Extremal Black Holes in String Theory

Motivation:

Low energy limit of string theory gives rise to gravity coupled to other fields.

These theories typically have black hole solutions.

Thus string theory gives a framework for studying classical and quantum properties of black holes.

Classically black holes are solutions of Einstein's equations with special properties.

They have a hypothetical surface – event horizon – surrounding them such that no object inside the event horizon can escape the black hole.

In quantum theory however the black hole behaves as a black body with finite temperature, entropy etc.

One of the important properties characterizing a black hole is the Bekenstein-Hawking entropy S_{BH}.

In the low energy limit

$$S_{BH} = A/(4G_N)$$

A: Area of the event horizon

G_N: Newton's constant

Question: Can we understand this entropy from statistical viewpoint i.e. as logarithm of the number of quantum states associated with the black hole?

Much of the work on string theory black holes have been carried out for **extremal** black holes.

Extremal black holes are black holes with zero temperature.

As a result they do not radiate and are usually stable.

Often, but not always, extremal black holes are also invariant under certain number of supersymmetry transformations.

In that case they are called BPS black holes.

In string theory one finds that for a wide class of extremal black holes

$$S_{BH} = S_{stat}, \qquad S_{stat} \equiv \ln(Degeneracy)$$

Strominger, Vafa; . . .

This gives a good understanding of this entropy from microscopic viewpoint.

Given this success, it is natural to carry out our study of black holes to finer details.

String theory leads to

Einstein gravity + higher derivative terms

Typical higher derivative terms: Square and higher powers of Riemann tensor

What are the effects of higher derivative corrections on the black hole entropy?

Does the agreement continue to hold even after taking into account the effects of higher derivative corrections?

In order to attack this problem we need to open two fronts.

First of all we need to learn how to take into account the effect of the higher derivative terms on the computation of black hole entropy.

$\rightarrow$ topic of first set of lectures.

A.S. hep-th/0506177, 0508042

B. Sahoo, A.S., hep-th/0601228, 0603149, 0608182

D. Astefanesei, K. Goldstein, R. Jena, A.S., S. Trivedi, hep-th/0606244

But we also need to know how to calculate the statistical entropy to greater accuracy.

$\rightarrow$ involves precise computation of the degeneracy of states with a given set of charges.

$\rightarrow$ topic of last set of lectures

D. Jatkar, A.S., hep-th/0510147

J. David, D. Jatkar, A.S., hep-th/0602254, 0607155

J. David, A.S., hep-th/0605210

Earlier related work:

Dijkgraaf, Verlinde, Verlinde, hep-th/9607026

Shih, Strominger and Yin, hep-th/0505094

A general frameork for computing higher derivative corrections to black hole entropy has been developed by Wald. gr-qc/9307038

We shall try to use Wald's result to calculate higher derivative corrections to S_{BH} for **extremal** black holes.

How do we define extremal black holes in a higher derivative theory?

Take the clue from usual (super-)gravity.

We shall restrict our analysis to spherically symmetric extremal black holes in $D = 4$ although the results can be generalized to

1. Rotating black holes

2. Black holes in higher dimensions

Spherically symmetric black holes in $D = 4$:

Take Einstein-Maxwell theory

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

This theory has charged black hole solutions, known as Reissner-Nordstrom solution:

$$\begin{aligned} ds^2 &= -(1 - \rho_+/\rho)(1 - \rho_-/\rho) dt^2 \\ &\quad + \frac{d\rho^2}{(1 - \rho_+/\rho)(1 - \rho_-/\rho)} \\ &\quad + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2) \\ F_{\mu\nu} &= \cdots \end{aligned}$$

$$
\begin{aligned}
ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)dt^2 \\
&\quad +\frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\
&\quad +\rho^2(d\theta^2+\sin^2\theta d\phi^2)
\end{aligned}
$$

Extremal limit: $\rho_+ = \rho_-$

Define $\tau = t/\rho_+^2, \qquad r = \rho - \rho_+,$

$$
\begin{aligned}
ds^2 &= -\frac{r^2\rho_+^4}{(\rho_+ + r)^2}d\tau^2 + \frac{(\rho_+ + r)^2}{r^2}dr^2 \\
&\quad +(\rho_+ + r)^2(d\theta^2+\sin^2\theta d\phi^2)
\end{aligned}
$$

$$ds^2 = -\frac{r^2\rho_+^4}{(\rho_+ + r)^2}d\tau^2 + \frac{(\rho_+ + r)^2}{r^2}dr^2 + (\rho_+ + r)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Take the ‘near horizon limit’ as follows:

1. Change coordinates $r \to \lambda\, r$, $\tau \to \tau/\lambda$

2. Take the limit $\lambda \to 0$.

$$ds^2 = \rho_+^2(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$\to$ near horizon geometry $AdS_2 \times S^2$

$$ds^2 = \rho_+^2(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

1. Since for any λ we have an exact classical solution, in the $\lambda \to 0$ limit also we have an exact classical solution.

2. Besides the usual SO(3) spherical symmetry, this background has an SO(2,1) symmetry that acts on the r, τ coordinate, generated by

$$L_1 = \partial_\tau, \quad L_0 = \tau\partial_\tau - r\partial_r$$

$$L_{-1} = \frac{1}{2}\left(\frac{1}{r^2 + \tau^2}\right)\partial_\tau - \tau\, r\, \partial_r$$

The complete near horizon solution:

$$ds^2 = \rho_+^2(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$F_{rt} = \frac{q}{4\pi}, \qquad F_{\theta\phi} = \frac{p}{4\pi}\sin\theta$$

$$\rho_+^2 = G_N \, \frac{q^2 + p^2}{4\pi}$$

q, p: label electric and magnetic charges

The full background has $SO(2,1)\times SO(3)$ isometry.

All known extremal non-rotating black holes in four dimensions with non-singular horizon have near horizon geometry $AdS_2 \times S^2$.

(These include some solutions in the presence of certain higher derivative terms.)

As a consequence the near horizon field configuration has isometry

$$SO(2,1) \times SO(3)$$

We shall take this as the definition of extremal black holes.

In $D = 4$ we define an extremal non-rotating black hole to be one whose near horizon geometry and other field configurations have

$$SO(2,1) \times SO(3)$$

isometry.

Generalizations:

1. A rotating extremal black hole in D=4 has near horizon geometry with $SO(2,1) \times U(1)$ isometry

2. A non-rotating extremal black hole for general D has near horizon geometry with $SO(2,1) \times SO(D-1)$ isometry.

The entropy of an extremal black hole

$\equiv$ entropy of a non-extremal black hole in the extremal limit.

We shall now use Wald's formula for the entropy for a non-extremal black hole with regular horizon.

