Lecture XXX

Renormalization, Regularization and Electrostatics

Let us calculate the potential due to an infinitely large object, e.g. a uniformly charged wire or a uniformly charged sheet. Our main interest in this class of problems is in tackling divergences. There are divergences in these calculations because of our original assumption of the potential at infinity to be zero. In order to make sense of these divergent integrals, we shall actually make use of the arbitrariness of the zero of the potential. This is the approach of *regularization* and *renormalization*. This is not a problem of electrostatics alone - identical problems occur in Newtonian gravity also.

1 Potential due to a charged wire/sheet

Consider the two cases: (a) a uniformly charged line with charge density λ per unit length, and (b) a planer sheet with a uniform charge density λ per unit area. We want the potential at P. Draw OP perpendicular to the source and choose O as the origin of the co-ordinate system. We may also consider a three-dimensional case where the whole space is filled with a uniform charge density and P is any point in space. The potential due to a small element S, (a) dl at length l, or, (b) $l\, dl\, d\theta$, (see Fig. 1) as

$$(i)\quad \frac{\lambda\, dl}{4\pi\varepsilon_0\sqrt{l^2+r^2}} \text{ or, } (ii)\frac{\lambda\, l\, dl\, d\theta}{4\pi\varepsilon_0\sqrt{l^2+r^2}}, \tag{1}$$

because SP $=\sqrt{l^2+r^2}$. The potential at P is then obtained by integrating over the source co-ordinates, i.e. $l \in (-\infty,\infty)$ in (i) or $l \in (0,\infty), \theta \in [0,2\pi]$ in (ii) in Eq. (1). We are in 3-dimensions but the source is of $d = 1, 2$ dimensions in the two cases.

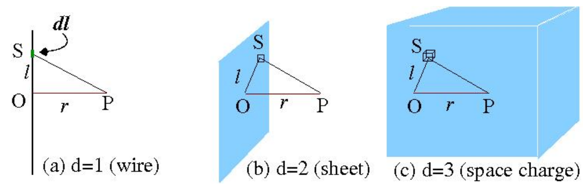

Figure 1: (a) A Line charge and (b) a planar charge problem. The potential at P due to a small infinitesimal source element at A is given by Eq. (1). (c) A uniformly charged space and P is a point anywhere in the space. This could be a uniformly distributed mass as in the universe in a gravity problem. The origin in (a) and (b) can be chosen unambiguously but not in (c).

We may also consider a third case where the whole space is filled with a uniform density λ of charge and P is any point in space. Unlike (a,b), there is no obvious choice for origin and so it is chosen arbitrarily., with OP as the z-axis. The potential is then given by slightly different form,

$$(iii)\frac{\lambda\, l^2\ \sin\theta\, dl\, d\theta d\phi}{4\pi\varepsilon_0\sqrt{l^2+r^2-2lr\cos\theta}}, \tag{2}$$

where (r,θ,ϕ) is the location of the charged volume element.

Q : How is d (dimensionality) defined?

This procedure is based on two prejudices:
(P1) The potential due to a point charge q at a distance r from the charge is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}.$$

and,
(P2) the superposition principle i.e. the idea of additivity of the potentials due to individual charges.

There is no immediate reason to question (P2), otherwise we would not get the simple q dependence in (P1), but the form in (P1) is not obvious or sacrosanct. To get this particular form, we have tacitly assumed that $V(r) \to 0$ as $r \to \infty$. This is, to some extent, an arbitrary choice - a choice of convenience, not necessity. But we know it works.

Remember that the potential is defined as the work done to bring a unit positive charge from infinity to point P - but why form infinity? Compare, e.g., with the earth's gravitational field. The potential is $g\,h$ where h is the height from the earth's surface. No reference to infinity. This is because the absolute value of the potential has no significance; it is the potential difference between two points that's important. In that case, the potential of any arbitrary point can be chosen as the reference, and set to zero, as per our whim or taste. The choices are obvious: $V = 0$ as $r \to \infty$ for the Coulomb potential, but $V = 0$ for $h = 0$ for the earth's gravitational field case.

> Q : The earth's field is also Newtonian gravitational, then why it is linear and not inverse of distance? Can we afford this arbitrariness (choice of origin of potential) if the potential is $V(h) = gh + bh^2$?

1.1 Problem of divergence

The whole problem lies with the integrals as we now see for the three cases separately.

