One-Dimensional Chain Collisions for Different Intermediate Mass Systems

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Abstract. It is known that one can transfer the bulk of the kinetic energy of a body to another body of smaller mass by arranging a large number of collisions with intermediate masses. In this project we explore the transfer of kinetic energy for masses arranged in arithmetic and harmonic progression. We also take into account inelastic collisions and find that there is an optimum number of intermediate masses which will ensure maximal transfer of kinetic energy. We have discovered interesting duality relations. Irrespective of the fact that the collisions are elastic or inelastic, we find that the results of arithmetic progression map onto that of the harmonic progression.

1. INTRODUCTION

Collision is one of the simplest mechanical interaction between two bodies and in the process energy and momentum are exchanged. One-dimensional collision as a means of transferring energy and ensuring velocity amplification is of interest because it provides a simple model for understanding natural phenomena where sequential collisions come into play. For example supernova explosion can be understood with the help of one-dimensional chain collision of vertically stacked masses [1]. Kerwin has explained the phenomena of super-ball collisions using an analytical method [2]. The dynamics of a queue, chain accidents in traffic, systems with narrow passage to allow for a single particle etc. can be modelled as one-dimensional chain collision systems.

The quantity of interest in studying such systems is the fraction of energy or momentum that is transferred. The exchange of kinetic energy and momentum depends mainly on the coefficient of restitution e and the ratios of colliding masses. The coefficient of restitution is a property of the colliding masses and for masses made of similar material we shall assume that it is a constant. What one can manipulate is 'mass' because one can extract a particular amount of mass from the bulk. Brilliantov and Pöschel considered viscoelastic particles where e is a function of colliding masses and their relative velocity [3]. Recently, Ricardo and Lee showed that the maximum transfer of

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kinetic energy takes place if the intermediate masses are geometric means of final and initial mass [4]. They have also considered the case of inelastic collision with fixed coefficient of restitution e.

In our work, we take two different intermediate mass systems and compare the numerical values of kinetic energy and velocity transfer ratios for a given value of initial and final mass. Unlike Ricardo and Lee, we take masses in arithmetic and harmonic progression. We consider the general case of inelastic collision.

2. BASIC EQUATIONS

We assume that the two colliding masses are spheres placed on the x- axis in such a way that the distance between their centres is greater than the sum of their radii. Let a mass M moving with velocity V, collide with a mass m which was intially at rest and due to which it's velocity changes to V' while the mass m gains a velocity v. This is shown in Fig. 1.

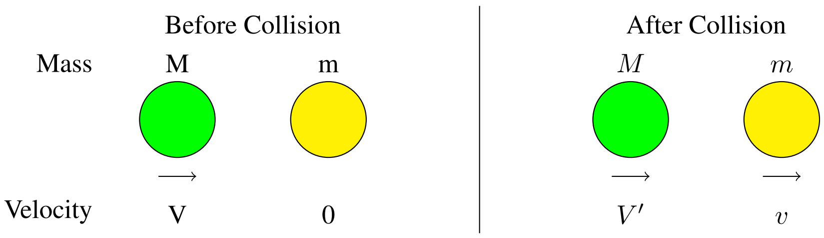

Figure 1. Collision of two masses

We define the velocity transfer ratio r_v as

$$r_v = \frac{v}{V}$$

Similarly kinetic energy transfer ratio is defined as

$$r_K = \frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2}$$

Since the momentum is conserved we have

$$MV = MV' + mv \tag{1}$$

The coefficient of restitution is

$$e = (v - V')/V \tag{2}$$

Using eqns. (1) and (2) we get the expression for velocity transfer ratio

$$r_v = \frac{(e+1)M}{(M+m)} \tag{3}$$

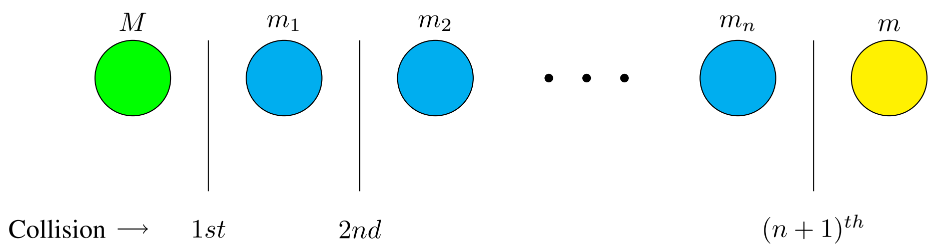

Figure 2. There are n intermediate masses between M and m and $(n+1)$ colisions.