Consider an arbitrary general coordinate invariant theory of gravity coupled to a set of Maxwell fields $A_\mu^{(i)}$ and neutral scalar fields $\{\phi_s\}$.

The most general form of the near horizon geometry of an extremal black hole consistent with $SO(2,1) \times SO(3)$ isometry:

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$\phi_s = u_s$$

$$F_{rt}^{(i)} = e_i, \qquad F_{\theta\phi}^{(i)} = \frac{p_i}{4\pi} \sin\theta \,,$$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right)$$
$$+ v_2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$
$$\phi_s = u_s \qquad F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \,,$$

v_1, v_2: sizes of AdS_2 and S^2

u_s: scalar field values at the horizon.

$p_i/4\pi$: near horizon radial magnetic field

e_i: near horizon radial electric field

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right)$$
$$+v_2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$
$$\phi_s = u_s \qquad F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi}\sin\theta\,,$$

$$R_{\alpha\beta\gamma\delta} = -v_1(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \quad \alpha,\beta,\gamma,\delta = r,t$$

$$R_{mnpq} = v_2(g_{mp}g_{nq} - g_{mq}g_{np}), \quad m,n,p,q = \theta,\phi$$

For this background covariant derivatives of the Riemann tensor, scalar fields and gauge field strengths vanish.

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$$\phi_s = u_s$$

$$F^{(i)}_{rt} = e_i, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \,,$$

Let $\sqrt{-\det g}\,\mathcal{L}$ be the Lagrangian density.

Define:

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \, \sqrt{-\det g}\,\mathcal{L}$$

$$\mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) \equiv 2\,\pi(e_i\, q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p}))$$

Results:

For an extremal black hole of electric charge $\vec{q}$ and magnetic charge $\vec{p}$,

1. the values of $\{u_s\}$, $\{e_i\}$, v_1 and v_2 are obtained by extremizing $\mathcal{E}(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p})$ with respect to these variables.

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0$$

2. $S_{BH} = \mathcal{E}$ at the extremum.

1. $\dfrac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \dfrac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \dfrac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \dfrac{\partial \mathcal{E}}{\partial e_i} = 0$

follow from applying equations of motion on the near horizon background and the definition of electric charge:

$$q_i = \int d\theta d\phi \sqrt{-\det g} \left(\delta \mathcal{L} / \delta F^{(i)}_{rt} \right)$$

2.

$$S_{BH} = \mathcal{E}$$

follows from manipulation of Wald's formula for black hole entropy.

To summarize, the single ‘entropy function’ $\mathcal{E}$ determines

– the near horizon values $\{u_s\}$ of the scalar fields,

– the sizes v_1, v_2 of AdS_2 and S^2

– the gauge field strengths $\{e_i\}$

– the entropy S_{BH}

These results are useful for explicit calculations as well as proving general results.

Computation of the entropy of extremal Reissner-Nordstrom black holes

Take Einstein-Maxwell theory in $D = 4$:

$$\mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Consider an extremal black hole solution with near horizon geometry:

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$$F_{rt} = e, \qquad F_{\theta\phi} = p \sin\theta / 4\pi$$

Then

$$
\begin{aligned}
f(v_1, v_2, e, p) &= \int d\theta d\phi \sqrt{-\det g}\, \mathcal{L} \\
&= 4\pi\, v_1 v_2 \left[\frac{1}{16\pi G_N} \left(-\frac{2}{v_1} + \frac{2}{v_2} \right) \right. \\
& \qquad \left. + \frac{1}{2}\, v_1^{-2} e^2 - \frac{1}{2} v_2^{-2} \left(\frac{p}{4\pi} \right)^2 \right] .
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(v_1, v_2, e, q, p) &= 2\pi (q\, e - f) \\
&= 2\pi \left[q\, e - \frac{1}{4 G_N} (2 v_1 - 2 v_2) \right. \\
& \left. -2\pi\ v_2\, v_1^{-1}\, e^2 + 2\pi\, v_1\, v_2^{-1} \left(\frac{p}{4\pi} \right)^2 \right] .
\end{aligned}
$$

$$\mathcal{E}(v_1, v_2, e, q, p) = 2\pi \left[q\, e - \frac{1}{4G_N}(2v_1 - 2v_2) \right.$$
$$\left. -2\pi\ v_2\, v_1^{-1}\, e^2 + 2\pi\, v_1\, v_2^{-1} \left(\frac{p}{4\pi}\right)^2 \right] .$$

$\partial\mathcal{E}/\partial e = 0, \quad \partial\mathcal{E}/\partial v_1 = 0, \quad \partial\mathcal{E}/\partial v_2 = 0$ gives

$$q = 4\pi\, v_2 v_1^{-1} e, \qquad v_1 = v_2 = G_N\, \frac{q^2 + p^2}{4\pi} .$$

$$S_{BH} = \mathcal{E} = \frac{1}{4}(q^2 + p^2)$$

$\rightarrow$ correct answer for the entropy of extremal charged black holes.

The entropy function formalism has been successfully used for calculating higher derivative corrections to the entropy of many extremal black holes.

Exercise: Add a term to $\mathcal{L}$ of the form

$$\lambda\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Find the change in the entropy and the solution to first order in λ.

The entropy function leads to a general proof of the 'attractor mechanism' for extremal black holes.

If we have a theory with scalar fields which have no potential then asymptotically the scalar fields can have arbitrary values.

Attractor mechanism $\leftrightarrow$ the entropy of a black hole with a given set of charges is independent of this asymptotic data.

This was initially observed in cases of certain supersymmetric black holes.

Proof of attractor mechanism:

If $\mathcal{E}$ has no flat directions then the extremization of $\mathcal{E}$ determines the near horizon parameters $\vec{u}$, $\vec{v}$, $\vec{e}$ completely in terms of $\vec{q}$, $\vec{p}$.

In this case the complete near horizon field configuration as well as $S_{BH} = \mathcal{E}$ is independent of all other asymptotic data *e.g.* values of the moduli scalar fields.

$\rightarrow$ attractor behaviour

If $\mathcal{E}$ has flat directions, then extremization of $\mathcal{E}$ does not determine $\vec{u}$, $\vec{v}$, $\vec{e}$ uniquely and there is a continuous family of extrema.

Thus these could depend on additional asymptotic data *e.g.* values of the moduli scalar fields.

But since $\mathcal{E}$ does not depend on the flat directions, $S_{BH} = \mathcal{E}$ is still determined in terms of $\vec{q}$, $\vec{p}$ and is independent of the asymptotic values of the scalar fields.

An open issue

The entropy function calculates the entropy assuming that there is an extremal black hole solution.

However it does not address the issue of whether there is really an interpolating solution between the asymptotically flat geometry and the near horizon $AdS_2 \times S^{D-2}$ geometry.

