Chapter 1

Lorentz and Poincare

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1.1 Lorentz Transformation

Consider two inertial frames $S$ and $S'$, where $S'$ moves with a velocity $v$ with respect to $S$ along $x$-axis\(^1\). Let an event occur at point $P$ whose space time coordinates are

$$S: (x, y, z, t), \quad S': (x', y', z', t') \quad (1.1)$$

We seek for a relationship

$$\begin{align*}
x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t, \\
y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t, \\
z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t, \\
t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t. \quad (1.2)
\end{align*}$$

In the following, we determine $a_{ij}$s appealing to the following properties:

1. Space-time is isotropic and homogeneous
2. All physical laws are invariant (no preferred frame of reference)
3. Speed of light is frame independent.

\(^1\)There are loads of excellent books for the stuff discussed here.
First, note that transformations in (1.2) have to be linear. Assume it is not and is given by \( x' = ax^2 \). Then

\[
x'_2 - x'_1 = a(x_2^2 - x_1^2).
\] (1.3)

Suppose the rod is of unit length in \( S \), i.e., say, \( x_2 = 2, x_1 = 1 \). Then \( x'_2 - x'_1 = 3a \). However, when \( x_2 = 5, x_1 = 4, x'_2 - x'_1 = 9a \). Result depends on where we are in space and hence assumption of homogeneity is violated.

Note that the coefficients \( a_{ij} \) depend on \( v \) such that as \( v \to 0, a_{ii} \to 1 \) and rest go to zero.

Also note that following planes and axes must be continuously connected.

1. \( x \) and \( x' \) axes (look at the figure)
   i.e. \( (y = 0, z = 0) \) should give \( (y' = 0, z' = 0) \)
   So \( a_{21}, a_{24}, a_{31}, a_{34} \) must be zero.

2. \( x'y' \) and \( xy \) plane
   i.e. \( z = 0 \) should give \( z' = 0 \)
   So \( a_{32} = 0 \).

3. \( x'z' \) and \( xz \) plane
   i.e. \( y = 0 \) should give \( y' = 0 \)
   So \( a_{23} = 0 \).

Now we show \( a_{22}, a_{33} = 1 \). Suppose a rod of unit length lying on the \( y \) axis. From \( S' \), the length of this rod is \( a_{22} \). Consider the same rod in rest in \( y' \).

In \( S \) frame the size of the rod is \( 1/a_{22} \). However reciprocal nature tells:

\[
a_{22} = 1/a_{22}, \rightarrow a_{22} = 1
\] (1.4)

Similarly one can show \( a_{33} = 1 \).

Now we consider the remaining equations of \( x', t' \) in (1.2). First we note that \( t' \) must be independent of \( y \) and \( z \). Focus on \( yz \) plane and suppose clocks are placed at \( y \) and \( -y \). From \( S' \), \( t' \) would look different for two clocks. So isotropy (direction independence) would be violated. Similarly for \( z \). Same exercise can be performed for \( x' \) also. From these arguments we get \( a_{42}, a_{43}, a_{12}, a_{13} \) are all zero.

Note now that point with \( x' = 0 \) must be same as \( x = vt \) this means \( a_{41} = va_{11} \).

We now use the postulate of the independence of the speed of light on frames. Consider at \( t = t' = 0 \), light emits from the origin (at \( t = t' = 0 \),
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origin of $S, S'$ are same). So

$$x^2 + y^2 + z^2 = c^2t^2, \quad x'^2 + y'^2 + z'^2 = c^2t'^2.$$  \hfill (1.5)

From the second of these equation

$$a^2_{11}(x - vt)^2 + y^2 + z^2 = c^2(a_{41}x + a_{44}t)^2.$$  \hfill (1.6)

Simplifying

$$(a^2_{11} - c^2 a^2_{41})x^2 + y^2 + z^2 - 2(va^2_{11} + c^2 a_{41} a_{44})xt = (c^2 a^2_{44} - v^2 a^2_{11})t^2.$$  \hfill (1.7)

