Directed walks and vertex models: renormalization, multicriticality and beyond

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We discuss the exact renormalization of interacting directed random walks (DRW) (using dimensional regularization), and the exact results for the multicritical points. We discuss how these exact results of the DRW problem can give exact critical exponents of the ferroelectric five vertex model in all dimensions. The multicritical DRW's lead to a new class of vertex models whose multicritical exponents can be computed exactly. A random vertex model and its connection to the DRW problem in a random medium is also discussed.

1. Directed walks

1.1. Introduction

A directed random walk (DRW) in \( d' = d + 1 \) dimensions is a random walk that is forbidden to take steps in one particular direction, say the \(-z\) axis. An equivalent description would be that of a \( d\)-dimensional random walk with the step length as an extra dimension. Such a walk is rather simple, thanks to the simplicity of ordinary random walks [1,2]. However, in most situations where DRW's occur, complications arise either because of interactions of many chains or because of disorder (or randomness). (Chain, walk, string, line, polymer etc will be used interchangeably). We address some of these problems here.

DRW's occur in various problems like vertex models [3–7], flux lattice melting in high \( T_c \) superconductors [8,9], biomembrane phase transitions [10–12], dimer models [10–13], electrorheological fluids [14,15], polymeric nematics [15], commensurate–incommensurate (CI) transition [16,17], world lines of nonrelativistic particles etc. For many of these problems, disorder quite often becomes important especially in vertex models [18], flux lattice melting [9,19], and CI transition [20]. In such a situation, even a single chain in a random environment is highly nontrivial, let alone the many chain system. Despite its simplicity, this problem of DRW in a random medium (to be called “Ran-D-Walk”) has attracted a lot of attention in the last few years because of
its richness to show most, if not all, the complexities of a disordered system. Furthermore, Ran-D-Walk occurs, as such, in spin glass, random magnets [21], surface growth ("KPZ" equation [22]) and other areas [23].

In this paper we review the renormalization group approach to the pure interacting DRW problem and show how many of the physical properties can be calculated exactly through RG. In the process we make contact with the various properties of the disorder problem. We next review the application of these exact results for vertex models and then, show how one can define a new class of anisotropic models whose critical behaviour can be obtained exactly through DRW.

1.2. Hamiltonian and thermodynamics

Our approach will be to use a continuum version for walks with interactions among points of various chains if and only if they have the same z coordinate (consistent with the directed nature) – an approach borrowed from the flux lattice melting problem [8]. In the path integral approach, the Hamiltonian for n chains can be written as

$$\mathcal{H}_n = \frac{1}{2} \sum_{a=1}^{n} \int_0^N dz \left( \frac{\partial r_a(z)}{\partial z} \right)^2 + \sum_{a<\beta} \int_0^N dz \; V(r_a(z) - r_\beta(z)), \quad (1.1)$$

where \(r_a(z)\) is the (d-dimensional) coordinate of the point at contour length \(z\) of chain \(a\), \(n\) the total number of chains, and \(V(r)\) is the interaction potential. The first term on the r.h.s. of eq. (1.1) is the usual entropic contribution (or the elastic energy) of the polymers, each of length \(N\), while the second term ensures the mutual repulsion at the same \(z\) of any pair of chains \(a, \beta\). The two noteworthy features are (1) the absence of any intrachain interaction and (2) the special nature of interaction vis-a-vis conventional polymers. So far as the interaction is concerned the simplest situation is a delta function potential \(V(r) = v_2 \delta(r)\), \(v_2 > 0\). One may consider various possibilities, like attractive potentials [24–26], higher order interactions [25,26], long range interaction [28], etc. Motivations for many of these come partly from the disorder problem (see below).