Physically, the moduli fields reach the attractor values due to spatial evolution along the infinite throat of AdS_2 according to the equations of motion.

Is the $AdS_2 \times S^{D-2}$ geometry a stable attractor?

For two derivative actions this issue has been answered by Goldstein, Iizuka, Jena, Trivedi

Answer:

$$\frac{\partial^2 \mathcal{E}}{\partial u_s \partial u_{s'}}$$

must be a positive definite matrix at the critical point.

Is there a generalization of this result to higher derivative action?

Derivation of the results:

1. $f = \int d\theta d\phi \sqrt{-\det g}\,\mathcal{L}$

$\rightarrow$ requiring the action to be stationary along the directions of v_1, v_2 and u_s deformations gives

$$\frac{\partial f}{\partial u_s} = 0, \qquad \frac{\partial f}{\partial v_1} = 0\,, \qquad \frac{\partial f}{\partial v_2} = 0\,.$$

The $SO(2,1) \times SO(3)$ invariance of the background

$\rightarrow$ these are the only independent components of the metric and the scalar field equations.

2. Non-trivial part of the gauge field equations and Bianchi identities:

$$\partial_r \left(\frac{\delta \mathcal{S}}{\delta F^{(i)}_{rt}} \right) = 0, \qquad \partial_r F^{(i)}_{\theta\phi} = 0 \ .$$

Evaluate the integration constants at $r \to \infty$

$\to$ they are proportional to the electric charges q_i and the magnetic charges p_i.

When evaluated on the near horizon geometry, this gives

$$\frac{\partial f}{\partial e_i} = q_i, \qquad \int d\theta \, d\phi F^{(i)}_{\theta\phi} = p_i$$

Thus for given $\vec{q}$ and $\vec{p}$ the equations determining the background are:

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_1} = 0\,, \quad \frac{\partial f}{\partial v_2} = 0, \quad \frac{\partial f}{\partial e_i} = q_i$$

f is a function of $\vec{u}$, $\vec{v}$, $\vec{e}$ and $\vec{p}$.

Now recall the definition of $\mathcal{E}$:

$$\mathcal{E}(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p}) = 2\,\pi\,(\vec{q}\cdot\vec{e} - f(\vec{u},\vec{v},\vec{e},\vec{p}))$$

The equations of motion are equivalent to:

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0, \quad \frac{\partial \mathcal{E}}{\partial v_1} = 0\,, \quad \frac{\partial \mathcal{E}}{\partial v_2} = 0, \quad \frac{\partial \mathcal{E}}{\partial e_i} = 0$$

Computation of entropy:

Higher derivative terms $\rightarrow$ the entropy is no longer given by the area law.

Wald; Iyer and Wald; Jacobson, Kang, Myers

For spherically symmetric black holes:

$$S_{BH} = -8\pi \int_H d\theta \, d\phi \, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr} \, g_{tt}} \, ,$$

In computing $\delta\mathcal{S}/\delta R_{\mu\nu\rho\sigma}$

1. express the action $\mathcal{S}$ in terms of symmetrized covariant derivatives of fields

2. treat $R_{\mu\nu\rho\sigma}$ as independent variables.

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \frac{\delta \mathcal{S}}{\delta R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

For our background $D_\mu \phi_s$, $D_\rho F^{(i)}_{\mu\nu}$ and $D_\tau R_{\mu\nu\rho\sigma}$ all vanish.

$\rightarrow$ can ignore all terms in $\mathcal{L}$ which involve covariant derivatives of ϕ_s, $F^{(i)}_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$.

Thus:

$$S_{BH} = -8\pi \int_H d\theta\, d\phi\, \sqrt{-\det g}\, \frac{\partial \mathcal{L}}{\partial R_{rtrt}} \sqrt{-g_{rr}\, g_{tt}}\,,$$

Define:

1. $\mathcal{L}_\lambda = \mathcal{L}$ with each factor of $R_{\alpha\beta\gamma\delta}$ in $\mathcal{L}$ multiplied by λ. $(\alpha, \beta, \gamma, \delta = r, t)$

2.

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta \, d\phi \sqrt{-\det g} \, \mathcal{L}_\lambda$$

Then $f_{\lambda=1} = f$.

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta \, d\phi \sqrt{-\det g} \, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}} R_{rtrt}$$

$$\lambda \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} = 4 \int d\theta\, d\phi\, \sqrt{-\det g}\, \frac{\partial \mathcal{L}_\lambda}{\partial R_{rtrt}}\, R_{rtrt}$$

For our background $R_{rtrt} = \sqrt{-g_{rr}\, g_{tt}}$

This gives

$$S_{BH} = -2\pi \left. \frac{\partial f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p})}{\partial \lambda} \right|_{\lambda=1} .$$

We shall now try to write $\partial f_\lambda / \partial \lambda$ in terms of derivatives of f_λ with respect to v_1 and e_i.

Collect all possible dependence of f on v_1

1. Every factor of λ in $\mathcal{L}$ must appear in the combination

$$\lambda\, g^{rr} g^{tt} R_{rtrt} = \lambda v_1^{-1}$$

2. Every $F^{(i)}_{rt}$ in $\mathcal{L}$ must appear in the combination:

$$\sqrt{-g^{rr} g^{tt}}\, F^{(i)}_{rt} = e_i v_1^{-1}$$

3. $R_{\theta\phi\theta\phi}$, $F^{(i)}_{\theta\phi}$ and u_s do not have any accompanying factor of v_1 (size of AdS_2).

4. There are no covariant derivatives contracting with the metric.

5. the only other v_1 dependence of f_λ comes from $\sqrt{-\det g}$ multiplying $\mathcal{L}$.

Result:

$$f_\lambda(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = v_1 g(\vec{u}, v_2, \vec{p}, \lambda\, v_1^{-1}, e_i v_1^{-1})$$

for some function g.

$$f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p}) = v_1 g(\vec{u}, v_2, \vec{p}, \lambda v_1^{-1}, e_i v_1^{-1})$$

$$\downarrow$$

$$\begin{aligned} \lambda\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial\lambda} + v_1\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial v_1} & \\ + e_i\frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial e_i} - f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p}) &= 0\,. \end{aligned}$$

For $\lambda = 1$, $f_\lambda \to f$.

Equation of motion $\to \partial f/\partial v_1 = 0$.

$$\lambda\partial f_\lambda/\partial\lambda|_{\lambda=1} = f - e_i\partial f/\partial e_i$$

Thus

$$
\begin{aligned}
S_{BH} &= -2\pi \left. \frac{\partial f_\lambda(\vec{u},\vec{v},\vec{e},\vec{p})}{\partial \lambda} \right|_{\lambda=1} \\
&= 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f \right) = 2\pi (e_i q_i - f) \\
&= \mathcal{E}(\vec{u},\vec{v},\vec{e},\vec{q},\vec{p}) \,.
\end{aligned}
$$

$\rightarrow$ the desired result.

