d_A$$

singular value decomposition

$$C_{ij} = U_{ik} \lambda_k V_{kj} = U \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} V$$

U, V : unitaries

$\lambda_k \geq 0$, $k = 1, \cdots, \min(d_A, d_B)$

$\sum_k \lambda_k^2 = 1$

- Schmidt decomposition

$$|\Psi\rangle = \sum_k \lambda_k |\tilde{\phi}_A^k\rangle |\tilde{\phi}^k_B\rangle$$

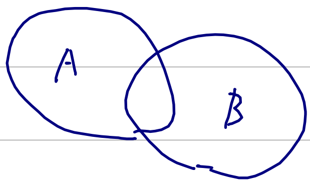
$$\Rightarrow \quad \boxed{S_A = -\sum_k \lambda_k^2 \log \lambda_k^2 = S_B}$$

- S_A is maximized for $\lambda_k = 1/\sqrt{\min(d_A, d_B)}$

$$\boxed{S_A\big|_{max} = \log \min(d_A, d_B)} \qquad \text{maximally entangled}$$

2.2 Some properties and variants

- Strong subadditivity [Lieb-Ruskai '73]



$$\boxed{S_{A\cup B} + S_{A\cap B} \leq S_A + S_B}$$

Follows from unitarity

$$\Rightarrow \quad |S_A - S_B| \leq S_{A\cup B} \quad (\text{subadditivity})$$

- Rényi entropy

Def

$$S_n(A) = \frac{1}{1-n} \log \operatorname{tr}_A \rho_A^n \qquad \text{for } n \in \mathbb{N}$$

$$S_A = \lim_{n\to 1} S_n(A)$$

- Rényi inequalities

$$
\begin{cases}
\partial_n S_n \le 0 \\
\partial_n \left(\frac{n-1}{n} S_n \right) \ge 0 \cdots (1) \\
\partial_n \left((n-1) S_n \right) \ge 0 - (2) \\
\partial_n^2 \left((n-1) S_n \right) \le 0 \cdots (3)
\end{cases}
$$

These can be interpreted as "thermodynamic" stability

$$\underset{(2)}{\langle H \rangle}, \quad \underset{(1)}{\langle S \rangle}, \quad \underset{(3)}{\langle C \rangle} \ge 0$$

if we identify

$$n \longleftrightarrow \beta \quad \text{inv temperature}$$

$$H \equiv -\log \rho \longleftrightarrow \text{Hamiltonian}$$

$$Z(n) = \mathrm{tr}\left(e^{-nH}\right) \longleftrightarrow \text{P.F.}$$

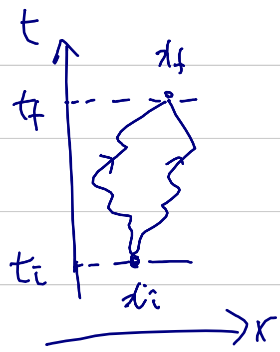
$$S = n^2 \partial_n \left(\frac{n-1}{n} S_n \right) \longleftrightarrow \text{entropy}$$

2.3 Path integral rep of $\mathcal{P}_A$

In QM, the transition amp. from x_i at time t_i to x_f at time t_f

$$\langle x_f ; t_f | x_i ; t_i \rangle = \langle x_f | e^{-i\hat{H}(t_f - t_i)} | x_i \rangle$$

$$= \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, e^{iI(x)}$$



- Wick rot : $t = -i\tau$

$$\langle x ; 0 | y ; -T \rangle = \langle x | e^{-\hat{H}T} | y \rangle$$

$$= \sum_n \psi_n(x) \psi_n^*(y) e^{-E_n T}$$

$$\xrightarrow[T\to\infty]{} \psi_0(x)\psi_0^*(y) \qquad (E_0 = 0)$$

$$\parallel$$

$$\int_{x(-\infty)=y}^{x(0)=x} \mathcal{D}x(\tau)\, e^{-I_E(\tau)}$$

$$\Rightarrow \boxed{\underset{\text{at } \tau=0}{\Psi(x)} \equiv \psi_0(x) = \underbrace{\int dy \int_{x(-\infty)=y}^{x(0)=x} \mathcal{D}x\, e^{-I_E(x)} \psi_0(y)}_{\displaystyle \parallel \; \int_{\tau=-\infty}^{x(0)=x} \mathcal{D}x \, e^{-I_E(x)}}}$$