One

Note Title 10/3/2006

Introduction to AdS/CFT

Advance string school 2006

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We would like to see if there is any relation between string theory and gauge theory ?

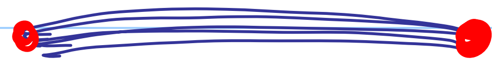
To see this Let's start from a simple, well-known fact about QCD. Consider a quark - anti - quark pair

$$\psi \bar{\psi} \qquad \psi \ \bar{\psi}$$

Let's separate them

$$\bar{\psi}(0) \qquad \bar{\psi}(x) \qquad \equiv \psi(0)\, \bar{\psi}(x)$$

$\psi(0)\, \bar{\psi}(x)$ is not a gauge invariant operator. But one can find a gauge invariant as follows



$$\psi(0)\, \underbrace{P\left[\exp\left(i\int_0^x A_\mu\, dx^\mu\right)\right]}_{\text{flux tubes}} \bar{\psi}(x) \quad = \text{gauge inv,}$$

This is the original discovery of strings.

Is there any other indication for "string theory" which can come out of gauge theory?

Yes - And it is 't Hooft Large N expansion of gauge theory.

QCD has two parameter g^2_{YM} and N

(mass scale Λ_{QCD})

We would like to study QCD around Λ_{QCD} where there is no obvious perturbative physics.

on the other hand there is "3" in QCD

one may use it as an expansion parameter "$\frac{1}{3}$" which is not bad!!!

$1, \frac{1}{3}, \frac{1}{9}, \cdots$

$A = 1 + \frac{1}{3} + \frac{1}{9} + \cdots = \frac{3}{2} = 1.5$

- free $A = 1$ % 33 err.
- one-loop $A = 1 + \frac{1}{3} = 1.33$ % 11 err.

• two-loop $A = 1 + \frac{\lambda}{3} + \frac{\lambda}{9} = 1.44$ % 4 err.

So "$\frac{1}{3}$" is a good expansion parameter.

Therefor in general one may study QCD with $\frac{1}{N}$ as the expansion parameter.

For Large N it may even simplify the life.

So we would like to consider Large N QCD when $\frac{1}{N}$ is expansion parameter:

First since we are taking $N \to \infty$ one may want to know how the other parameters scale ?

the beta function of pure QCD $SU(N)$ is

$$\mu \frac{d g_{YM}}{d\mu} = -\frac{11}{3} N \frac{g_{YM}^3}{16\pi^2} + \mathcal{O}(g_{YM}^4)$$

So

$$N g^2_{YM} = \frac{1}{\ln \Lambda_{QCD}}$$

It is natural to keep Λ_{QCD} fixed so the limit is

$N \to \infty$ $\quad g^2_{YM} N \equiv \lambda = \text{fix}$

↳ 't Hooft coupling

This is called 't Hooft limit.

λ is fixed but could be large (strong coupling) or small (weak coupling).

Let's see what happen for QCD in this limit. In QCD we have gauge field A^a which is in adj rep. of $SU(N)$ the action is

$$\frac{1}{g^2_{YM}} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu})$$

schematicly one has

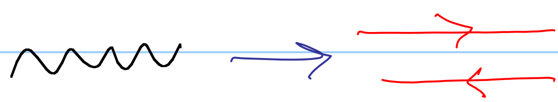
$$\mathcal{L} \sim \frac{1}{g^2_{YM}} \mathrm{Tr}\left[\partial A \partial A + \partial A A A + AAAA\right]$$

$$\sim \frac{N}{\lambda} \mathrm{Tr}\left[\partial A \partial A + \partial A A A + AAAA\right]$$

Now the trick is double line notation

$$N \otimes \bar{N} = adj \oplus 1$$

$$A^a \longrightarrow \phi^{i}{}_{j} \qquad i \to N, \quad j \to \bar{N}$$



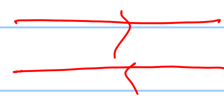
of cource one need to consider "1"

