Two

Note Title 10/20/2006

Now we would like to understand the dynamics of AdS/CFT correspondence. or to see how the dynamics of one side can be obtained from the other side.

To do this let's start with AdS-space

AdS_{n+1} is a sphere in $R^{2,n-1}$ flat space

$$-X_0^2 - X_{n+1}^2 + X_1^2 + \cdots + X_n^2 = -X_0^2 - X_{n+1}^2 + \vec{X}^2 = R^2$$

To get the metric one needs to solve this equation in terms of $(n+1)$-parameters and plugging into

$$ds^2 = -dX_0^2 - dX_{n+1}^2 + d\vec{X}^2$$

Different solutions give differen metric which one useful for different purpose.

Globale coordinates

$$ds^2 = -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega^2_{n-1}$$

Poincaré coordinates

$$ds^2 = -r^2 d\vec{x}^2 + \frac{dr^2}{r^2}$$

or $$ds^2 = \frac{1}{z^2}\left(dz^2 + d\vec{x}^2\right)$$

Let's define light cone coordinates

$$u = X_0 + i X_{n+1} \qquad v = -X_0 + i X_{n+1}$$

so that

$$uv - \vec{x}^2 = R^2$$

now define ξ^a as $\xi^a = \frac{x^a}{u}$

so $uv - u^2 \vec{\xi}^2 = R^2$

$$v = \xi^2 u + \frac{R^2}{u}$$

$$dv = 2\xi \cdot d\xi\, u + \xi^2 du - \frac{R^2}{u^2} du^2$$

$$dX_a = u\, d\xi^a + \xi^a du$$

Therefore $ds^2 = \frac{R^2}{u^2} du^2 + u^2 d\vec{\xi}^2$

$\xi^0 = u^{-1}$

$$ds^2 = \frac{1}{\xi^{0\,2}} \left(d\xi^{0\,2} + d\vec{\xi}^2 \right)$$

Let's start from $(X_0, \dots X_{n+1})$ system and rescale by λ

$$X_a = \lambda \tilde{X}_a \quad , \; u = \lambda \tilde{u} , \quad v = \lambda \bar{v}$$

$$\tilde{u}\,\tilde{v} - \tilde{X}^2 = \frac{R^2}{\lambda^2}$$

The boundary is at $\lambda \to \infty$ so

The boundary manifold is

$$v u - \vec{X}^2 = 0$$

But $\beta\lambda$ (for a constant β) is as good

as λ. so the boundary is

$v u - \vec{x}^2 = 0$ with $(u, v, \vec{x}) \sim \beta(u, v, \vec{x})$

"Boundary is n-dimensional space"

The constraint can be solved as

$$u v = 1 \qquad \vec{y}^2 = 1$$

so the boundary is $S^1 \times S^{n-1}$

one may solve it as follows.

if $v \neq 0$ $\quad v = 1 \quad u = x^2$

and $\vec{x}$ can be used to parametrize the boundary. the same for $u \neq 0$ $\quad u = 1$

$v = \vec{x}^2$.

when either of v or u is zero we have:

For $v = 0$ we may think it as the point at infinty in x coordinates. so

The boundary is automatically compactified (as needed to get CFT)

Note also that the conformal group in d- dimensions is

$SO(2,d)$ Minkowski

$SO(1,d+1)$ Euclidean

The isometry of AdS_{n+1} is

$SO(2,d)$ Minkowski

$SO(1,d+1)$ Euclidean

One can see: $SO(2,d)$ $[SO(1,d+1)]$ has the following action

- on the AdS_{n+1} it is isometry of space
- on the boundary of AdS_{n+1} it is the conformal group.

on the other hand $N=4$ SYM in four dimensions is a conformal field theory and therefore it may consistently live at the boundary.

our other considerations confirm this too.