1.1.1 d=1,d=2

Look at the integrals from Eqs. (1). The denominator for the integrand is never zero so long $r \neq 0$. To analyze these, we therefore need to look at large l behaviour (for $l \gg r$):

$$V_1 \sim \int^{\mathcal{L}} \frac{dl}{l} \sim \ln \mathcal{L} \to \infty, \text{ as } \mathcal{L} \to \infty, \text{ (log divergence)} \tag{3}$$

Similarly

$$V_2 \sim \int^{\mathcal{L}} \frac{l\ dl}{l} \sim \mathcal{L}, \text{ (power law divergence).} \tag{4}$$

Therefore, both $V_1(r)$ and $V_2(r)$ are infinity for all r including $r \to \infty$. But by definition [prejudice (P1), and Eq. 1], the potential at $r \to \infty$ by every small element is zero. We find, after superposition (prejudice (P2)), that addition of zero's yield infinity (potential at $r \to \infty$).

These are not ill defined or ill posed problems, because the electric field can be calculated without any ambiguity. By using symmetry and choosing appropriate gaussian surfaces, we get

$$\text{(a)}\mathbf{E_1} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}, \text{ and } \quad \text{(b)}\mathbf{E_2} = \frac{\lambda}{2\pi\varepsilon_0}\hat{k},$$

where $\hat{r}$ is the radial vector in the cylindrical coordinates for (a) while $\hat{k}$ is the z-direction unit vector in (b). (In both cases the vectors are along OP in Fig 1.) By integrating, one finds

$$V_1 \sim \ln r, \text{ and} V_2 \sim r, \tag{5}$$

with $V_{1,2} \to \infty\ as\ r \to \infty$. We have deliberately used the " $\sim$ " sign because the log term is problematic. Here, r, as a length, is a dimensionwala quantity and there is no way we can take log of a dimensionwala object. We need something of the form $\ln(r/L)$ but who orders L?

In a nut-shell, we seem to know the answer to the potential problem but a direct calculation gives infinity from some divergent integrals. Is it possible to tackle the divergence and make sense out of it by getting finite answers?

1.1.2 d=3

The three-dimensional case is more trickier. The problem was recognized even in the Newton-era itself.

To do the integrals, we face the possibility of the denominator becoming zero (for $l = r, \theta = 0$), but that is not very serious, because the θ-part is integrable. This can be seen by expanding around $\theta = 0$ keeping

$l = r$, and to leading order the integral is just $\int d\theta$ by using $\sin\theta \sim \theta$ and $\cos\theta \sim 1 - \theta^2/2$. One therefore has to check the large l behaviour which is

$$V_3 \sim \int^{\mathcal{L}} \frac{l^2\, dl}{l} \sim \mathcal{L}^2, \text{ (power law divergence)}. \tag{6}$$

Going back to the Gaussian surface approach, we may choose a sphere of radius OP, centered at O. As per Gauss' theorem, only the charges inside the surface contributes to the field. And the spherical symmetry due to the infinite extent of the system, suggests a radial electric field only. Combining the two,

$$\mathbf{E_3} = \frac{4}{3\varepsilon_0}\,\lambda\mathbf{r},$$

so that the potential is

$$V_3(r) \sim \lambda r^2. \tag{7}$$

But O is arbitrary - so by choosing a different origin we may change both the electric field and the potential. In fact both can be made to vanish by choosing the origin at P - the most logical, symmetrical answer.

So the problem in this case is more deep-rooted than for $d = 1, 2$. Newtonian gravity case is ultimately solved by declaring that a revision of the force law is needed, but there is no such panacea for electrostatics. We cannot also ignore the problem because one has to face this situation in the case of an ionic solution in contact with buffers, or the jellium model of charged electrons moving in the infinite space of uniformly charged ionic background.

1.2 Generalizing the problem

In order to handle the divergent integral, we noted in Eqs. (3,4) that the nature of divergences depend on the dimensionality d of the source. This prompts us to consider the nature of the integral as a general function of the dimensionality d. We therefore consider a wider class of problems where the source is not just one or two dimensional but a d-dimensional one. The problem then is to find

$$V_d(r) = \frac{\lambda}{4\pi\epsilon_0} \int_0^\infty \Omega_d l^{d-1} \frac{dl}{\sqrt{l^2 + r^2}} \tag{8}$$

where Ω_d is the surface area of a d-dimensional sphere.

Agreeably, this does not include the three dimensional case, but the nature of divergence of this integral is the same as that in Eq. (6). Therefore the essential problem is captured by this.