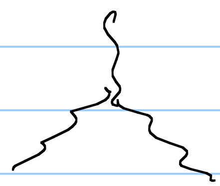
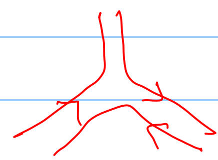
but in Large N we one in the safe side

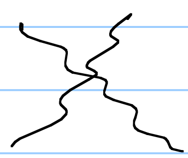
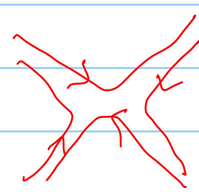
$$\langle \phi^i{}_j \, \phi^k{}_\ell \rangle \propto \delta^i_\ell \, \delta^j_k - \frac{1}{N} \underbrace{\delta^i_j \, \delta^k_\ell}_{\text{"1"}}$$

for Large N is ok!

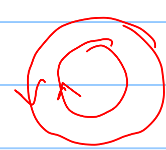
The Feynmann rules in doubl line notation

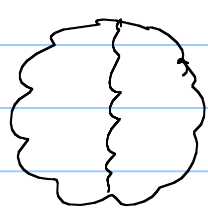
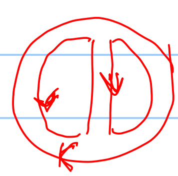
  $\frac{\lambda}{N}$

  $\frac{N}{\lambda}$

  $\frac{N}{\lambda}$

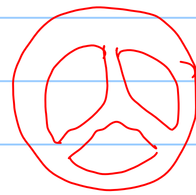
what is the power of λ and N for given diagram.

  N^2

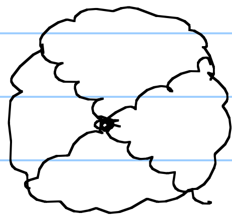
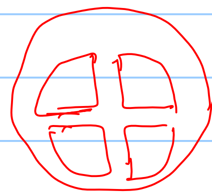
 

(vertex) (propagator) (loop)

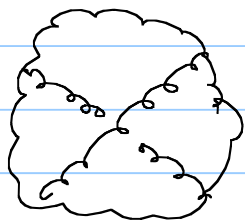
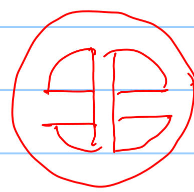
$$\left(\frac{N}{\lambda}\right)^2 \left(\frac{\lambda}{N}\right)^3 N^3 = \lambda N^2$$

$$\left(\frac{N}{\lambda}\right)^4 \left(\frac{\lambda}{N}\right)^6 N^4 = \lambda^2 N^2$$

$$\left(\frac{N}{\lambda}\right)^5 \left(\frac{\lambda}{N}\right)^8 N^5 = \lambda^3 N^2$$

$$\left(\frac{N}{\lambda}\right)^4 \left(\frac{\lambda}{N}\right)^6 N^2 = \lambda^2 N^0$$

In general

V vertices E propagators F loops

$$\left(\frac{N}{\lambda}\right)^V \left(\frac{\lambda}{N}\right)^E N^F = N^{V-E+F} \lambda^{V-E} = N^{\chi} \lambda^{E-V}$$

$\chi = V - E + F$ Euler number

for closed oriented surface $\chi = 2 - 2g$

any diagram $\sim \sum N^{2-2g} f_g(\lambda)$

$f_g(\lambda)$ is some polynomial of λ.

It has exactly the same form as string theory provided if we identify N with string coupling $\boxed{e^{\phi} \sim \frac{1}{N}}$

There are two expansion

λ expansion $\longrightarrow$ α' expansion

$\frac{1}{N}$ expansion $\longrightarrow$ string loop expansion.

Let's for the moment stop here and go to another topic

"Holography"

suppose we know what the fundamental theory is. Now we would to ask the following question;

How many degrees of freedom do we need to fully describe a system in the fundamental level?

A nive answer is just to consider a quantum field Theory as a fundamental framework which is defined on a classical background which satisfies Einstein's equations.