So the physical observables of $\mathcal{N}=4$ are local operators live on the boundary which characterized by spin, dimension and R-charges

$$SO(2,4)\times SO(6) \supset \underset{\substack{\Psi \\ S_1,\, S_2}}{SO(4,3)} \times \underset{\substack{\Psi \\ \Delta}}{SO(4,1)} \times \underset{\substack{\Psi \\ J_1,\, J_2,\, J_3}}{SO(6)}$$

on the other hand in the bulk $AdS_5 \times S^5$ we have isometry and therefore one can define conserved charges as follows

$$AdS_5 \begin{cases} \partial_t \longrightarrow E \\ \left.\begin{matrix}\partial_{\varphi_1} \\ \partial_{\varphi_2}\end{matrix}\right\} \longrightarrow S_1\,,\, S_2 \end{cases}$$

$$S^5 \left\{ \begin{matrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{matrix} \right\} \longrightarrow J_1\,,\, J_2\,,\, J_3$$

Therefore in general we have a field in the bulk with $(E, s_1, s_2, J_1, J_2, J_3)$ which corresponds to an operator on the boundary with $(\Delta, s_1, s_2, J_1, J_2, J_3)$

So one needs to start from some field and try to identify it with some operator on the boundary.

To procced let's consider a scalar field

It is more convinet to work with the following metric.

$$ds^2 = \frac{1}{z^2}\left(dz^2 + d\vec{x}^2\right)$$

where the boundary is at $z = 0$

For simplicity consider a massive scalar on AdS_{d+1}

$$\frac{1}{2}\int d^{d+1}x\,\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+m^2\phi^2\right]$$

$$=\frac{1}{2}\int d^d x\,dz\;z^{-d+1}\left[(\partial_z\phi)^2+(\partial_i\phi)^2+\frac{m^2}{z^2}\phi^2\right]$$

one can easily write the equations of motion and solve it to find classical solution for ϕ. There are two independent solutions which are charactrized by thier behaviour near $z\to 0$

$$\phi(z)\longrightarrow\begin{cases} z^{\Delta} \\ z^{d-\Delta}\end{cases}\quad \text{as}\quad z\to 0$$

where Δ is a solution of

$$\Delta(\Delta-d)=m^2\;\Rightarrow\;\Delta=\frac{d}{2}+\sqrt{\frac{d^2}{2}+m^2}$$

Note that we set $R=1$

$z^{\Delta} \longrightarrow$ normalizable modes

$z^{d-\Delta} \longrightarrow$ non-normalizable modes

There is a particular combination of them which is physical. So in general we have

$$\Phi(z,\vec{x}) \longrightarrow z^{d-\Delta}\,\Phi_0(x) + z^{\Delta} A(x)$$

$\Phi_0(x)$ and $A(x)$ live on the boundary. Given $\Phi_0(x)$, $A(x)$ means we are fixing the boundary condition and therefore we get a solution on the bulk $\Phi(z,x)$

So one may solve Φ and find it such that at the boundary we have

$$\phi(z, x)\Big|_{\text{boundary}} = \phi_0(x)$$

So changing the boundary condition we get different solutions. Since $\phi_0(x)$ is on the boundary one would expect that $\phi_0(x)$ has to do something with the gauge theory lives on the boundary. To make it cleare consider the simplest case which could give an insight about the $\phi_0(x)$.

In the gauge theory we have a gauge coupling and we can change it by making use of a marginal operator.

But changing the coupling constant means we are changing the Dilaton at infinity so we are changing the boundary condition so deforming the gauge theory by an operator leads to changing the boundary condition. Therefore one may assume the following term on the boundary gauge theory

$$\int d^d x \, \phi_0(x) \, \mathcal{O}(x)$$

an operator

boundary value of a field

Using this notation the AdS/CFT says:

Consider an operator $\mathcal{O}(x)$ on the boundary with following term

$$\int d^d x\, \phi_0(x)\, \mathcal{O}(x)$$

such that there is a field in the bulk whose value on the boundary is $\phi_0(x)$

$$\phi(z,x)\Big|_{\text{boundary}} = \phi_0(x)$$

Then

$$\left\langle e^{\int d^d x\, \phi_0(x)\mathcal{O}(x)} \right\rangle_{YM} = Z_{\text{bulk}}\left[\phi\big|_{\text{bou}} = \phi_0\right]$$

so in the expansion $\phi(z, x)$ at $z \to 0$

$$\phi(z,x) \sim z^{d-\Delta}\phi_0(x) + z^{\Delta} A(x)$$

$\phi_0(x)$ is source for operator $\mathcal{O}$.