Application: Heterotic string theory on T^6.

$\{y^n\} \equiv$ coordinates along T^6 ($4 \leq n \leq 9$)

$\{x^\mu\} \equiv$ coordinates along non-compact directions ($0 \leq \mu \leq 3$)

Massless fields at a generic point in the moduli space:

1. The string metric $G_{\mu\nu}$ $\qquad 0 \leq \mu, \nu \leq 3$

2. 28 U(1) gauge fields $A_\mu^{(i)}$, $\qquad 1 \leq i \leq 28$

3. 28×28 matrix valued scalar field M satisfying

$$M^T L M = L, \qquad M^T = M,$$

$$L = \begin{pmatrix} & I_6 & \\ I_6 & & \\ & & -I_{16} \end{pmatrix}$$

I_k: $k \times k$ identity matrix

4. Dilaton-axion field (S, a)

$\langle S \rangle$ = inverse string coupling constant^2

Canonical metric $g_{\mu\nu} = S\, G_{\mu\nu}$

Physical origin of various fields:

Denote by A^I_M the 16 U(1) gauge fields in the ten dimensional theory

M comes from G_{mn}, B_{mn} and A^I_n

$(4 \leq m, n \leq 9, \quad 1 \leq I \leq 16)$

a comes from dualization of $B_{\mu\nu}$

$S \equiv e^{-2\Phi}$, where Φ is the dilaton

$A^{(i)}_\mu$ come from components $G_{n\mu}$, $B_{n\mu}$, A^I_μ

$$
\begin{aligned}
\text{Action} &= \int d^4x\,\sqrt{-\det G}\,\mathcal{L} \\
\mathcal{L} &= \frac{1}{32\pi}\,S\,\Big[R_G + \frac{1}{S^2}\,G^{\mu\nu}(\partial_\mu S\partial_\nu S - \frac{1}{2}\partial_\mu a\partial_\nu a) \\
&\quad + \frac{1}{8}G^{\mu\nu}Tr(\partial_\mu ML\partial_\nu ML) \\
&\quad - G^{\mu\mu'}G^{\nu\nu'}\,F^{(i)}_{\mu\nu}(LML)_{ij}F^{(j)}_{\mu'\nu'} \\
&\quad - \frac{a}{S}G^{\mu\mu'}G^{\nu\nu'}\,F^{(i)}_{\mu\nu}L_{ij}\widetilde{F}^{(j)}_{\mu'\nu'}\Big] + \cdots
\end{aligned}
$$

$$
F^{(i)}_{\mu\nu} \equiv \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu
$$

Near horizon field configuration:

$$ds^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) + v_2 (d\theta^2 + \sin^2\theta \, d\phi^2) \,,$$

$$S = u_S, \qquad a = u_a, \qquad M_{ij} = u_{Mij}$$

$$F^{(i)}_{rt} = e_i, \quad F^{(i)}_{\theta\phi} = \frac{p_i \sin\theta}{4\pi}$$

This gives

$$\begin{aligned} & f(u_S, u_a, u_M, \vec{v}, \vec{e}, \vec{p}) \equiv \int d\theta d\phi \sqrt{-\det G} \, \mathcal{L} \\ = \; & \frac{1}{8} v_1 \, v_2 \, u_S \left[-\frac{2}{v_1} + \frac{2}{v_2} + \frac{2}{v_1^2} e_i (L u_M L)_{ij} e_j \right. \\ & \left. -\frac{1}{8\pi^2 v_2^2} p_i (L u_M L)_{ij} p_j + \frac{u_a}{\pi u_S v_1 v_2} e_i L_{ij} p_j \right] \end{aligned}$$

$$q_i \equiv \frac{\partial f}{\partial e_i} = \frac{v_2 u_S}{2v_1}(L u_M L)_{ij} e_j + \frac{u_a}{8\pi} L_{ij} p_j \,,$$

and

$$\begin{aligned}
&\mathcal{E}(u_S, u_a, u_M, \vec{v}, \vec{q}, \vec{p}) \\
\equiv\ & 2\pi\left(e_i q_i - f(u_S, u_a, u_M, \vec{v}, \vec{e}, \vec{p})\right) \\
=\ & 2\pi\Big[\frac{u_S}{4}(v_2 - v_1) + \frac{v_1}{v_2 u_S}\, q^T u_M q \\
& + \frac{v_1}{64\pi^2 v_2 u_S}(u_S^2 + u_a^2) p^T L u_M L p \\
& - \frac{v_1}{4\pi v_2 u_S}\, u_a\, q^T u_M L p\Big]
\end{aligned}$$

Define:

$$Q_i = 2q_i, \qquad P_i = \frac{1}{4\pi} L_{ij} p_j,$$

Then

$$\begin{aligned}\mathcal{E} &= \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2 u_S}\Big(Q^T u_M Q \\ &\quad + (u_S^2 + u_a^2)\, P^T u_M P - 2\, u_a\, Q^T u_M P\Big)\Big]\end{aligned}$$

We shall now try to find a solution by extremizing $\mathcal{E}$ with respect to u_M, u_S, u_a, v_1 and v_2.

$$\mathcal{E} = \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2 u_S}\Big(Q^T u_M Q + (u_S^2 + u_a^2)\, P^T u_M P - 2\, u_a\, Q^T u_M P\Big)\Big]$$

Note: $\mathcal{E}$ is invariant under

$$Q \to \Omega Q, \quad P \to \Omega P, \quad u_M \to (\Omega^T)^{-1} u_M \Omega^{-1}$$

for Ω satisfying

$$\Omega L \Omega^T = L$$

$\to$ continuous T-duality transformation.

$$Q \to \Omega Q, \quad P \to \Omega P, \quad u_M \to (\Omega^T)^{-1} u_M \Omega^{-1}$$

Thus after extremization with respect to u_M, the entropy function will depend on P and Q only through the T-duality invariant combinations:

$$Q^2 \equiv Q^T L Q, \qquad P^2 \equiv P^T L P, \qquad Q{\cdot}P \equiv Q^T L P$$

We can carry out the analysis for various ranges of P^2, Q^2 and $Q \cdot P$ by choosing suitable representative Q and P.

Result of elimination of u_M for $(Q{\cdot}P)^2 < Q^2P^2$:

$$\mathcal{E} = \frac{\pi}{2}\left[u_S(v_2 - v_1) + \frac{v_1}{v_2}\left(\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q \cdot P\right)\right]$$

Now eliminate v_1, v_2, u_S, u_a.