The quantum field theory consists of one or more oscillators at every point in space and therefore we get infinity!

But it is not correct. Because in the fundamental level we can't ignor the effects of gravity.

Taking into accunt the gravity means one can't resolved distance smaller than plank length in quantum gravity. So we need to discretize space time and assume that in each box there is one oscillator. So we get

# of degrees $\sim \hat{V}$ (voulme of system)

But still we were not Careful enough:
we need to take into account the stability of the system which will put an extra Condition on excitation of any oscillators.

To answer the question we use what we know from black hole physics.

we know the entropy of a black hole is given by

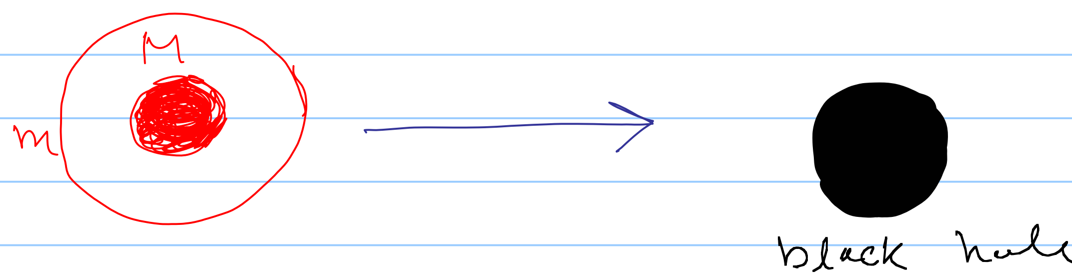
$$S = \frac{A}{4\pi}$$

where A is the area of horizon.

Now Consider a sphercally symmetric weakly coupled gravity. And Consider a system with mass M and entropy S_m. Let's denote by A the area of the smallest sphere that fits the system.

To get a stable system M must be less than the mass of a black hole of the same area.

Now consider a shell with mass m and entropy S_{sh} such that adding to the system it can be converted to a black hole.



the initial entropy $S_{total} = S_M + S_{sh}$

final entropy $S_{total} = S_{BH} = \frac{A}{4\pi}$

since initial entropy can't exceed the final one we get

$$S_M + S_{sh} \leq \frac{A}{4G}$$

So $\boxed{S_M \leq \frac{A}{4G}}$

on the othere hand from thermodynamical point of view the entropy has a statistical interpretation :

The number of independent quantum states $= e^S$

So the entropy bound will put a bound on the number of degrees of freedom.

This leads to the following statement for holography :

A region with boundary of area A is fully described by no more than $\frac{A}{4G}$ degrees of freedom or about 1 bit of information per plank area.

Most of our intution about the holography principle has come from AdS/CFT Correspondence.

What is AdS/CFT Correspondence?

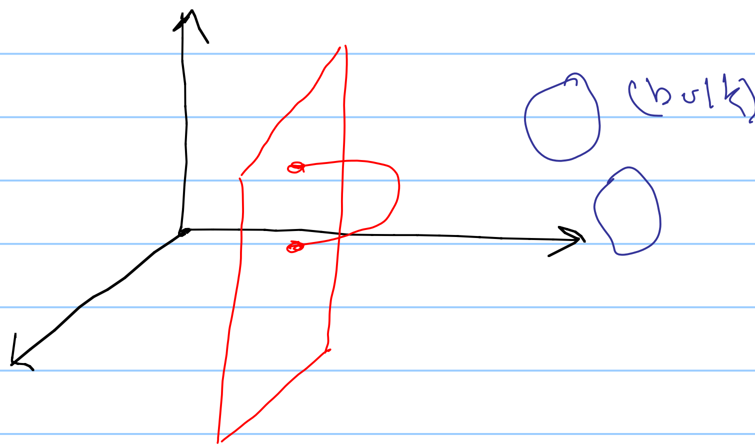
In fact AdS/CFT is an explicit example where one can see how the holography works and also how to relate a string theory to a gauge theory as we studied before.