Using eqn. (3) in the definition of kinetic energy transfer ratio for inelastic collision we get

$$r_K = \frac{(e+1)^2 Mm}{(M+m)^2} \tag{4}$$

Now consider the situation with n intermediate masses $m_1, m_2, m_3 \ldots m_n$ between M and m. For the transfer of kinetic energy from M to m there has to be $n+1$ collisions. Initially the sphere of mass M was moving with velocity V towards the above described assembly of set of stationary masses. After the first collision let the velocity of M be V' and that of m_1 be v_1. Then after the second collision between m_1 and m_2 the velocity of m_1 becomes v_1' and m_2 gains a velocity v_2. Generalising the notation, the i^{th} collision is between m_{i-1} and m_i. Just after $(i-1)^{th}$ collision m_{i-1} gets a velocity v_{i-1} and after the i^{th} collision it becomes v_{i-1}'. The velocity of mass m_i is v_i after i^{th} collision. Let the velocity transfer ratio and kinetic energy transfer ratio at i^{th} collision be denoted by r_{vi} and r_{Ki} respectively.

It can be shown that the velocity transfer ratio for i^{th} collision is

$$r_{vi} = \frac{v_i}{v_{i-1}} = \frac{(e+1)m_{i-1}}{m_{i-1}+m_i} \tag{5}$$

and the kinetic energy transfer ratio is

$$r_{Ki} = \frac{\frac{1}{2}m_i v_i^2}{\frac{1}{2}m_{i-1}v_{i-1}^2} = \frac{(e+1)^2 m_{i-1}m_i}{(m_{i-1}+m_i)^2} \tag{6}$$

Given the initial velocity of mass M we want to find r_v and r_K. From the definition of velocity transfer ratio

$$r_v = \frac{v}{V}$$

Multiplying and dividing with v_i where $i = 1, 2, 3 \ldots n$ we get

$$\begin{aligned} r_v &= \frac{v_1}{V}\frac{v_2}{v_1}\ldots\frac{v_n}{v_{n-1}}\frac{v}{v_n} \\ &= \prod_{i=1}^{n+1} r_{vi} \end{aligned}$$

Similarly

$$r_K = \prod_{i=1}^{n+1} r_{Ki} \tag{7}$$

Next we consider two intermediate mass systems, arithmetic and harmonic.

3. INTERMEDIATE MASS SYSTEMS

3.1 *Intermediate Masses in Arithmetic Progression*

In this case the intermediate masses m_i are such that $M > m_1 > m_2 \ldots m_n > m$ and the magnitude of difference of any two consecutive masses is a constant for the system. That is

$$M - m_1 = m_{i-1} - m_i = m_n - m$$

where $i = 2, 3 \ldots n$ Let the common difference be denoted by d.

$$d = \frac{M - m}{n + 1} \tag{8}$$

Mass of i^{th} intermediate sphere is $m_i = M - id$. Substituting the value of d from eqn. (8) we get

$$m_i = \frac{(n + 1 - i)M + im}{n + 1} \tag{9}$$

Similarly

$$m_{i-1} = \frac{(n + 2 - i)M + (i - 1)m}{n + 1}$$

Using eqns. (5), (7) and (9), the velocity transfer ratio is

$$r_v = \prod_{i=1}^{n+1} \frac{(e + 1)[(n + 2 - i)M + (i - 1)m]}{[2(n - i) + 3]M + (2i - 1)m} \tag{10}$$

The momentum transfer ratio r_{pi} for the i^{th} collision is obtained by multiplying the mass ratio m_i/m_{i-1} with r_{vi}. So, the momentum transfer ratio is

$$r_p = \prod_{i=1}^{n+1} \left(\frac{(e + 1)[(n + 1 - i)M + im]}{[2(n - i) + 3]M + (2i - 1)m} \right) \tag{11}$$

Similarly, using eqns. (6), (7) and (9), the kinetic energy transfer ratio is

n	$e = 1$	$e = 0.99$	$e = 0.95$	$e = 0.90$
0	0.7462	0.7388	0.7094	0.6735
1	0.8508	0.8361	0.7688	0.6929
2	0.8953	0.8705	0.7691	0.6581
3	0.9196	0.8835	0.7510	0.6101
4	0.9348	0.8891	0.7257	0.5597
5	0.9453	0.8901	0.6976	0.5108
6	0.9528	0.8882	0.6685	0.4647
7	0.9585	0.8847	0.6393	0.4219
8	0.9630	0.8799	0.6106	0.3825
9	0.9666	0.8784	0.5826	0.3465

Table 1. The energy transfer r_K for varying number (n) of intermediate masses is depicted in this table. Here the mass ratio $x = m/M = 0.33$. See text for discussion.

x	$e = 1$		$e = 0.99$		$e = 0.95$		$e = 0.90$	
	n_{opt}	r_K	n_{opt}	r_K	n_{opt}	r_K	n_{opt}	r_K
0.10	∞	1	13	0.7539	5	0.5394	3	0.4261
0.33	∞	1	5	0.8901	2	0.7691	1	0.6929
0.50	∞	1	3	0.9313	1	0.8498	0	0.8022

Table 2. Optimum number of intermediate masses (n_{opt}) and corresponding energy transfer (r_K) for various mass ratios (x) and coefficient of restitution (e).

$$r_K = \prod_{i=1}^{n+1} \frac{(e+1)^2[(n+1-i)M + im][(n+2-i)M + (i-1)m]}{[(2(n-i)+3)M + (2i-1)m]^2} \tag{12}$$

The above expression for kinetic energy transfer is displayed for $x = 0.33$ in Table 1. For nearly elastic collision (e.g. e = 0.99), the optimum number of collisions is $n_{opt} = 5$. As the collision becomes increasingly inelastic, n_{opt} shifts to lower values. In fact for $e = 0.9$, n_{opt} is 1. For realistic scenarios, the exercise of introducing intermediate masses is counter-productive.