The thermodynamics of the interacting polymer system is described by the osmotic pressure \(\Pi\). This can be computed in a canonical ensemble with the Hamiltonian of eq. (1.1) or equivalently in the grand canonical ensemble with a chemical potential. In fact, the connection with the vertex model, to be discussed later on, would require \(T - T_c\) as the chemical potential. In any case, the osmotic pressure will have the conventional virial expansion
\[
\frac{\Pi}{kT \rho} = 1 + A_2(N) \rho + \cdots + A_m(N) \rho^{m-1} + \cdots,
\]  

(1.2)

where \( k \) is the Boltzmann constant, \( T \) the temperature, \( \rho = n/V_\perp \) the density of lines, \( V_\perp \) being the volume of the transverse \( d \) dimensional space, \( A_2 \) (\( A_m \)) is the second (\( m \)th) virial coefficient. As is known for conventional polymers, the crossover from dilute to semidilute solutions is determined by the virial coefficients – in particular by \( A_2 \) for two body interactions of eq. (1.1) [29].

We may generalize the Hamiltonian of eq. (1.1) as

\[
\mathcal{H}_n = \frac{1}{2} \sum_{\alpha=1}^{n} \int_0^N dz \left( \frac{\partial r_\alpha(z)}{\partial z} \right)^2 + \sum_{p=2}^{m} v_p \sum_{\text{chains}} \int_0^N dz \delta(r_{\alpha_1}(z) - r_{\alpha_2}(z)) \cdots \delta(r_{\alpha_{p-1}}(z) - r_{\alpha_p}(z)) ,
\]  

(1.3)

with \( v_m > 0 \). This general case is rather clumsy to study, but several special cases can be considered. The simplest situation is the multicritical case \( v_2 = \cdots = v_{m-1} = 0 \) but \( v_m > 0 \). This pure \( m \) body interaction will lead to \( A_2 = \cdots = A_{m-1} = 0 \) in eq. (1.2), but \( A_m \neq 0 \). The thermodynamics is then determined by \( A_m \). We will show how to get these virial coefficients for \( N \to \infty \) – in fact they can be computed exactly.

1.3. Directed walks in random medium ("Ran-D-Walks") and equivalent pure problems

A Ran-D-Walk can be described by a Hamiltonian [21,23]

\[
\mathcal{H} = \frac{1}{2} \int_0^N dz \left( \frac{\partial r(z)}{\partial z} \right)^2 + \int_0^N V_D(r(z), z) dz ,
\]  

(1.4)

where \( V_D(r, z) \) is a random potential which we take to be Gaussian (other situations have also been discussed [30]), with \( \langle V_D \rangle = 0 \) and \( \langle V_D(r, z) V_D(r', z') \rangle = \Delta \delta(r-r') \delta(z-z') \) if the randomness is uncorrelated, or \( = \Delta |r-r'|^{-2\alpha} \delta(z-z') \) if there is correlation in space [21–23].

One of the methods to study such disordered problems is to use the replica trick [31]: \( \langle \ln Z \rangle = \lim_{n \to 0} (\langle Z^n \rangle - 1)/n \), where \( Z \) is the partition function,
and \((\ldots)\) denotes the disorder averaging. This procedure avoids the log averaging at the cost of a controversial \(n \to 0\) limit – a controversy we will not get into.

With the distribution of \(V_D\) given above, one can perform \((Z^n)\) and write a Hamiltonian for \(n\) chains ("replicas") very similar to eq. (1.1) with \(V(r) = -\Delta \delta(r)\) for uncorrelated noise or \(V(r) \sim |r - r'|^{-2\sigma}\) for correlated noise. In other words, the correlation of disorder dictates the range of the interaction of the equivalent pure many chain problem. Furthermore, an attractive interaction will, for stability, necessitate consideration of higher order interaction as in eq. (1.3).

In the replica approach, the quantities one wants to calculate are the overlaps of the different replicas [23,32]. For example, the overlap of two replicas would mean the fraction of the lengths two chains of the Hamiltonian in eq. (1.1) spend together. This quantity is obtained from the connected two chain partition function [32] and, for the pure problem, the latter is precisely the second virial coefficient. Similarly one wants to calculate the \(m\)-chain overlap and that would come from the \(m\)th virial coefficient case.