Result:

$$v_1 = v_2 = 2P^2 \,, \quad u_S = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2} \,,$$

$$u_a = \frac{Q\cdot P}{P^2}$$

$$S_{BH} = \mathcal{E} = \pi \sqrt{Q^2P^2 - (Q\cdot P)^2}$$

Note: Large $P^2 \to v_1, v_2 >> 1$

$\to \alpha'$ expansion is valid.

$Q^2P^2 - (Q\cdot P)^2 >> (P^2)^2 \to u_S >> 1$

$\to$ string loop expansion is valid.

The extremal black holes for $P^2Q^2 > (Q \cdot P)^2$ are known to be supersymmetric.

Using the entropy function one can also study black holes for which $P^2Q^2 < (Q \cdot P)^2$.

Result for the entropy:

$$S_{BH} = \mathcal{E} = \pi \sqrt{(Q \cdot P)^2 - Q^2P^2}$$

Generalization to CHL string theory:

These theories are obtained by modding out heterotic on T^6 by an appropriate orbifold group which preserves $\mathcal{N} = 4$ supersymmetry.

Result: The massless field content gets modified.

CHL models based on $\mathbb{Z}_N$ orbifolds

1. Begin with heterotic string theory on

$$T^4 \times S^1 \times \widehat{S}^1$$

T^4: A four torus

S^1, $\widehat{S}^1$: two circles with period 2π

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by

$2\pi/N$ shift along S^1 together with an order N internal symmetry of heterotic string theory on T^4.

Dual description

1. Begin with type IIA string theory on

$$K3 \times S^1 \times \widehat{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by

   $2\pi/N$ shift along S^1 together with an order N internal symmetry of type IIA string theory on $K3$.

A special class of values of N:

$$N = 1, 2, 3, 5, 7$$

$N = 1$: heterotic string theory on T^6.

For these theories the rank of the gauge group is

$$r = 2k + 8, \qquad k = \frac{24}{N+1} - 2$$

At a generic point in the modul space we have

1. The string metric $G_{\mu\nu}$.

2. r U(1) gauge fields $A_\mu^{(i)}$

3. $r \times r$ matrix valued scalar field M satisfying

$$M^T L M = L, \qquad M^T = M,$$

L: A matrix with six eigenvalues $+1$ and $r-6$ eigenvalues -1.

4. Dilaton-axion field (S, a)

The supergravity effective action has the same form as for toroidal compactification.

Thus we get the same entropy function and same values of the near horizon field configuration and same entropy.

$$u_S = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}\,, \qquad u_a = \frac{Q\cdot P}{P^2}\,,$$

$$v_1 = v_2 = 2P^2\,,$$

$$S_{BH} = \mathcal{E} = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}$$

Now consider the effect of a special type of higher derivative correction to the action:

$$\sqrt{-\det G}\Delta\mathcal{L}$$

$$= \frac{S}{16\pi}\sqrt{-\det g}\left\{R_{g\mu\nu\rho\sigma}R_g^{\mu\nu\rho\sigma} - 4R_{g\mu\nu}R_g^{\mu\nu} + R_g^2\right\}$$

$\rightarrow$ the Gauss-Bonnet term.

This term appears at the heterotic string tree level and is the same for toroidal compactification and CHL models.

Quantum corrected effective action contains additional terms.

A special class of corrections modify the tree level Gauss-Bonnet term to

$$\sqrt{-\det G}\,\Delta\mathcal{L}$$

$$= \phi_k(a,S)\,\sqrt{-\det g}\left\{R_{g\mu\nu\rho\sigma}R_g^{\mu\nu\rho\sigma} - 4R_{g\mu\nu}R_g^{\mu\nu} + R_g^2\right\}$$

$\phi_k(a,S)$: an S-duality invariant function which for large S behaves as $S/16\pi$.

$$
\begin{aligned}
\phi_k(a,S) &= -\frac{1}{64\pi^2}\Big((k+2)\ln S \\
&\quad + \ln f^{(k)}(a+iS) + \ln f^{(k)}(-a+iS)\Big) \\
f^{(k)}(\tau) &= \eta(\tau)^{k+2}\,\eta(N\tau)^{k+2} \\
k &= \frac{24}{N+1} - 2
\end{aligned}
$$

The addition of Gauss-Bonnet term in the action induces the following changes in f and $\mathcal{E}$:

$$\Delta f = -32\,\pi\,\phi_k(u_a, u_S)\,, \quad \Delta\mathcal{E} = 64\,\pi^2\,\phi_k(u_a, u_S)$$

Thus

$$\begin{aligned}\mathcal{E} + \Delta\mathcal{E} \;&=\; \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2 u_S}\Big(Q^T u_M Q \\ &+(u_S^2 + u_a^2)\,P^T u_M P - 2\,u_a\,Q^T u_M P\Big) \\ &+128\,\pi\,\phi_k(u_a, u_S)\Big]\end{aligned}$$

Elimination of u_M, v_1, v_2 can be done as before.

Elimination of u_M, v_1, v_2 gives, for $P^2Q^2 > (P\cdot Q)^2$

$$\mathcal{E}_{total} = \frac{\pi}{2}\Big[\left(\frac{Q^2}{u_S} + \frac{P^2}{u_S}(u_S^2 + u_a^2) - 2\,\frac{u_a}{u_S}\,Q\cdot P\right) + 128\,\pi\,\phi_k(u_a, u_S)\Big]$$

Final result for entropy is obtained by eliminating u_a and u_S by extremizing $\mathcal{E}_{total}$.

e.g. for $\phi_k(a, S) = S/16\pi$ as in tree level heterotic string theory, we get

$$S_{BH} = \pi\sqrt{Q^2P^2 - (Q \cdot P)^2}\sqrt{1 + \frac{8}{P^2}}$$

Recall: Tree level approximation is valid when $u_S >> 1$, i.e. $Q^2P^2 - (Q \cdot P)^2 >> (P^2)^2$.

Question: What is the effect of other four derivative terms on the entropy?

First focus on tree level terms.

Even in this case Gauss-Bonnet term is only a subset of all tree level four derivative terms.

What is the effect of other tree level four derivative terms?

This problem can be solved by including the set of all tree level four derivative correction terms in the Lagrangian. Sahoo,Sen;Exirifard

Result: Same as the one obtained by just using the Gauss-Bonnet term.

One can also give a general argument based on supersymmetry that tree level higher derivative terms do not modify the result. Kraus, Larsen

When Q and P are of same order, then keeping only tree level terms is not a useful approximation scheme.

Thus we need to include the full $\phi_k(a, S)$ as coefficient of the Gauss-Bonnet term.

However there are other four derivative corrections to the effective action.

What is their effect on the entropy?

Is there a non-renormalization theorem similar to that for the tree level result?

As of now there is no known entropy non-renormalization theorem for loop corrections in the heterotic theory.