In Table 2 we display the optimum number of collisions for varying mass ratios. Even for an almost elastic collision ($e = 0.99$), the kinetic energy transfer is sub-optimal varying from 93% to 75%. For a realistic case like $e = 0.90$ we find that the exercise of introducing intermediate masses is not beneficial.

3.2 *Intermediate Masses in Harmonic Progression*

For the intermediate masses to be the harmonic means of M and m, their reciprocals have to be the arithmetic means of $1/M$ and $1/m$. Let us denote the common difference by d'. So,

$$d' = \left(\frac{\frac{1}{m} - \frac{1}{M}}{n+1}\right) \tag{13}$$

The reciprocal of i^{th} mass is

$$\frac{1}{m_i} = \frac{1}{M} + id'$$

On simplifying further we get

$$m_i = \frac{Mm(n+1)}{(n+1-i)m + iM} \tag{14}$$

Similarly the mass of $(i-1)^{th}$ sphere will be

$$m_{i-1} = \frac{Mm(n+1)}{(n+2-i)m + (i-1)M}$$

Using eqns. (5), (7) and (14) the velocity transfer ratio is

$$r_v = \prod_{i=1}^{n+1} \left(\frac{(e+1)[(n+1-i)m + iM]}{[2(n-i)+3]m + (2i-1)M}\right) \tag{15}$$

The momentum transfer ratio in this case is

$$r_p = \prod_{i=1}^{n+1} \frac{(e+1)[(n+2-i)m + (i-1)M]}{[2(n-i)+3]m + (2i-1)M} \tag{16}$$

Now, using eqns. (6), (7) and (14) the kinetic energy transfer ratio is

$$r_K = \prod_{i=1}^{n+1} \left(\frac{(e+1)^2[(n+2-i)m + (i-1)M][(n+1-i)m + iM]}{[(2(n-i)+3)m + (2i-1)M]^2}\right) \tag{17}$$

3.3 *Symmetry*

A numerical exercise for the mass ratio m/M=0.33 in the harmonic case yields results identical to the arithmetic case of Table 2. This is not surprising since an interesting symmetry relation can be discerned by examining the relevant expressions. For a system with n intermediate masses, it is seen that the velocity transfer ratio in the $(n+2-i)^{th}$ collision for the arithmetic mean system is the same as that in i^{th} collision of the harmonic mean system. Replacing i by $(n+2-i)$ we get the self-same expression for the velocity transfer ratio of i^{th} collision in harmonic mean system.

$$(r_{vi})_{AP} = \frac{(e+1)[(n+2-i)M + (i-1)m]}{[2(n-i)+3]M + (2i-1)m} \tag{18}$$

Replacing $i \to n+2-i$ in eqn. 18

$$\left(r_{v(n+2-i)}\right)_{AP} = \frac{(e+1)[(n+2-(n+2-i))M + ((n+2-i)-1)m]}{[2(n-(n+2-i))+3]M + (2(n+2-i)-1)m}$$

This simplifies and we get

$$\left(r_{v(n+2-i)}\right)_{AP} = \frac{(e+1)[iM + (n+1-i)m]}{[(2i-1)M + (2(n-i)+3)m]} \qquad = (r_{vi})_{HP} \tag{19}$$

Now it immediately follows that the kinetic energy transfer ratio will be same for $(n+2-i)^{th}$ collision in arithmetic mean system and i^{th} collision in harmonic mean system.

$$\left(r_{K(n+2-i)}\right)_{AP} = (r_{Ki})_{HP} \tag{20}$$

It is easy to see that the final velocity transfer ratio and kinetic energy transfer ratio will be same for both progressions.

Consider the expression for momentum transfer for i^{th} collision in the harmonic case.

$$(r_{pi})_{HP} = \frac{(e+1)[(n+2-i)m + (i-1)M]}{[2(n-i)+3]m + (2i-1)M} \tag{21}$$

An interesting relation exists. If we switch the masses $M \leftrightarrow m$ for the velocity gain (eqn. (18)), we obtain the corresponding momentum gain for the harmonic case (eqn. (21)). The reverse is also true.

4. CONCLUSION

We began by discussing that full transfer of kinetic energy from one body to another is not possible, if their masses are unequal. However, a judicious introduction of intermediate masses may ensure optimum transfer of kinetic energy. We have taken two different intermediate mass systems, arithmetic and harmonic. An interesting duality relation between arithmetic and harmonic was observed (section 3.3). We find that for realistic scenarios the exercise of introducing intermediate masses yields limited benefit. This scheme is a paradigm for similar exercises, such as impedance matching in electrical circuits. We hope to explore such connections in the near future.

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