In short, a Ran-D-walk problem can be formulated as a pure interacting many chain problem (with \(n \to 0\)) and many of the properties that characterize the disordered system can be thought of as generalizations of appropriate pure quantities, though the language used is different.

1.4. Second virial coefficient: renormalization

The second virial coefficient is obtained from

\[ \mathcal{H}_2 = \frac{1}{2} \int_0^N dz \left( \frac{\partial r_1}{\partial z} \right)^2 + \frac{1}{2} \int_0^N dz \left( \frac{\partial r_2}{\partial z} \right)^2 + v_2 \int_0^N dz \delta(r_1(z) - r_2(z)) \]

as

\[ A_2 = -\frac{Z_2(N,N,v_2) - Z_1^2(N)}{Z_1^2(N)}, \]

(1.6)

where \(Z_2(N,N,v_2)\) is the partition function of two interacting DRWs and \(Z_1(N)\) is that of one DRW [27,7]. The partition function is defined as

\[ Z_2(N,N,v_2) = \int Dr_1 Dr_2 e^{-\mathcal{H}_2}, \]

where the integral is over all possible configurations ("paths") of the chains. \(Z_1\) is also defined similarly.
A simple dimensional analysis of the parameters in the Hamiltonian in eq. (1.5) shows that \([N] \sim L^2\) and \([v_2] \sim L^{d-2}\), identifying \(d = 2\) as the upper critical dimension.

A perturbation expansion in \(v_2\) for \(A_2\) can be developed and thanks to the normalization in \(A_2\), only ladder type diagrams occur [27]. The resulting series is

\[
A_2 = NV_\perp v_2 \left( 1 + \sum_{n=1}^{\infty} (-)^n \frac{v_2^n}{(4\pi)^{nd/2}} N^{n(1-d/2)} \frac{\Gamma^n(1-d/2)}{\Gamma(2 + n(1-d/2))} \right) .
\]

(1.7)

Such a Mittag-Leffler type series also occurs in other problems [33].

As expected, there are divergences in every order, and these divergences can be tamed through dimensional regularization with minimal subtraction of the poles. Since there is no self-interaction (unlike polymers) \(N\) does not require any renormalization. We introduce an arbitrary length \(L\) to define a dimensionless coupling constant \(u_0 = v_2 L^{2-d}\) and write the renormalized coupling as

\[
u_0 = u \left( 1 + D_1 u + D_2 u^2 + \cdots \right) .
\]

(1.8)

An analysis of eq. (1.7) shows that \(D_p = (2\pi \epsilon)^{-p}\) \((\epsilon = 2 - d)\) for all \(p\), so that

\[
u_0 = \frac{u}{1 - u/2\pi \epsilon} = \frac{2\pi \epsilon u}{2\pi \epsilon - u} .
\]

(1.9)

One can define the \(\beta\) function

\[
\beta(u) = L \left( \frac{\partial u}{\partial L} \right)_{v_2} = \epsilon u_0 \frac{\partial \ln u_0}{\partial u} = u \epsilon \left( 1 - \frac{u}{2\pi \epsilon} \right) ,
\]

(1.10)

which is known exactly, and is the same as the one loop result (a consequence of the geometric series). The fixed point is, therefore, \(u^* = 2\pi \epsilon\), true to all orders of \(\epsilon\). By an integral representation \([7,33]\) \(A_2\) can be computed exactly at the fixed point, and is given by

\[
A^* = V_\perp (4\pi N)^{1-\epsilon/2} \frac{\sin(\pi \epsilon/2)}{\pi (1-\epsilon/2)} .
\]

(1.11)

The important point to note, apart from the exactness of the result, is that \(A_2 \sim N^{d/2}\) \((\epsilon = 2 - d)\) which is what dimensional analysis predicts. In fact, no anomalous dimension is required for DRW's because of the absence of any "self-" interaction.
1.5. Many chain interaction: exact renormalization, multicriticality

We now consider the m chain Hamiltonian of eq. (1.3) with pure m-body interaction:

\[ H = H_0 + v_m \int_0^N dz \delta [r_1(z) - r_2(z)] \cdots \delta [r_{p-1}(z) - r_p(z)], \]  

(1.12)

\( H_0 \) being the free chain part. The quantity to calculate is \( A_m = -(Z_m - Z_m^\to)/Z_m^\to. \) This definition of the mth order virial coefficient would require lower order cummulants if lower order interactions were included in the above Hamiltonian.