Q : Find Ω_d. $\left(\text{Ans}: \dfrac{2\pi^{d/2}}{\Gamma(d/2)}\right)$

1.3 Dimensional analysis

Let us do a dimensional analysis first. Since λ is the charge density it follows

$$[\lambda] = \frac{[\text{charge}]}{\mathsf{L}^d} \quad \text{and} \quad [l] = [r] = \mathsf{L} \qquad (\mathsf{L}: \text{dimension of length}).$$

Define $v_d(r) = 4\pi\varepsilon_0 V(r)$, so that $v_d(r)] = [\text{charge}]\mathsf{L}^{-1}$. Since r is a running variable, it is better to introduce another arbitrary length scale L - this scale L, so to say, is the scale at which we are studying the problem. Define dimensionless quantities

$$\tilde{r} = \frac{r}{L},\ \tilde{l} = \frac{l}{L}, \tag{9}$$

and maintaining charge,

$$\tilde{\lambda} = \lambda L^d,\ \tilde{v}_d = v_d L.$$

Equating the charge dimensions, we can then write

$$\tilde{v}_d = \tilde{\lambda}\ f_d(r/L)$$

where the unknown function f_d needs to be determined. The potential from Eq. (8) can now be rewritten as

$$\tilde{v}_d(r) = \lambda\, L^d\, L^{1-d}\, \Omega_d \int_0^\infty \frac{l^{d-1}\, dl}{\sqrt{l^2+r^2}} \tag{10}$$

$$= \tilde{\lambda}\, \Omega_d \int_0^\infty \frac{\tilde{l}^{d-1}\, d\tilde{l}}{\sqrt{\tilde{l}^2+\tilde{r}^2}} \tag{11}$$

$$= \frac{\tilde{\lambda}}{\tilde{r}}\, \tilde{r}^d\, \Omega_d \int_0^\infty \frac{l^{d-1}\, dl}{\sqrt{l^2+1}} \tag{12}$$

$$= \tilde{\lambda}\, \tilde{r}^{d-1}\, \Omega_d \int_0^\infty l^{d-1}(1+l^2)^{-1/2} dl \tag{13}$$

The integral is r-independent but it is infinite for $d \geq 1$ though it is finite, for $d < 1$. The $\tilde{r}$ dependence is consistent with our expectations in Eqs. (5,7). It better be because it is pure dimensional analysis.

1.4 Dimensional Regularization

Such divergent integrals can however be handled by looking for the source of divergence in d and then by analytic continuation. In other words we treat d as a complex variable and study the behaviour of the integral in the complex d-plane.

By substituting $l = \tan\, x$, the integral in Eq. (13) can be written as

$$I(d) = \int_0^{\pi/2} (\sin x)^{d-1}\, (\cos x)^{-d}\, dx = \frac{1}{2}\, \frac{\Gamma(d/2)\Gamma(\frac{1-d}{2})}{\Gamma(1/2)}.$$

Since d enters as a parameter here, there is no obstacle in treating d as a continuous, even complex, variable. By defining in terms of the beta functions and gamma functions, we have actually located the singularities and done the analytic continuation. No further continuation is needed.

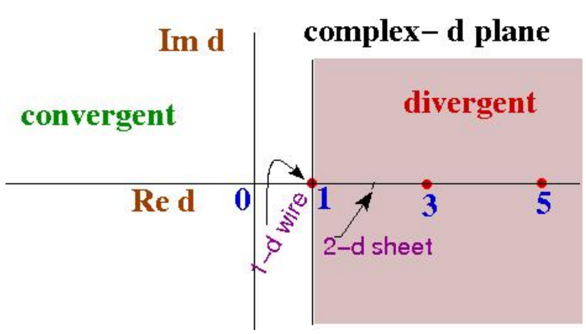

Figure 2: The integral $I(d)$ is divergent in the shaded region Re$d \geq 1$ in the complex d plane while convergent on the left of Re$d < 1$. Analytic continuation via beta- and gamma-function shows poles at $d = 1, 3, 5...$, indicated by the solid circles..

$\Gamma(x)$ has poles at $x = -1, -2, ...$, all negative integers. Therefore $I(d)$ is actually convergent for $d < 1$ but has poles $d = 1, 3, 5...$, i.e. all positive odd integers. See Fig. 2. The reason we see infinity for all $d > 1$, is because the Riemann integral as a sum of individual contributions is convergent only for $d < 1$. To make sense, we need to go beyond the singularity at $d = 1$. Also, there is no singularity for $d = 2$. By analytic continuation in d, we have obtained a finite answer or regularized the divergent integral, though a few d remained singular (wait for the next step). This procedure of rendering a singular problem finite is called *regularization*. There are in fact many ways of regularizing integrals. Here we adopted the procedure by modifying the dimensionality of the source and so it is called *dimensional regularization*.