Comparing this with the last equation of (1.5), we get

$$a^2_{11} - c^2 a^2_{41} = 1$$

$$va^2_{11} + c^2 a_{41} a_{44} = 0$$

$$c^2 a^2_{44} - v^2 a^2_{11} = c^2.$$  \hfill (1.8)

Solving this

$$a_{44} = \frac{1}{\gamma}, a_{11} = \frac{1}{\gamma}, a_{41} = -\frac{\beta}{c\gamma},$$  \hfill (1.9)

where we have defined

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \beta = \frac{v}{c}.$$  \hfill (1.10)

So finally, the transformations are

$$x' = \frac{(x - vt)}{\sqrt{1 - v^2/c^2}}, y' = y, z' = z, t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}.$$  \hfill (1.11)

First, note that inverting (1.11), we get

$$x = \frac{(x' + vt')}{\sqrt{1 - v^2/c^2}}, y = y', z = z', t = \frac{t' + vx/c^2}{\sqrt{1 - v^2/c^2}}.$$  \hfill (1.12)

This is what is expected as (1.11) and (1.12) should be related by $v \rightarrow -v$. Secondly, note that in $v/c << 1$ limit, (1.11) reduces to the Galilean transformation

$$x' = x - vt, y' = y, z' = z, t' = t.$$  \hfill (1.13)

Velocities in different frames can be related using (1.12).

$$\frac{dx}{dt} = \frac{dx}{dt'} \frac{dt'}{dt} \quad \text{giving} \quad u = \frac{u' + v}{1 + vu'/c^2}.$$  \hfill (1.14)
where \( u, u' \) have their usual meaning.

A more physical way to get to this velocity transformation rule is to consider the train problem (due to David Mermin). Let us attach the frame \( S' \) with a train moving in a straight line with a velocity \( v \) and having length \( L \) seen from a frame \( S \). One shoots a photon with velocity \( c \) and a particle with velocity \( u \) measured with respect to \( S \). Photon goes, hits the front of the train in time \( T_1 \), bounces back and meets after time \( T_2 \) the forward moving particle at a fraction \( f \) distance away from the front. We can then construct following equations they obey. First, total distance the particle moved is equal to the distance the photon moved in going from the rear to the front minus the distance the photon moved in going from the front to the particle. This gives

\[
\frac{u(T_1 + T_2) = c(T_1 - T_2)}{(1.15)}
\]

Furthermore, the distance the photon moved in time \( T_1 \) is

\[
cT_1 = L + vT_1 \quad (1.16)
\]

Finally, the distance photon moved in \( T_2 \) is

\[
cT_2 = fL - vT_2 \quad (1.17)
\]

Eliminating \( T_1, T_2 \) from (1.15), (1.16) and (1.17), we have

\[
f = \frac{(c + v)(c - u)}{(c - v)(c + u)}. \quad (1.18)
\]

Since, in deriving this, we have not used the fact that velocity of the train \( v \) is non-zero, the above result must be true in frame \( S' \) (note that \( f \) is dimensionless frame independent number). Setting \( v = 0 \) then we reach

\[
f = \frac{c - u'}{c + u'} \quad (1.19)
\]

where \( u' \) is the velocity of the particle measured from \( S' \) frame. Equating the above two expression for \( f \), we get

\[
u = \frac{u' + v}{1 + u'v/c^2}. \quad (1.20)
\]

Note that for \( u' = v \) (particle does not move forward), we get \( f = 1 \) and for \( u' = c \) we have \( f = 0 \) as expected.
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**Problem 1**: Two particle of equal charge $q$ moving with equal uniform velocity $v$. The separation is $r$. Show that $|F| = q^2 \sqrt{1 - v^2/c^2}/4\pi\varepsilon_0 r^2$.