Dimensional analysis shows that the upper critical dimension is \( d_m^* = 2/(m-1). \) An RG analysis would require a double expansion in \( v_m \) and \( \epsilon = d_m^* - d. \) However, it turns out that the natural variable is \( \epsilon_m = (m-1)(d_m^* - d) \) [25,26].

Just like the second virial coefficient case, only ladder type diagrams appear and all the diagrams can be evaluated. The technical reason for the exact solvability is the directed nature so that the product of m distribution functions can be expressed as a single distribution function with extra m dependent factors. To bring out the simplicity of the model, we define the following dimensionless variables (different from section 1.4):

\[ g_m = A_m [V_{\perp} (2\pi N)^{(m-1)d/2} m^{d/2}]^{-1}, \]  

(1.13a)

\[ \zeta_m = [ (2\pi)^{(m-1)d/2} m^{-d/2}] v_m N^{1-(m-1)d/2}. \]  

(1.13b)

In terms of these variables,

\[ g_m = \sum_{n=0}^{\infty} (-\zeta_m)^{n+1} \frac{\Gamma^n (\epsilon_m/2)}{\Gamma(2 + n\epsilon_m/2)}. \]  

(1.14)

again a Mittag-Leffler series. The renormalization follows the same path as in section 1.4. The \( \beta \) function is (\( \zeta_{mR} \) is the renormalized \( \zeta_m \))

\[ \beta(\zeta_{mR}) = N \frac{\partial \zeta_{mR}}{\partial N} = \frac{\epsilon_{mR}}{2 \zeta_{mR}} \left( 1 - \frac{2\zeta_{mR}}{\epsilon_m} \right), \]  

(1.15)

which is exact, and is the same as the one loop equation. The fixed point is \( \zeta_m^* = \epsilon_m/2. \) This agrees with the result of section 1.4 for \( m = 2. \) Note that the \( \beta \) function has no explicit \( m \) dependence – \( m \) is hidden in the variables.
Hence, in terms of these variables, all quantities will have similar behaviour, like

\[ g_m^* = \frac{\sin(\pi \epsilon_m/2)}{\pi (1 - \epsilon_m/2)}, \]  

(1.16)

with \( m \) in the “metric”. This is a case of grand universality.

Since the \( \beta \) function is known exactly, the correction to scaling exponent, the special log correction at the upper critical dimension can also be computed, and all of these show the grand universality [25,26].

We might add that the \( O(n) \phi^4 \) model for \( n \to \infty \) also has similar simplicity but the simplicity is lost for multicritical points. In contrast, the multicritical hierarchy can be studied exactly in the directed walk models.

To discuss the multicritical nature of these fixed points, we just consider the \( v_2, v_3 \) case. For \( v_3 > 0 \), if \( v_2 \) changes sign (i.e. there is attraction between two chains), one expects a “fusion” process to take place, in contrast to the jiggling of the chains with occasional collisions for \( v_2 > 0 \). In fact, for two chains, one can go over to the relative coordinate and get a “relative” chain attracted to origin. For a similar lattice model of a chain attracted to the origin, Majumdar has shown that for \( d < 2 \), this relative chain, in the limit \( N \to \infty \), is always coiled around the origin (size \( \sim O(1) \)) [24]. This change in behaviour is a manifestation of the third order multicritical point. If we take a many chain problem, then there will be a phase separation between semidilute solution and a dilute phase of “fused” chains, with \( v_2 = 0 \) as a special end point (\( N \to \infty \)) [25,34]. The behaviour is very similar, but not identical, to the tricritical (or theta point) phenomenon [29].