The main point in understanding AdS/CFT is the fact that D-branes can be considered from two different points of view

- D-brane is an object where open strings can end (open string picture)
- D-brane is a solution of low energy effective action of string theory (closed string picture)

Open string picture

Consider N D_3-branes on top of each other in type IIB



Two kinds of perturbative excitations

- bulk closed string
- open strings end on D3-brane

at low energy $\omega \ll \frac{1}{l_s}$ we have only massless modes

Closed string $\longrightarrow$ gravity (SUGRA) open string $\longrightarrow$ gauge theory ($N=4$ SYM)

So the complet action for the system is

$$S = S_{bulk} + S_{brane} + S_{intraction}$$

The action just has only massless modes but the effect of massive modes are also taken into accunt. one may think about it as Wilsonian effective action.

S_{bulk} = SGRA

S_{brane} = $N=4$ SYM + higher order correction

S_{int} = gravity + gauge field.

In the lowest level we may start from

Covariantizing $F_{\mu\nu} F^{\mu\nu}$ term

$$S \sim \int \frac{\sqrt{g}}{g_{YM}^2} g^{\mu\mu'} g^{\nu\nu'} F_{\mu\nu} F_{\mu'\nu'} + \int \frac{1}{2\kappa^2} \sqrt{g} R$$

$$g = \eta + \kappa h \qquad \kappa = g_s l_s^4 \qquad g_{YM}^2 \sim g_s l_s^{p-3}$$

$\hookrightarrow$ canonically normalized.

$$S \sim \frac{1}{g_{YM}^2} \int d^4x \sqrt{g}\, g_{\mu\mu'} g_{\nu\nu'} F^{\mu\nu} F^{\mu'\nu'}$$

$$\sim \frac{1}{g_{YM}^2} \int d^4x \left(F_{\mu\nu} F^{\mu\nu} + \kappa \eta_{\mu\mu'} h_{\nu\nu'} F^{\mu\nu} F^{\mu'\nu'} + \mathcal{O}(h^2) \right)$$

so

$$S_{brane} \sim \frac{1}{g_{YM}^2} \int d^4x\, F_{\mu\nu} F^{\mu\nu}$$

$$S_{int} \sim \frac{1}{g_{YM}^2} \int d^4x \left(\kappa \eta_{\mu\mu'} h_{\nu\nu'} F^{\mu\nu} F^{\mu'\nu'} + \mathcal{O}(h^2) \right)$$

$$S_{bulk} \sim \frac{1}{2\kappa^2} \int \sqrt{g} R \sim \int (\partial h)^2 + \kappa (\partial h^2) h + \mathcal{O}(h^4)$$

In low energy $l_s \to 0$ $(\omega_s \ll \frac{1}{l_s})$ and keeping all other dimensionless parameters fixed $(g_s = \text{fixed})$ one find

$$\underbrace{S = \frac{1}{g^2_{YM}} \int d^4x \, F_{\mu\nu} F^{\mu\nu}}_{N=4 \text{ SYM } U(N)} + \underbrace{\int (\partial h)^2}_{\text{free gravity}}$$

Therefor we see that in a limit of $\underbrace{\alpha' \to 0 \quad g_s = \text{fixed}}$ we have two decoupled theory $\longrightarrow$ (decoupling limit)

- free gravity
- $N=4$ $U(N)$ SYM theory

Now let's look at the system from another point of view.

Closed string picture

Let's look at the system from closed string point of view.

D_3 - brane can be thought of as massive charged object which acts as a source for the various supergravity fields.

