A path integral approach to the Langevin equation

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Reference:

- *A path integral approach to the Langevin equation*, A. Das, S. Panda and J. R. L. Santos, arXiv:1411.0256 (to be published in Int. J. Mod. Phys. **A**).

Outline of the talk

- Langevin equation
- Path integral approach
- Lagrangian and Hamiltonian
- Generating functional
- Fokker-Planck equation
- Conclusion

Langevin equation for the Brownian motion of a free particle

- The random collisions with the Brownian particle are represented by a random force (noise) in the evolution equation

$$\dot{x}(t) = v(t), \quad \dot{v}(t) + \gamma v(t) = \frac{\eta(t)}{m}.$$

- The random noise is described by a probability distribution, the simplest of which is a Gaussian leading to

$$P(\eta) = e^{-\frac{1}{4B}\int \mathrm{d}t\, \eta^2(t)}, \quad B > 0,$$
$$\langle \eta(t_1)\eta(t_2)\cdots\eta(t_{2n+1})\rangle = 0, \quad \langle \eta(t_1)\eta(t_2)\rangle = 2B\delta(t_1 - t_2).$$

- This is known as a Gaussian noise or a "white" noise.

- The equation for v can be easily solved to give

$$v(t) = v_0 e^{-\gamma t} + \frac{1}{m}\int_0^t \mathrm{d}s\, e^{-\gamma(t-s)}\eta(s).$$

which shows that the dynamical variable becomes "stochastic" because of the presence of the random noise.

- We can now calculate the velocity correlations which lead to

$$\langle v^2(t)\rangle = \left(v_0^2 - \frac{B}{\gamma m^2}\right) e^{-2\gamma t} + \frac{B}{\gamma m^2} \xrightarrow{t\to\infty} \frac{B}{\gamma m^2}.$$

- On the other hand, from equipartition theorem we know that in equilibrium $(k=1)$

$$\langle v^2(t)\rangle = \frac{T}{m}, \quad \Rightarrow B = \gamma m T.$$

- The position can also be obtained by integrating the velocity

$$x(t) = x_0 + \int_0^t \mathrm{d}t'\, v(t').$$

- This leads to (the Fluctuation-Dissipation theorem)

$$(\Delta x)^2 = \langle x^2(t)\rangle - \langle x(t)\rangle^2 \xrightarrow{t\to\infty} \frac{2Bt}{\gamma^2 m^2} = \frac{2Tt}{\gamma m} = 2Dt,$$

$$D = \frac{T}{\gamma m}.$$

General Langevin equation

- One can generalize the Langevin equation to describe the Brownian (random) motion of other physical systems

$$\dot{x} = v, \quad \dot{v} + \frac{\partial S(v)}{\partial v} + \frac{1}{m}\frac{\partial V(x)}{\partial x} = \frac{\eta}{m}.$$

- For $V(x) = 0$ and $S(v) = \frac{1}{2}\gamma v^2$, this corresponds to the free particle motion we have discussed.

- For $V(x) = \frac{1}{2}m\omega^2 x^2$ and $S(v) = \frac{1}{2}\gamma v^2$, this describes the damped harmonic oscillator.

- For $V(x) = \frac{1}{2}m\omega^2 x^2 - \frac{1}{3}\nu x^3$ and $S(v) = \frac{1}{2}\gamma v^2$, the system corresponds to the nonlinear damped oscillator and so on.

Markovian and non-Markovian processes

- Langevin equation opens up the branch of study known as stochastic differential equations. It is a simple way of studying nonequilibrium phenomena (approaching equilibrium).
- When the noise is Gaussian ("white"), the process is called Markovian or memoryless. This is the simplest of the nonequilibrium phenomena.
- When the noise is not Gaussian ("colored"), the process is called non-Markovian or with memory and describes a general nonequilibrium phenomenon which is harder to solve.
- We note that when the x, v equations are coupled, the system develops a "colored" noise induced by the coupling.

- For example, for the damped harmonic oscillator, the velocity equation can be integrated to yield

$$\dot{x}(t) = v(t) = -\omega^2 \int^t \mathrm{d}s\, e^{-\gamma(t-s)} x(s) + \frac{\overline{\eta}}{m},$$

$$\overline{\eta}(t) = \int^t \mathrm{d}s\, e^{-\gamma(t-s)} \eta(s).$$

- This leads to a "colored" noise in the x equation with

$$\langle \overline{\eta}(t) \overline{\eta}(t') \rangle = \frac{B}{\gamma} e^{-\gamma|t-t'|} = K(t-t').$$

- Langevin equation can also be extended to field theories and forms the basis for stochastic quantization.

Motivation for a path integral description

- In the case of Brownian (random) processes, the dynamical equations are first solved and then individual correlation functions are calculated by taking the ensemble average. This is a tedious process.

- We know that the path integrals lead to generating functionals for correlation functions and indeed contain all the correlation fuctions. Individual correlation functions are simply calculated by taking derivatives with respect to appropriate sources and setting the sources to zero.

- If we have a path integral description of the Langevin equation, we would have all the correlation functions contained in the generating functional and do not have to calculate them individually. Also perturbative calculations can be facilitated enormously.

What was known earlier

- There was no generating functional constructed from first principles. Rather functional methods were developed as practical calculational methods using the diagrammatic techniques of quantum field theory.

- The dynamical equations were studied as functional equations leading to Schwinger-Dyson equations in order to facilitate a diagrammatic evaluation of correlation functions. But, Schwinger-Dyson equations do not define a closed set of equations.

- To have a manageable closed set, extra fields were introduced which do not commute with the original dynamical variables of the theory and satisfy additional equations.