1.6. Many chain case

The osmotic pressure, eq. (1.2), for \( N \to \infty \), can also be computed to \( O(\epsilon = 2 - d) \) for two body interaction. Now, using the identity \( \delta (r_1 - r_2) = \int dr \delta (r - r_1) \delta (r - r_2) \), the \( v_2 \) part of the Hamiltonian of eq. (1.5) can be written as

\[ H_n = H_0 + v_2 \int dz \int dr \rho^2(r, z), \]  

(1.17)

where \( \rho (r, z) \) is the local density variable \( \rho (r, z) = \sum_\alpha \delta (r - r_\alpha (z)) \).

In a mean field approximation, ignoring fluctuations, we get

\[ \frac{\Pi}{kT} = \rho + v_2 \rho^2. \]  

(1.18)
However, for $d \leq 2$, fluctuations are important and since the fixed point is known exactly (section 1.4), one can get the the right $\rho$ dependence from the $O(\epsilon)$ results [7]. The result in the limit $N \to \infty$ limit is

$$ II \sim \rho^{(d+2)/d}, $$

which agrees with the exact result known for $d = 1$ from vertex and dimer models [11-13].

2. Vertex models

2.1. Models and exponents

The ferroelectric five vertex model, fig. 1, is an exactly solvable model in two dimensions [3], the model can be defined in any dimension $d'$ on “diamond” type lattices [5–7]. The ground state is a vacuum state and the excited states are obtained by creating strings which are just $d' = d + 1$ dimensional DRW. The critical point is $T_c = \epsilon/k \ln 2$ for all $d$, and the critical behaviour is determined by the behaviour of a finite density DRW with density $\rho \to 0$. The relevant exponents are $\alpha$, $\beta$, $\nu_\parallel$, and $\nu_\perp$ defined as

$$ c \sim t^{-\alpha}, \quad \rho \sim t^\beta, \quad \xi_\parallel, \perp \sim t^{-\nu_\parallel, \perp}, $$

where $t = (T - T_c)/T_c > 0$, $c$ is the specific heat, $\rho$ is the density of the lines, and $\xi_\parallel (\xi_\perp)$ is the length scale parallel (perpendicular) to the $z$ direction. In two dimensions, $\alpha = \beta = 1/2$, and $\nu_\parallel = 2\nu_\perp = 1$ are known exactly. Note that the system remains frozen in the ground state for $T < T_c$. The problem is to get the exponents in arbitrary dimensions.

A clarification regarding $\xi_\parallel (\xi_\perp)$ is probably needed. These models in two dimensions show algebraic decay of correlations [35,13] for large distances. However, the low temperature frozen phase indicates that there is a short distance “frozen” type strong correlation which goes over to the power law

Fig. 1. Five vertices for the five vertex model. It costs an energy $\epsilon$ to create a DRW (thicker line) at a site and each DRW traverses the whole lattice in the $z$ direction. For the random model, $\epsilon (> 0)$ is a random number of finite variance.
decay at large distances. \( \xi_\perp, \xi_\parallel \) define the corresponding crossover lengths in the two directions and not the conventional exponential decay. Again, such length scales are enough to explain the exact finite size scaling known in two dimensions [36,13] and their exponents do satisfy the anisotropic hyperscaling relation [36,16]. These are, therefore, the natural length scales for the critical behaviour.

2.2. Formulation

In the transfer matrix approach, the free energy is obtained from the function

\[
f(\rho) \sim -t\rho + s(\rho),
\]

(2.2)

where \( s(\rho) \) is the loss of free energy of the DRWs due to interaction. Minimizing \( f(\rho) \) with \( \rho \) (i.e., choosing the largest eigenvalue of the transfer matrix), we can obtain the exponents \( \alpha \) and \( \beta \) [6,7,11,12].