In fact the D3-brane solution is given by

$$ds^2 = f^{-\frac{1}{2}}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)+f^{1/2}\left(dr^2+r^2d\Omega_5^2\right)$$

$$F_5 = (1+*)\, dt\wedge dx_1\wedge dx_2\wedge dx_3\wedge df^{-1}$$

$$f = 1+\frac{R^4}{r^4} \quad \text{with} \quad R^4 = 4\pi g_s \alpha'^2 N$$

This solution will change the background from flat to some curved background. With this solution we have two modes

- near $r=0$ modes
- bulk modes $(r\to\infty)$

we are intersted in interaction between to modes. This can be done for example by study scattering of $r\to\infty$ modes from branes. Suppose we are sending some particle (graviton) from infinty to the branes and look at the absorption cross section.

We will perturb the metric of the background as follows

$$g_{ab} = \bar{g}_{ab} + h_{ab} \qquad a,b=0,1,\dots,9$$

$\hookrightarrow$ D_3-bane solution

Consider S-wave with momenta along the brane

$$h_{ab} = G_{ab}\, h(r)\, e^{ik_\mu x^\mu} \qquad \mu = 0, \dots, 6$$

$$h_{a\mu} k^\mu = 0 \qquad (\text{gauge fixing})$$

useing the linearized equation of motion one finds for $\begin{cases} G_{ab} = 0 & a,b = 6, \dots, 9 \\ k_\mu = \omega\, \delta_{0,\mu} \end{cases}$

$$\partial_\mu \left[\sqrt{g}\, g^{\mu\nu}\, \partial_\nu h(r) \right] = 0$$

$$h(r) = \phi(r)\, e^{i\omega t}$$

$$\partial_r (r^5 \partial_r \phi) + \omega^2 f r^5 \phi = 0$$

$$\Rightarrow \partial_r^2 \phi + \frac{5}{r} \partial_r \phi + \omega^2 f\, \phi = 0$$

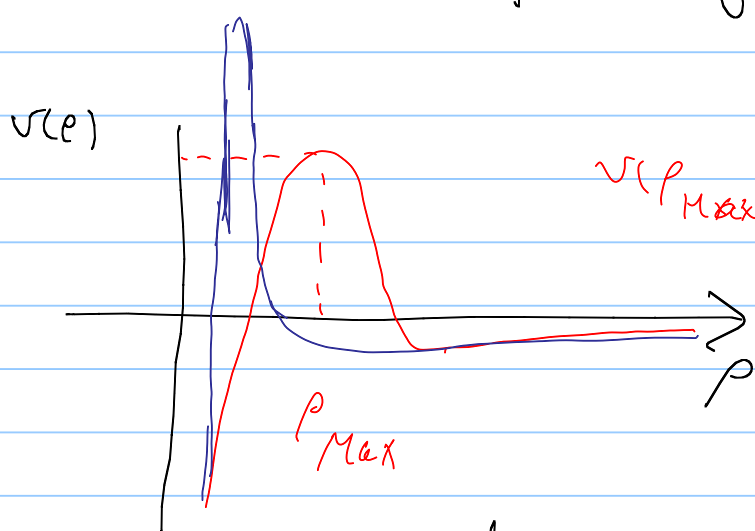
$$\phi(r) = a(r)\, \psi(r) \qquad \text{for} \quad a(r) = -\frac{5}{2r}$$

we get

$$\partial_\rho^2 \psi + V(\rho)\psi(\rho) = 0 \qquad \rho = \omega r$$

$$V(\rho) = -\left(1 + \frac{(R\omega)^4}{\rho^4}\right) + \frac{15}{4\rho^2} + \underbrace{\frac{(l+4)\,l}{\rho^2}}_{\text{for } l\text{-wave}}$$