We shall proceed with the assumption that at least at the level of four derivative terms, the result for entropy obtained by including the Gauss-Bonnet term is exact.

Question: Can we find an exact formula for the degeneracy $d(Q,P)$ of these dyonic back holes using a microscopic description and compare the black hole entropy with $\ln d(Q,P)$?

Computation of statistical entropy

1. Quarter BPS states in CHL models

2. Half BPS states in CHL models

CHL models based on $\mathbb{Z}_N$ orbifolds

1. Begin with heterotic string theory on

$$T^4 \times S^1 \times \widehat{S}^1$$

T^4: A four torus

S^1, $\widehat{S}^1$: two circles with period 2π

This theory has $\mathcal{N} = 4$ supersymmetry.

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 together with an order N internal symmetry of heterotic string theory on T^4 which commutes with $\mathcal{N} = 4$ susy.

Dual description

(based on heterotic on T^4 $\leftrightarrow$type IIA on K3)

1. Begin with type IIA string theory on

$$K3 \times S^1 \times \widehat{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 together with an appropriate order N internal symmetry of type IIA string theory on $K3$.

The resulting theory is $\mathcal{N} = 4$ supersymmetric.

A third description

(based on type IIA on $\widehat{S}^1 \leftrightarrow$ type IIB on $\widetilde{S}^1$)

1. Begin with type IIB string theory on

$$K3 \times S^1 \times \widetilde{S}^1$$

2. Take the orbifold by a $\mathbb{Z}_N$ group generated by $2\pi/N$ shift along S^1 together with an appropriate order N internal symmetry of type IIB string theory on $K3$.

The resulting theory is $\mathcal{N} = 4$ supersymmetric.

A special class of values of N:

$$N = 1, 2, 3, 5, 7$$

$N = 1$: heterotic string theory on T^6.

These theories have r different U(1) gauge fields where

$$r = 2k + 8, \qquad k = \frac{24}{N+1} - 2$$

Thus a generic state will be characterized by an r dimensional electric charge vector Q and r dimensional magnetic charge vector P.

Typically the notion of electric and magnetic charges get exchanged when we consider different descriptions of the theory.

In our convention we shall classify charges as electric or magnetic according to the heterotic description.

Similarly non-perturbative S-duality symmetry relating strong and weak coupling regions in one description may be realised as a perturbative T-duality symmetry in another description.

We shall classify symmeries as S- or T-dualities depending on their action in the heterotic description.

S-duality symmetry of CHL models

This is a non-perturbative symmetry from the heterotic view point since it maps the weak coupling region to finite or strong coupling.

$\rightarrow$ difficult to guess.

But this has a simple form in the third description based on type IIB on $K3 \times S^1 \times \widetilde{S}^1$

Consider the torus $S^1 \times \widetilde{S}^1$ labelled by σ_1 and σ_2, each with period 1.

This torus has a global diffeomorphism symmetry:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

$$ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z}$$

Only a subgroup of this symmetry that commutes with the $\mathbb{Z}_N$ transformation survives in the orbifold theory.

$$a, d = 1 \text{ mod } N, \quad c = 0 \text{ mod } N$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

$$ad - bc = 1, \qquad a, b, c, d \in \mathbb{Z}$$

$$a, d = 1 \text{ mod } N, \quad c = 0 \text{ mod } N$$

$-$ defines the group $\Gamma_1(N)$.

This appears as S-duality symmetry in the heterotic description.

Knowing the action of this transformation on the fields in type IIB theory we can find their action on the fields in the heterotic theory and hence their action on the electric and magnetic charges Q and P.

S-duality symmetry acts on (Q, P) as

$$\begin{pmatrix} Q \\ P \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

$$ad - bc = 1$$

$$a, b, c, d \in \mathbb{Z}, \quad a, d = 1 \text{ mod } N, \quad c = 0 \text{ mod } N$$

T-duality symmetry:

This is associated with $R \to 1/R$ and various other symmetries of the conformal field theory describing heterotic on $T^6/\mathbb{Z}_N$.

Under this symmetry transformation

$$Q \to \Omega Q, \qquad P \to \Omega P$$

where Ω is an $r \times r$ matrix satisfying

$$\Omega^T L \Omega = L$$

L is a fixed $r \times r$ matrix with 6 eigenvalues 1 and $(r-6)$ eigenvalues -1.

$$Q \to \Omega Q, \qquad P \to \Omega P$$

$$\Omega^T L \Omega = L$$

This describes a continuous group.

One gets further constraints on Ω by requiring that acting on a vector in the charge lattice it produces another vector in the charge lattice.

$\to$ makes this into a discrete group.

$$Q \to \Omega Q, \qquad P \to \Omega P$$

$$\Omega^T L \Omega = L$$

As a result

$$P^2 \equiv P^T L P, \qquad Q^2 \equiv Q^T L Q, \qquad P \cdot Q \equiv P^T L Q$$

are invariant under T-duality transformation.

Dyon Spectrum of CHL string theory

Consider a generic 1/4 BPS dyonic state in CHL string theory carrying r dimensional electric charge vector Q and magnetic charge vector P.

What is the degeneracy $d(Q,P)$ of these states?

I shall outline the general steps and give the result, but not describe the details of the calculation.

Significance of BPS states

1/4 BPS: States which are invariant under 4 of the 16 supersymmetry generators of the theory.

We use BPS states because the degeneracy of BPS states is robust, insensitive to the coupling constant and other parameters of the theory.

Classical field configuration produced by a BPS state typically describes an extremal black hole solution.

We shall derive the formula for $d(Q, P)$ for a specific class of charge vectors (Q, P).

Then we shall express the formula in terms of T-duality invariant combinations P^2, Q^2 and $Q \cdot P$.

i.e. we assume, but not prove the T-duality invariance of the formula.

We would like to verify that

for large charges $\ln d(Q, P)$ **matches black hole entropy.**

We shall also verify S-duality invariance of our formula.

$\rightarrow$ a consistency check.

The result for $d(Q,P)$

$$d(Q,P) \;=\; \frac{1}{N}\int_C d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right] ,$$

ρ, σ, v: complex parameters

The integration 'contour' C is defined to be the three real dimensional subspace:

$$Im\,\rho = M_1, \quad Im\,\sigma = M_2, \quad Im\,v = M_3,$$

$$0 \le Re\,\rho \le 1, \quad 0 \le Re\,\sigma \le N, \quad 0 \le Re\,v \le 1\,.$$

M_1, M_2, M_3: Large real constants

Expression for Φ_k:

$$\Phi_k(\rho,\sigma,v) = \exp\left(2\pi i\left(\frac{1}{N}\sigma+\rho+v\right)\right)$$

$$\prod_{r=0}^{N-1} \prod_{\substack{l,b\in Z, k'\in Z+\frac{r}{N} \\ k',l,b>0}} \left\{1-\exp(2\pi i(k'\sigma+l\rho+bv))\right\}^{\sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c^{(r,s)}(4lk'-b^2)}$$

$k',l,b > 0$: $(k' > 0, l \geq 0, b \in \mathbb{Z})$ or $(k'=0, l>0, b\in\mathbb{Z})$ or $(k'=0, l=0, b<0)$

$c^{r,s}(n)$: known coefficients, given in terms of jacobi ϑ-functions and Dedekind η-functions.