The planar sheet, $d = 2$, problem is then solved. The cases of $d = 1, 3$ are problematic because of the pole of $I(d)$. Define $d = 1 + \epsilon$. Then,

$$\tilde{v}_d(r) = \tilde{\lambda}\, \tilde{r}^{d-1} \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{\Gamma(d/2)\Gamma(-\epsilon/2)}{2\Gamma(1/2)} \tag{14}$$

$$= \tilde{\lambda}\tilde{r}^{\epsilon}\, \pi^{\epsilon/2}\Gamma(-\epsilon/2) \tag{15}$$

Doing an ϵ-expansion, we have ($\gamma = 0.57721566490...$ is the Euler gamma, a constant)

$$\tilde{v}_d(\tilde{r}) = \tilde{\lambda}\left(1 + \epsilon \ln \frac{r}{\sqrt{\pi}L} + 0(\epsilon)\right)\left(-\frac{2}{\epsilon} - \gamma + 0(\epsilon)\right) \tag{16}$$

$$= \tilde{\lambda}\left(-\frac{2}{\epsilon} + 2\ln \frac{r}{\sqrt{\pi}L} - \gamma + 0(\epsilon)\right) \tag{17}$$

$$= 2\tilde{\lambda}\left(-\frac{2}{\epsilon} + \ln \frac{r}{L'} + 0(\epsilon)\right) \tag{18}$$

At the end, we have a singularity in ϵ but that's a "constant" term. The residue is independent of $\tilde{r}$.

Note the miracle. The singularity ($\sim 1/\epsilon$) from the integral is completely removed by ϵ from the other factors so perfectly that only a pole term with a constant residue is left over. This is no magic and happened because v_d/λ is dimensionless at $d = 1$. For example, There is also a pole at $d = 3$ but if we want to expand around $d = 3 + \tilde{\epsilon}$, such a cancellation will not occur. Even if we write $\tilde{r}^{d-1} = \tilde{r}^2\, \tilde{r}^{\tilde{\epsilon}}$, the left-over pole $O(\tilde{\epsilon}^{-1})$ term will have $\tilde{r}^2$ factor, not a case of constant residue.

1.5 Additive Renormalization

Now we remove the singular part namely $-2\tilde{\lambda}/\epsilon$ by an additive renormalization of the potential:

$$\tilde{v}_R = \tilde{v}_d(\tilde{r}) - \tilde{v}_{\text{sing}} = 2\tilde{\lambda}\ \ln r/L' \text{ for } \epsilon \to 0, \tag{19}$$

where $\tilde{v}_{\text{sing}} = -2\tilde{\lambda}/\epsilon$.

For $\epsilon = 0$ or $d = 1$, we have the renormalized potential

$$\tilde{v}_R = 2\lambda \ln \frac{r}{L'}, \quad \text{or}, V_{1R} = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r}{L'}, \tag{20}$$

with $v_{1R} = 0$ at $r = L'$ and not at $r \to \infty$!

The whole regularization and renormalization procedure was to isolate the singularity and then take advantage of the arbitrariness of the origin of potential (additive constant) to get a finite answer at least in $d = 1$. No harm was done in higher dimensions and one gets sensible results for all $d < 3$. This is called renormalization. Ultimately the reason we could do it is that the divergent part is additive and independent of r so that at the end we have a definite point (or line or surface) where the potential is zero. We could not have done this around $d = 3$ because the additive constant would then depend on r. This is to be called a *non-renormalizable* case.

The lesson is to recognize three different types of situations:, (i) the problem is trivially solvable, i.e. can be reduced to a quadrature or outsourced to silicon-valley, (ii) there are divergences but can be handled by renormalization, (iii) potentially hard problems - these are nonrenormalizable.

1.6 Callan-Symanzik/GellMann-Low RG equation

Next is to rewrite the renormalization in terms of the Callan-Symanzik or GellMann-Low equation.

For $\epsilon \neq 0$, we note that the original problem had no L and so

$$L\frac{\partial v_d}{\partial L} |= 0 \Rightarrow \Lambda\, \frac{\partial \tilde{v}_R}{\partial \Lambda} = \Lambda\frac{\partial \tilde{v}_d}{\partial \Lambda} - \Lambda\frac{\partial \tilde{v}_{sing}}{\partial \Lambda} = \tilde{\varphi}_d + 2.d\frac{\tilde{\lambda}}{\epsilon} = \tilde{\varphi}_R + 2\tilde{\lambda}$$

Need to be checked, *** to be completed**

2 Finite size effect

Consider now a finite one-dimensional wire of total length $\mathcal{L}$. A scaling or rather a finite-size scaling analysis given below can predict the behaviour in this finite case.

For a finite wire, various limiting cases can be considered because r is a running variable. These are,

1. $r/\mathcal{L} \ll 1$ which can happen for any $\mathcal{L}$ if point P is very close to the wire;
2. $r/\mathcal{L} \gg 1$ where the point is far away from the wire.

In the first case, the wire would look like an infinite one and the infinite length result should follow, i.e., $V_d(r, \mathcal{L}) \sim V_d(r)$. In the other case, from a far away point, the wire may look like a point of total charge $\lambda\mathcal{L}$ and the potential should look like a point charge Coulomb case, *i.e.*, $V_d(r, \mathcal{L}) \sim \frac{\lambda\mathcal{L}}{4\pi\varepsilon_0 r}$. **to be completed**