one can also see that $A(x)$ describes the expectation value of the corresponding operator $\langle \mathcal{O} \rangle = A$

Since $\phi(z, x)$ is a scalar on AdS_{n+1} has dimension zero. so that $[\phi_0] = L^{\Delta - d}$

$\int d^d x\, \phi_0(x)\, \mathcal{O}(x) \Longrightarrow \mathcal{O}$ has dimension Δ

so that $\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle \sim \frac{1}{L^{2\Delta}}$

In Large N and Large λ one may use saddle point approximations so that

$$\left\langle e^{\int d^d x\, \phi_0(x)\, \mathcal{O}(x)} \right\rangle \sim e^{I_{SUGRA}[\phi|_b = \phi_0]}$$

Therefore one can easily compute n-point function by computing the classical action on the solution with given boundary condition.

Let's see how it works in an explicit example.

Consider a massive scalar on AdS_{d+1}

$$I = \frac{1}{2}\int d^{d+1}x \sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right)$$

$$ds^2 = \frac{1}{z^2}\left(dz^2 + d\vec{x}^{\,2}\right)$$

e.o.m: $-z^{d+1}\frac{d}{dz}\left(z^{d+1}\frac{d\phi}{dz}\right) + m^2\phi = 0$

As in the electrodynamic let's first find

The Green function. In other words the boundary condition is

$$\phi(z, x) \longrightarrow z^{d-\Delta} \delta(x)$$

$\delta(x-y)$

$\hookrightarrow$ some point on the boundary

The corresponding solution is $K(z, x; y)$

$$K(z, x; y) \longrightarrow z^{d-\Delta} \delta(x-y)$$

having had K one has

$$\phi(z, x) = \int d^d y \, K(z, x, y) \, \phi_0(y)$$

K is called bulk to boundary propagator

In our case it is easy to see

$$K(z, x; y) = \frac{z^{\Delta}}{\left[z^2 + (x-y)^2\right]^{\Delta}}$$

so that

$$\phi(z, x) = \int d^d y \, \frac{z^{\Delta}}{\left(z^2 + (x-y)^2\right)^{\Delta}} \, \phi_0(y)$$

On the other hand we have

$$I = \frac{1}{2} \int d^{d+1}x \sqrt{g} \, \left(\partial_\mu \phi \, \partial^\mu \phi + m^2 \phi^2\right)$$

$$= \frac{1}{2} \int d^{d+1}x \sqrt{g} \left\{ \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} \, \phi \, \partial^\mu \phi\right) \right.$$

$$\left. - \phi \left(\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \, \partial^\mu - m^2 \right) \phi \right\}$$

$$= -\frac{1}{2}\int_{z\to 0} d^d x \; z^{-d+1} \underbrace{\phi(z,x)}_{\downarrow \atop z^{d-\Delta}\phi_0(x)} \partial_z \phi(z,x)$$

$$= -\frac{1}{2}\int_{z\to 0} d^d x \; z^{-d+1} z^{d-\Delta} \phi_0(x)\, \partial_z \phi(z,x)$$

on the other hand one finds

$$\partial_z \phi(z,x) = z^{\Delta-1}\int d^d y \frac{\phi_0(y)}{\left(z^2+(x-y)^2\right)^{\Delta}} + \cdots$$

so altogether we find

$$I = -\frac{1}{2}\int_{z\to 0} d^d x\, d^d y \; z^{-d+1} z^{d-\Delta} z^{\Delta-1} \frac{\phi_0(x)\phi_0(y)}{\left(z^2+(x-y)^2\right)^{\Delta}}$$

$$+ z(\cdots)$$

in the limit of $z \to 0$

$$I = -\frac{1}{2}\int d^d x \, d^d y \; \frac{\phi_0(x)\,\phi_0(y)}{(x-y)^{2\Delta}}$$

so we get

$$\left\langle e^{\int d^d x \, \phi_0(x)\, \mathcal{O}(x)} \right\rangle_{FM} = e^{-I_{SUGRA}}$$

$$= \exp\left[\frac{1}{2}\int d^d y \, d^d x \; \frac{\phi_0(x)\,\phi_0(y)}{(x-y)^{2\Delta}}\right]$$