Equivalently we can express $d(Q,P)$ in terms of Fourier coefficients of $1/\Phi_k$.

Define $g(m,n,p)$ through

$$\frac{1}{\Phi_k(\rho,\sigma,v)} = \sum_{\substack{m,n,p \\ m\geq -1, n\geq -1/N}} e^{2\pi i(m\rho+n\sigma+pv)} g(m,n,p)\,.$$

Then

$$d(Q,P) = g\left(\frac{1}{2}P^2, \frac{1}{2}Q^2, Q\cdot P\right)$$

Derivation of the formula

We consider a configuration in type IIB string theory on $K3 \times S^1 \times \widetilde{S}^1/\mathbb{Z}_N$ with

1) Q_5 D5-branes wrapped on $K3 \times S^1$,

2) Q_1 D1-branes wrapped on S^1, and

3) one Kaluza-Klein monopole associated with $\widetilde{S}^1$ compactification

carrying $-n$ units of momentum along S^1 and J units of momentum along $\widetilde{S}^1$

In the heterotic description this corresponds to a state with

$G_{5\mu}$ electric charge $= n/N$

$B_{5\mu}$ electric charge $= 1$

$G_{4\mu}$ magnetic charge $= Q_5$

$B_{4\mu}$ magnetic charge $= (Q_1 - Q_5)$

$B_{5\mu}$ magnetic charge $= J$

y^4: coordinate along $\widehat{S}^1$

y^5: coordinate along S^1

T-duality invariants:

$$P^2 = 2Q_5(Q_1 - Q_5), \quad Q^2 = 2n/N, \quad Q\cdot P = J$$

We take $Q_5 = 1$ and denote by $h(Q_1, n, J)$ the degeneracy of states carrying charges (Q_1, n, J).

The 'partition function' is defined as

$$f(\rho,\sigma,v) = \sum_{Q_1,n,J} h(Q_1,n,J) e^{2\pi i(\rho\,(Q_1-1)+\sigma\, n/N + v\, J)}$$

Then

$$d(Q,P) \equiv h(Q_1,n,J) = \frac{1}{N}\int_C d\rho d\sigma dv f(\rho,\sigma,v) \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right],$$

In the weakly coupled type IIB description the low energy dynamics of the system is described by several pieces:

1) The dynamics of the Kaluza-Klein monopole

2) The dynamics of the D1-D5 center of mass coordinate in the Kaluza-Klein monopole background

3) The relative motion between the D1 and the D5-brane

Each of these systems carry certain amount of momenta along S^1 and $\widetilde{S}^1$.

The ‘partition function’ $f(\rho, \sigma, v)$ of the spectrum of BPS states is given by the product of the contribution to the ‘partition function’ from each of these three different systems.

Low energy dynamics of KK monopole:

$$e^{-2\pi i\sigma/N}\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma/N})^{-\frac{24}{N+1}}(1-e^{2\pi in\sigma})^{-\frac{24}{N+1}}\right\}$$

D1-D5 center of mass motion in KK monopole background:

$$\prod_{n=1}^{\infty}\left\{(1-e^{2\pi in\sigma})^{4}\,(1-e^{2\pi in\sigma+2\pi iv})^{-2}\,(1-e^{2\pi in\sigma-2\pi iv})^{-2}\right\}$$
$$\times\, e^{-2\pi iv}\,(1-e^{-2\pi iv})^{-2}$$

Relative motion between the D1 and D5 branes:

$$e^{-2\pi i\rho}\prod_{r=0}^{N-1}\prod_{\substack{l,b\in Z,k'\in Z+\frac{r}{N}\\ k'\geq 0,l>0}}\left\{1-\exp(2\pi i(k'\sigma+l\rho+bv))\right\}^{-\sum_{s=0}^{N-1}e^{-2\pi ils/N}\,c^{(r,s)}(4lk'-b^2)}$$

Product $= 1/\Phi_k(\rho,\sigma,v)$

$$d(Q,P) = \frac{1}{N}\int_C d\rho d\sigma dv \frac{1}{\Phi_k(\rho,\sigma,v)} \exp\left[-i\pi(\rho P^2 + \sigma Q^2 + 2vQ\cdot P)\right],$$

C: $0 \le Re\,\rho, Re\,v \le 1$, $0 \le Re\,\sigma \le N$

S-duality invariance of this formula can be proved by using the invariance of Φ_k under

$$\rho' = a^2\rho + b^2\sigma - 2abv + n_1\,,$$
$$\sigma' = c^2\rho + d^2\sigma - 2cdv + n_2\,N\,,$$
$$v' = -ac\rho - bd\sigma + (ad+bc)v + n_3\,.$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$, $n_1, n_2, n_3 \in \mathbb{Z}$.

Statistical entropy $\ln d(Q,P)$

For comparing $\ln d(Q,P)$ to black hole entropy we need to estimate $d(Q,P)$ for large Q,P.

Strategy:

a) Do the v integral by picking up residues from the poles of $1/\Phi_k$

Result:

$$d(Q,P) = \int d\rho d\sigma e^{-F(\rho,\sigma)}$$

for some function $F(\rho,\sigma)$.

b) Then do the ρ and σ integral using saddle point approximation.

Define $W(\vec{J})$ through

$$e^{W(\vec{J})} = \int d\rho d\sigma e^{-F(\rho,\sigma)+J_1\rho+J_2\sigma}$$

Then

$$e^{W(\vec{0})} = d(Q, P)$$

$$e^{W(\vec{J})} = \int d\rho d\sigma e^{-F(\rho,\sigma)+J_1\rho+J_2\sigma}$$

Define

$$\widehat{\rho} = \partial W(\vec{J})/\partial J_1, \quad \widehat{\sigma} = \partial W(\vec{J})/\partial J_2$$

$$\Gamma(\widehat{\rho},\widehat{\sigma}) = J_1\widehat{\rho} + J_2\widehat{\sigma} - W(\vec{J})$$

Then

$$J_1 = \partial\Gamma/\partial\widehat{\rho}, \quad J_2 = \partial\Gamma/\partial\widehat{\sigma}$$

If $\partial\Gamma/\partial\widehat{\rho} = \partial\Gamma/\partial\widehat{\sigma} = 0$ at $(\widehat{\rho},\widehat{\sigma}) = (\widehat{\rho}_0,\widehat{\sigma}_0)$ then

$$\Gamma(\widehat{\rho}_0,\widehat{\sigma}_0) = -W(\vec{0}) = -\ln d(Q,P)$$

If $\partial\Gamma/\partial\widehat{\rho} = \partial\Gamma/\partial\widehat{\sigma} = 0$ at $(\widehat{\rho}, \widehat{\sigma}) = (\widehat{\rho}_0, \widehat{\sigma}_0)$ then

$$\Gamma(\widehat{\rho}_0, \widehat{\sigma}_0) = -W(\vec{0}) = -\ln d(Q, P)$$

Thus $\ln d(Q, P)$ is the value of $-\Gamma(\widehat{\rho}, \widehat{\sigma})$ at its extremum.