Use of thermodynamic relations shows that \( s(\rho) \) is related to the osmotic pressure discussed in the previous section [7]. Note that by construction, two DRW’s of fig. 1 cannot be at the same site – a two body repulsion at work! We have seen that this repulsive interaction leads to strong fluctuations for \( d' < 3 \) (i.e., \( d < 2 \)) that changes the \( \rho \) dependence of \( s(\rho) \) from \( \rho^2 \) (mean field) to \( \rho^{(d+2)/d} \). This \( \rho \) dependence leads to [7]

\[
\beta = \frac{d}{2} = \frac{d' - 1}{2} \quad \text{and} \quad \alpha = \frac{3 - d'}{2},
\]

(2.3)

which agree with exact results for the \( (d = 1) \) two-dimensional five vertex model or the Kasteleyn dimer model.

2.3. Finite size scaling: \( \nu_\perp \) and \( \nu_\parallel \)

So far we are considering a thermodynamic system that gave \( \beta \) and \( \alpha \) (eq. (2.3)). To obtain the length scale exponents, we use finite size scaling (FSS) [6,7].

Let us take a system which is finite (length \( N \)) in the \( z \) direction. The directed polymers are, therefore, of length \( N \). Now, near the critical point, FSS dictates that \( N \) should appear in the scaled form \( N/\xi_\parallel \) or \( N^{\nu_\parallel} \), and this variable would give a smooth connection between the low temperature and the high temperature phase – after all there is no critical point in this finite geometry.

For the polymer, the crossover from dilute \( (\rho = 0) \) to the semidilute regime is also described by a scaling form [29], where the density \( \rho \) is scaled by an
overlap density $\rho^*$. This $\rho^*$ gives a density at which the chains feel each other (i.e., $\rho^* \sim N/(R^2)^{d/2}$) and is quantitatively determined by the appropriate virial coefficient, $A_2$ in this case. The scaled variable is, therefore, $\rho/\rho^*$ or $A_2\rho$, and using eq. (2.1), that $\rho \sim t^\beta$, we can write the variable as $A_2t^\beta$. If $A_2 \sim N^\psi$, then the scaled combination variable is $Nt^\beta/\psi$. Comparison with FSS yields

$$\nu_\parallel = \beta/\psi = 1$$  \hspace{1cm} (2.4)

using eqs. (2.3) and (1.10).

The length scale in the transverse direction is just the average spacing of the lines, $\xi_{\perp} \sim \rho^{-1/d}$ or

$$\nu_{\perp} = \beta/d = 1/2$$  \hspace{1cm} (2.5)

using again eq. (2.3). However, note that

$$\nu_\parallel = \frac{d}{\psi} \nu_{\perp} = 2\nu_{\perp}$$  \hspace{1cm} (2.6)

is determined only by the virial coefficient [6,7].

We therefore see that the five vertex model has superuniversal $\nu_\parallel$, and $\nu_{\perp}$ exponents. The specific heat exponent is determined by the anisotropic hyperscaling

$$2 - \alpha = (d' - 1)\nu_{\perp} + \nu_\parallel$$  \hspace{1cm} (2.7)

and agrees with eq. (2.3).

It is also worth pointing out that FSS is based on the hypothesis that the length of the system does not require any renormalization, and in the present context this is automatically guaranteed by the nonrenormalization of the length of the chains.

2.4. Exponents: scaling relations

It is possible to connect the vertex model exponents to the geometric size exponent $(R^2)^{1/2} \sim N^\nu$ for a DRW. Such arguments were used in refs. [20,21] in a different context. One can think of a DRW in the many chain system as confined in a region of size $\xi_{\perp}^d$. Equating the elastic energy for confinement to the "chemical" potential, $t$ in this case, we get [7,20,21]

$$\beta = \frac{d\nu}{2(1-\nu)}, \quad \nu_{\perp} = \nu, \quad \nu_\parallel = \nu[2(1-\nu)]^{-1}. $$  \hspace{1cm} (2.8)
Of course, $\alpha = 1 - \beta$. For the pure problem $\nu = 1/2$, and we get back the exponents of section 2.3. Recently Natterman et al. [9] showed in connection with the flux lattice melting problem, that eq. (2.8) is true to one loop order for disordered systems also. See also section 2.6.