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle = \frac{\delta \langle \quad \rangle_{FM}}{\delta\phi_0(x)\,\delta\phi_0(y)} = \frac{1}{(x-y)^{2\Delta}}$$

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle = \frac{1}{(x-y)^{2\Delta}}$$ as expected

one may start with a scalar field with $\frac{1}{p!}\phi^p$ interaction and find p-point function.

one could do the same for r-form.

The procedure is the same but the dimension is given by

$$\Delta = \frac{1}{2}\left(d + \sqrt{(d-2r)^2 + m^2R^2}\right)$$

Back to Type IIB on $AdS_5 \times S^5$

one can identify some field operators and fields.

$$e^{\phi} \longrightarrow g^2_{YM}$$

$$\underset{\text{zero form}}{\chi} \longrightarrow \underset{\text{vacuum.}}{\theta}$$

$\chi + ie^{-\phi} = \tau$ in which $SL(2,\mathbb{Z})$ acts

maps to $\tau = \frac{\theta}{2\pi} + \frac{i}{g_{YM}^2}$ and

we have S-duality (Montonen-Olive)

using DBI action of finds

$e^{\phi} F^2 \Rightarrow$ from dilaton we find

$$\langle F^2 F^2 \rangle .$$

from C.S. term one gets $C_0 F\tilde{F}$

so χ corresponds to $F\tilde{F}$ operator

one can also see that

$$h_{ij} \longleftrightarrow T_{ij}$$

$$B_{\mu\nu} \longleftrightarrow F_{\mu\rho} F^{\rho\sigma} F_{\sigma\nu} + \cdots$$

From this we can compute $\langle T^i{}_i \rangle$ using the metric action.

$$\langle T^i{}_i \rangle = C \cdots$$

$\hookrightarrow$ centeral charge $= \frac{N^3}{4}$

Few Comments

1 - starting from D_3-brane solution one has $f = 1 + \frac{4\pi g_s N l_s^4}{r^4}$

The decoupling limit is

$$l_s \to 0 \qquad \frac{r}{l_s^2} = U \quad \text{fixed}$$

$$[U] = \text{energy}$$

U plays the role of energy in the gauge theory.

UV cut off "U" on radius of AdS_5 traslates into a UV cut off "ε" in dual CFT.

A simple way to see this is as follows. Consider Laplace equation

$$U^{-3}\partial_z\left(U^5\partial_u \Phi\right) + \frac{\omega^2 g^2_{YM} N}{U^2}\Phi = 0$$

from the scaling of the equation the solution is only a function of $\frac{\omega\sqrt{g^2_{YM}N}}{U}$ so that we get a relation of cut off

$$\omega \sim \frac{U}{\sqrt{g_{YM}^2 N}}$$

So if we are at energy U only those modes in CFT are excited which are in the region given by

$$\delta X = \frac{\sqrt{g_s N}}{U}$$

2 - We can also break susy.

$t \longrightarrow it = \tau \equiv$ compact circle with anti-periodic boundary condition.

So we get $N=4$ at finite temperature.

from gravity point of view it is $SAdS_5$ solution.

$\mathcal{N}=4$ at finit tem.

$|||$

Type II B on

$$\frac{U^2}{R^2}\left(-\left(1-\frac{U_0^4}{U^4}\right)dt^2+d\vec{x}_i^{\,2}\right)+\frac{R^2}{U^2}\,\frac{dU^2}{1-\frac{U_0^4}{U^4}}$$

$$+\ R^2\ d\Omega_5^2$$

U_0 is related to Hawking temp.

3 – Gravity modes $\longleftrightarrow$ $\Delta \sim N^0$

chiral operator

string modes $\longleftrightarrow$ some operators with

$\Delta \sim N^{\#}$

depends on $\#$ we get different operators