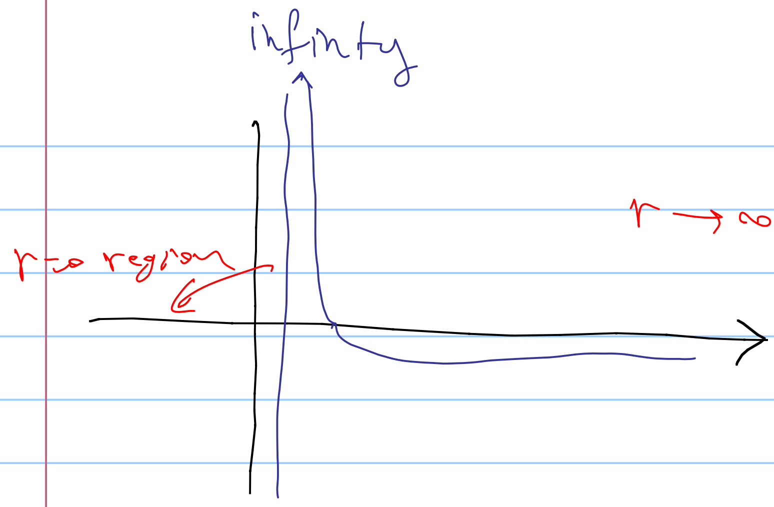
It has the following form



$$V(\rho_{Max}) = \frac{1}{(\omega R)^4}$$

In the limit of $\omega R \to 0$ $\quad V(\rho_{max}) \to \infty$

we get infinite barrier



We get an infint barrier which seprates two regions $r=0$ and $r\to\infty$ where we get free gravity.

Therefore the modes at infinty which are free gravity can freely propagate in the bulk but because of an infinte barrier they can't reach the brane ($r=0$ modes) or in other words they can't intract with $r=0$ modes.

on the other hand the near horizon modes can't escpe from near horizon region due to an infinit barrier.

So we get two decoupled system

- free gravity
- near horizon modes

[note that because of $f = (1 + \frac{R^4}{r^4})$ red shift geometry also decoupls]

Let's see what are the near horizon modes.

They are IIB which leaves at $r=0$ region in the D_3-brane background.

But at $r \to 0$ one gets

$$f = 1 + \frac{R^4}{r^4} \approx \frac{R^4}{r^4} \quad \text{at } r \to 0$$

So

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2$$

$+ \; N$ flux

It is $AdS_5 \times S^5$ with N-flux.

Now Let see what we got

we have two diffrent description

for D_3-brane

- open string (gauge theory)
- closed string (gravity)

In both descriptions we have decoupling lit

in which two diffrent systems decouple

from each other

1) $\mathcal{N}=4$ $U(N)$ SYM + free gravity

2) type IIB on $AdS_5 \times S^5$ + free gravity
+ N - flux

in both descriptions we have a system which is free gravity. therefore it is natural to assume that the other system is the same. So:

$\mathcal{N}=4$ $U(N)$ SYM $\equiv$ Type IIB SUGRA on $AdS_5 \times S^5$ + N-flux

In general the Maldacena's conjecture is

$\mathcal{N}=4$ $U(N)$ SYM theory in four dimensions is "the same as" type IIB superstring on $AdS_5 \times S^5$ with fluxes on S^5

we will see what is the meaning

of "the same as".

Let's study the decoupling limit better.

In both cases we have $\alpha' \to 0$

But we need to keep something fixed

to get nontrivial theory.

In open string picture we get

$$\alpha' \to 0 \qquad g_s = \text{fixed}$$

this means we are taking the gauge

coupling fixed.

In closed string we have $\alpha' \to 0$

but we want to keep the energy in

the string scale fixed. $E \sim \sqrt{\alpha'}\, E_{\text{near horizon}} = \text{fixed}$

but we have red shift as

$$E_{\text{infinity}} \sim f^{-1/4} E_{\text{near horizon}} = \text{fixed}$$

$$E_{\text{infinity}} \sim \frac{r}{l_s} E_{\text{near horizon}} = \frac{r}{l_s^2} \underbrace{l_s E_{\text{near horizon}}}_{\text{fixed}}$$

So $\frac{r}{l_s^2} = \text{fixed}.$

decoupling limit : $\alpha' \to 0$ $\quad \frac{r}{\alpha'} = \text{fixed}$

Question

In what sense these two theories are the same ?

To understand the duality (correspondence) let's look at diffrent parameters in both sides.