2.5. New multicritical vertex models

In the previous sections, we exploited the connection between the known five vertex model and the DRW with two chain repulsion. We can use this mapping to define a new class of vertex models that will allow up to $m - 1$ chains (see section 1.5) to intersect without any penalty but not $m$ chains. Such vertex models are obtained on triangular lattices ($m = 3$), union jack lattice ($m = 4$) and so on [37].

The formulation for any given $m$ is identical to sections 2.2 and 2.3, and the results follow from the exact beta functions of section 1.5. Note that since $d_m^* \leq 1$ for $m \geq 3$, a two dimensional model will be described by the mean field DRW results. Hence, for any $m \geq 3$,

$$s(\rho) \sim \rho^m, \quad (2.9)$$

which gives $\alpha = (m - 2)/(m - 1)$ and $\beta = 1/(m - 1)$. There will be an extra log factor for $m = 3$, as a signature of upper critical dimension [37].

For $d' < d_m^* + 1$, we can again use FSS argument of section 2.3, $\rho^*$ now being determined by $A_m$. The exact result of section 1.5, then shows that for all $m$, $\nu_\parallel = 2\nu_\perp = 1$ for all $d$. In fact $\alpha = (m - 2)/(m - 1)$ for $m > 3$ agrees with the anisotropic hyperscaling relation at $d' = d_m^* + 1$, as it should.

We, therefore, obtain exact multicritical exponents for a class of vertex models. A few questions remain: (1) Are these models exactly solvable in two dimensions?, (2) What happens to conformal invariance? The multicritical nature is similar to the $m$th order multicritical points of isotropic systems ($\phi^{2m}$ theories) – but the exponents are not the same. Note that the path integral formulation guarantees a unitary theory. (3) Is it possible to get the full multicritical description of these models?

2.6. Ran-D-walk and vertex models

Let us consider the vertex model of fig. 1 but with the energy parameter $\epsilon$ a positive random number. The critical behaviour is then determined by Ran-D-wlaks. The exponents are given by eq. (2.8). It is straightforward to check that the upper critical dimension ($\alpha = 0$) is 2, if the Kim-Kosterlitz formula $\nu = (d + 2)/(2(d + 1))$ is used [18,38,19].
A more interesting result is obtained from the CCFS [39] inequality, \( d\nu_\perp + \nu_\parallel > 2 \) (generalized for anisotropic systems), and using \( \nu_\perp = \nu \nu_\parallel = \nu [2(1 - \nu)]^{-1} \), we get [18]

\[
\nu \geq \frac{3}{d' + 3}.
\]

(2.10)

In other words, the Flory exponent of Ran-D-walk is a lower bound to the size exponent \( \nu \) [40].

The breakdown of the hyperscaling in random systems is connected to the anomalous growth of the free energy scale in a correlation volume [41]. Taking a modified hyperscaling relation as

\[
2 - \alpha = d\nu_\perp + \nu_\parallel + \Gamma,
\]

(2.11)

we find, from the exponents in eq. (2.8) that \( \Gamma = (1 - 2\nu)/2(1 - \nu) \). The breakdown exponent \( \Gamma \) vanishes if and only if \( \nu = 1/2 \) (i.e., a pure system). This turns out to be the same as for one and two chains [22,32]. This leads to an important conclusion that the anomalous growth of the free energy of a finite density many chain system is the same as for a single chain, i.e. does not require any new exponent.

3. Summary

We have shown how several "polymeric" properties of directed polymers can be computed exactly through RG – from many chain system to multicritical behaviour. We further showed how these can lead to a continuum formulation of vertex models and a new class of vertex models characterized by an integer \( m > 2 \). For the latter case several multicritical exponents can be computed exactly. The synthesis of the vertex model and directed walks with disorder leads to a lower bound to the geometric size exponent of the walk, and shows that the many chain system, as in the flux lattice melting problem, is described by the same free energy fluctuation exponent as a single chain.

Acknowledgements

This paper summarizes several pieces of work done in collaboration with B. Duplantier, J.J. Rajasekaran, and S. Suresh Rao. It is a great pleasure to thank them for enlightening discussions.
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