- The physical meaning of the additional fields and the equations were not clear and led to some unexpected behavior.

- This method could be further improved by combining with the renormalization group techniques, but the meaning of the additional fields continued to remain unclear.

- Some works tried to eliminate the additional fields at the cost of increasing the nonlinearities in the set of equations which is not practical.

- The issues with the nonlinearities have been addressed by appealing to the methods of stochastic quantization, but they, too, have their own difficulties.

The Lagrangian and the Hamiltonian

- The main obstacle in a first principle construction of the generating functional appears to have been the absence of a Lagrangian or Hamiltonian description for a (second order) dissipative system.

- Consider the Lagrangian (x, v are independent variables)

$$L = \lambda\left(\dot{v} + \frac{\partial S}{\partial v} + \frac{1}{m}\frac{\partial V}{\partial x} - \frac{\eta}{m}\right) + \xi(\dot{x} - v),$$

  where ξ, λ are (naively) Lagrange multiplier fields. The dynamical equations result from varying ξ and λ.

- This is a first order Lagrangian (like the Dirac theory) and, therefore, there are constraints. The constraint analysis leads to the (nontrivial) Dirac brackets and the Hamiltonian

$$\{x, \xi\}_D = 1 = \{v, \lambda\}_D\,,$$
$$H = -\lambda \left(\frac{\partial S}{\partial v} + \frac{1}{m}\frac{\partial V}{\partial x} - \frac{\eta}{m}\right) + \xi v.$$

- $\dot{x} = \{x, H\}_D$ and $\dot{v} = \{v, H\}_D$ lead to the dynamical equations, but now we also have

$$\dot{\xi} = \{\xi, H\}_D = \frac{\lambda}{m}\frac{\partial^2 V}{\partial x^2},$$
$$\dot{\lambda} = \{\lambda, H\}_D = -\xi + \lambda\frac{\partial^2 S}{\partial v^2}.$$

- If we identify the doublet of dynamical variables as $\psi_\alpha = (x, v)$ and introduce a second doublet as $\hat{\psi}_\alpha = (\xi, \lambda)$, then we can write $(\alpha, \beta = 1, 2)$

$$\left\{\psi_\alpha, \hat{\psi}_\beta\right\}_D = \delta_{\alpha\beta}.$$

- The doublet of fields $\hat{\psi}_\alpha$ coincides with the additional fields introduced earlier in the functional analysis together with the correct quantization condition as well as the additional equations.

- However, now their physical meaning is clear, they correspond to the pair of conjugate field variables and their dynamical equations.

Generating functional

- The generating functional can now be constructed in a straightforward manner.

- We define the Lagrangian with sources for the dynamical variables as
$$L^J = L + \widetilde{J}x + Jv,$$
which leads to the generating functional of the form
$$U^J = N \int \mathcal{D}\eta\mathcal{D}\lambda\mathcal{D}\xi\mathcal{D}v\mathcal{D}x\, e^{iS^J - \frac{1}{4B}\int \mathrm{d}t\,\eta^2}.$$

- If we are calculating correlation functions, it has to be remembered that the η integration needs to be done at the end in order to get the ensemble average. Otherwise, the integrations can be done in any order convenient.

- For example, in the case of the general Langevin equation, if we do the ξ and λ integrations, they lead to delta function constraints which impose the dynamical equations of motion for x, v respectively.

- The x equation can always be solved as $(\partial_t^{-1} v)$ and x can be integrated out. If the v equation can also be solved exactly (as in the case of the free particle or the harmonic oscillator), one can also integrate out v and then the noise variable η to yield a generating functional depending only on the sources. If the v equation is not exactly soluble (as will be the case for highly nonlinear $V(x)$), one has to solve the delta function constraint perturbatively and integrate out v order by order.

- In either case, the generating functional will depend only on sources and lead to any correlation function directly through functional derivation.

Fokker-Planck equation

- Fokker-Planck equation is another approach for handling nonequilibrium phenomena. Here one tries to determine directly the time evolution of the function $P(x, v, t)$ which describes the probability that a particle will have the coordinate x and velocity v at time t.

- This can also be determined from the path integral representation in a simple manner much like the Schrödinger equation is obtained from the path integral since time evolution is obtained from the difference in probabilities for infinitesimal time intervals.

- Here we are not calculating correlations and, therefore, sources can be set to zero and the probability at a later time (and coordinates) is given by the transition amplitude as

$$P(x, v, t) = N \int \mathrm{d}x' \mathrm{d}v' \, U(x, v, t; x', v', t') P(x', v', t').$$

- We are interested in infinitesimal time intervals

$$t = t' + \epsilon,$$

so that time derivatives inside the integral can be written as infinitesimal differences and the x equation requires

$$\dot{x} = \frac{x - x'}{\epsilon} = v.$$

- Therefore, making a Taylor expansion we can integrate out

x', v' to obtain the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{1}{m}\frac{\partial V}{\partial x}\frac{\partial P}{\partial v} + \frac{\partial}{\partial v}\left(\frac{B}{m^2}\frac{\partial P}{\partial v} + \frac{\partial S}{\partial v}P\right).$$

Future directions

- Our goal is to set up a formalism for studying general nonequilibrium phenomena within the context of quantum field theories.

- Having a path integral description of the Langevin equation is just the first step in this direction.

- Here temperature dependence is still brought in through the fluctuation-dissipation theorem.

- The next step is to define this path integral in a closed time path setting and see if the fluctutation-dissipation theorem will naturally result.

- If it does not, one has to incorporate this into the formalism in a natural way before any realistic application can be made.