$-\Gamma(\widehat{\rho}, \widehat{\sigma})$ can be called the statistical entropy function.

On the other hand Γ can be calculated by summing over 1PI Feynman diagrams in the 0-dimensional quantum field theory with action $F(\rho, \sigma)$.

Result for Γ after a suitable change of variables from $(\widehat{\rho},\widehat{\sigma})$ to (a,S):

$$
\begin{aligned}
-\Gamma(a,S) &= \frac{\pi}{2}\Big[\left(\frac{Q^2}{S}+\frac{P^2}{S}(S^2+a^2)-2\,\frac{a}{S}\,Q\cdot P\right) \\
&\quad +128\,\pi\,\phi_k(a,S)\Big]+\mathcal{O}(Q^{-2},P^{-2})
\end{aligned}
$$

$$
\begin{aligned}
\phi_k(a,S) &= -\frac{1}{64\pi^2}\Big((k+2)\ln S \\
&\quad +\ln f^{(k)}(a+iS)+\ln f^{(k)}(-a+iS)\Big)
\end{aligned}
$$

$$
f^{(k)}(\tau)=\eta(\tau)^{k+2}\,\eta(N\tau)^{k+2}
$$

Now recall that the entropy function for the black hole, after extremization with respect to all the near horizon parameters except the values of the axion-dilaton field, is given by:

$$\mathcal{E} = \frac{\pi}{2}\Big[\left(\frac{Q^2}{S} + \frac{P^2}{S}(S^2 + a^2) - 2\,\frac{a}{S}\,Q\cdot P\right) + 128\,\pi\,\phi_k(a,S)\Big]$$

$a + iS$: near horizon value of the axion-dilaton field.

$\mathcal{E}$ and $-\Gamma$ are identical functions to this order.

Thus extremization of $\mathcal{E}$ and $-\Gamma$ give the same answer.

$\rightarrow$ equality between black hole entropy and statistical entropy to first non-leading power of inverse charges.

Thus we see that the formula for the statistical entropy matches the black hole entropy to this order.

This result can be generalized for

1. CHL models with non-prime values of N.

2. $\mathcal{N} = 4$ supersymmetric $\mathbb{Z}_N$ orbifiolds of type IIA string theory on T^6.

Half BPS states in CHL models

For simplicity we shall focus on heterotic on $T^6 = T^5 \times S^1$ (the $N = 1$ model) but the analysis can be easily generalized to other heterotic string compactification.

Consider an elementary string wound w-times along S^1 and carrying n units of momentum along S^1.

There are left- and right-moving oscillator excitations on the string.

World-sheet as well as space-time supersymmetry acts on the right-moving sector.

If we put all the right-moving oscillators in the lowest state consistent with GSO projection, then we have a state that is invariant under half of the original 16 space-time supersymmetry transformations.

The left-moving oscillators can be excited without violating supersymmetry.

The $L_0 = \bar{L}_0$ constraint gives:

$$N_L = 1 + n\,w$$

N_L: contribution to L_0 from the left-moving oscillators.

$$N_L = 1 + n\,w$$

For large nw, i.e. large N_L, the degeneracy $d(n,w)$ of these states grows exponentially:

$$d(n,w) \sim \exp(4\pi\sqrt{N_L}) \sim \exp(4\pi\sqrt{nw})$$

Thus

$$S_{stat} = \ln d(n,w) \simeq 4\,\pi\,\sqrt{nw}\,.$$

Does this agree with the entropy of a black hole carrying the same charges?

What is the black hole description of this state?

Recall that for heterotic on T^6 we have a 28-dimensional electric charge vector Q and a 28-dimentional magnetic charge vector P.

$$Q^2 \equiv Q^T L Q, \quad P^2 \equiv P^T L P, \quad Q \cdot P \equiv Q^T L P$$

$$L = \begin{pmatrix} & I_6 & \\ I_6 & & \\ & & -I_{16} \end{pmatrix}$$

I_k: $k \times k$ identity matrix

In this case we have

$$Q^2 = 2nw\,, \qquad P^2 = 0, \qquad Q \cdot P = 0$$

$$Q^2 = 2nw\,, \qquad P^2 = 0, \qquad Q \cdot P = 0$$

We can use entropy function formalism to calculate the entropy of such a black hole.

In the leading supergravity approximation, the entropy function, after elimination of the scalar fields u_M, takes the form:

$$\mathcal{E} = \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2}\frac{Q^2}{u_S}\Big]$$

This has no extremum as a function of u_S, v_1 and v_2.

Now consider including the tree level Gauss-Bonnet term.

The entropy function gets modified to

$$\mathcal{E} = \frac{\pi}{2}\Big[u_S(v_2 - v_1) + \frac{v_1}{v_2}\frac{Q^2}{u_S}\Big] + 4\pi u_S$$

This has extremum at

$$u_S = \sqrt{Q^2/8} = \sqrt{nw/4}, \quad v_1 = v_2 = 8$$

At this extremum

$$\mathcal{E} = 4\pi\sqrt{Q^2/2} = 4\pi\sqrt{nw}$$

$\rightarrow$ agrees with statistical entropy!

What about other higher derivative / string loop corrections?

Note: $u_S = \sqrt{nw/4}$ is large for large nw.

Thus string coupling at the horizon is small.

$\rightarrow$ can ignore string loop corrections.

But $v_1 = v_2 = 8$

Thus sizes of AdS_2 and S^2 are of order 1.

$\rightarrow$ curvature is of order 1.

$\rightarrow$ cannot ignore other tree level higher derivative terms.

There is however a general argument due to Kraus and Larsen which shows that the tree level higher derivative corrections do not modify this result.

This is based on

1. Space-time supersymmetry

2. The observation that the AdS_2 factor and the compact direction S^1 along which the string wraps together form an AdS_3 space locally.

Thus for half BPS state we also have agreement between black hole and statistical entropy, at least in the leading order.