The Correspondence has two levels

- Kinematics - dynamics

In the first level we compare diffrent parameters in two sides. In the dynamical level we would like to see how one can find the dynamics of one side from the other.

Kinematic

Starting from D_3-bran we have type IIB with 16 supercharges. Taking near horizon limit we have susy enhancement; we get 32 supercharges. In fact $AdS_5 \times S^5$ is maximally supersymmetric solution.

we have sigma mode with $SU(2,2|4)$ target space. The bosonic part is

$SU(2,2) \times SU(4) \sim SO(2,4) \times SO(6)$

which is the isometry of $AdS_5 \times S^5$

$AdS_5 \to SO(2,4) \to -X_0^2 - X_5^2 + X_1^2 + \cdots + X_4^2 = R^2$

$S^5 \to SO(6) \to Y_1^2 + \cdots + Y_6^2 = R^2$

In string theory we have dilaton whose value at infinity is string coupling $g_s = e^{\phi}$

when $e^{\phi} \ll 1$ one can forget the stringy corrections and in fact the gravity is a good approximation.

for $e^{\phi} \sim 1$ one has to take into account

The string loop.

since we have a curve space there is a curvature R and therefore the classical gravity is valid for $\alpha' R \ll 1$.

for $\alpha' R \sim 1$ we have α'-correction

In string theory in differen level we have diffrent objects.

- at low energy we have massless modes and in our case we have KK modes (on S^5)
  (These are point like particles)
- classical string
- other classical objects like p-branes wrapped on some space.

Now look at gauge theory side.

we have $\mathcal{N}=4$ SCFT in four dimensions.

$\mathcal{N}=4$ susy in $d=4$ has 16 supercharges.

but since it is superconformal it is 32.

The superconformal group in $d=4$ is

$$SU(2,2|4) \xrightarrow{\text{bosonic part}} SO(2,4) \times SO(6)$$

super poincaré

R-symmetry

In gauge theory we have N (number of colors)

and 't Hooft coupling $\lambda = g_{YM}^2 N$.

As we have seen there is $\frac{1}{N}$ expansion and in large N only planer diagrams dominat.

one the other hand the gauge theory is strongly coupled when $\lambda \gg 1$ while it is weakly coupled for $\lambda \ll 1$

Finally in the gauge theory side we have gauge invariant operators which are labled by some number which are given by some representation of $SU(2,2|4)$.

In fact there is one to one correspondance between the objects of two sides. But one needs to identify them.

From BI action we can easily see

$$\boxed{g^2_{YM} = g_s}$$

on the other hand from $AdS_5 \times S^5$ solution we find

$$\alpha' R \sim \frac{1}{\sqrt{g_s N}} = \frac{1}{\sqrt{\frac{g^2 N}{4\pi}}} = \frac{1}{\sqrt{\lambda}}$$

so $\boxed{\alpha' R \sim \frac{1}{\lambda^{1/2}}}$

for fix λ one gets

$$e^{\Phi} \sim g_s = \frac{g_s N}{N} = \frac{\lambda}{N}$$

so $\boxed{e^{\Phi} \sim \frac{1}{N}}$

This is the same as what we saw in 't Hooft Large N limit expansion.

To summerize we got

String theory	Gauge theory
$SU(2,2\|4)$	$SU(2,2\|4)$
$SO(2,4) \times SO(6)$ (isometry)	$SO(2,4) \times SO(6)$
	SUSY R-symmetry
e^{ϕ}	$\frac{1}{N}$
$\alpha' R$	$\lambda^{1/2}$
$SL(2,2)$ duality	$SL(2,Z)$ duality
gravity modes, string modes, branes ----	gauge invariante operators

It is weakly – strongly coupled duality

$\alpha' R \ll 1$ weakly coupled gravity $\Longrightarrow$

$\lambda \gg 1$ strongly coupled gauge theory

Because of this, it is hard to check the duality.

To finish the conjecture we need to see what is the statment of the dynamics.

How one can read the dynamics of one side from the other side ?