**Problem 2**: For a frame $S'$ moving with arbitrary velocity $v$ with respect to frame $S$, the general forms of the coordinate, time and velocity transformations are

$$
r' = \gamma(r - vt) + (\gamma - 1)\frac{v \times (v \times r)}{v^2}
= r + v\left[\frac{(\gamma - 1)v.r}{v^2} - \gamma t\right]. \tag{1.21}
$$

$$
t' = \gamma(t - (v.r/c^2)) \tag{1.22}
$$

and

$$
u' = \frac{u - v}{1 - \frac{uv}{c^2}} + \frac{\gamma - 1}{\gamma v^2} \frac{v \times (v \times u)}{1 - \frac{uv}{c^2}}. \tag{1.23}
$$

Let us supplement Lorentz transformation by rotation of coordinate system, say about $z$ axis through an angle $\theta$.

$$
x' = x \cos \theta + y \sin \theta
y' = -x \sin \theta + y \cos \theta
z' = z
\tag{1.24}
$$

Writing in matrix form

$$
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix}. \tag{1.25}
$$

Similarly defining $\phi$ as

$$
cosh \phi = \gamma, \quad \sinh \phi = \beta \gamma, \quad \text{with} \quad \beta = \frac{v}{c}, \tag{1.26}
$$

we can write the Lorentz transformation as

$$
\begin{pmatrix}
ct' \\
x' \\
y' \\
z'
\end{pmatrix} =
\begin{pmatrix}
cosh \phi & -\sinh \phi & 0 & 0 \\
-\sinh \phi & \cosh \phi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
ct \\
x \\
y \\
z
\end{pmatrix}. \tag{1.27}
$$
Here are few points. First, the inverse of these four cross four matrices can be obtained by flipping signs of $\phi$ and $\theta$. Second, angles of rotation are not additive unless all rotations are about same axis. Similarly, velocities of successive Lorentz boosts are not additive unless boosts are all along same directions. Third, Lorentz transformation is a linear transformation $\Lambda$ on $(ct, x, y, z)$ keeping $-c^2t^2 + x^2 + y^2 + z^2$ invariant.

Four Vector:
Since Lorentz transformation mixes $x, ct$ among themselves and treats them on equal footing, we would like to do the same. We define four vectors representing four components of a vector. Let $\lambda, \mu, \nu, ...$ take values 0, 1, 2, 3, 4 and $i, j, k, ...$ take values 1, 2, 3.

Define a position four vector as

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x).$$  \hfill (1.28)

Like in three dimensions, define the distance in four dimensional space as

$$l^2 = -(x^0 - y^0)^2 + \sum_i (x^i - y^i)^2 = g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu).$$  \hfill (1.29)

where

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \hfill (1.30)$$

Note the contravariant vector $x^\mu$ is related to the covariant vector through $g_{\mu\nu}$ as $x_\mu = g_{\mu\nu}x^\nu$. Hence $x_\mu = (-ct, x)$. Define $g^{\mu\nu} = (g_{\mu\nu})^{-1}$. It has the property $x^\mu = g^{\mu\nu}x_\nu$.

**Problem 3:** Show that $g_{\mu\nu}$ satisfy $g^{\mu\lambda}g_{\lambda\nu} = \delta_\nu^\mu$.

**Lorentz Transformation in four vector space:**
It is a coordinate transformation of the form

$$x'^\mu = \Lambda_\mu^\nu x^\nu,$$ \hfill (1.31)

such that $x^\mu x_\mu$ remains invariant. That is:

$$x'^\mu x'_\mu = g_{\mu\nu}x'^\mu x'^\nu = g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma x^\rho x^\sigma = g_{\rho\sigma}x^\rho x^\sigma = x^\rho x_\rho,$$ \hfill (1.32)
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if
\[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma}. \]  
\((1.33)\)

**Problem 4:** For infinitesimal Lorentz transformation, we write
\[ \Lambda^\mu_\nu = g^{\mu \nu} + \omega_{\mu \nu}. \]  
\((1.34)\)

Show that it satisfies (1.33) if \( \omega_{\mu \nu} = -\omega_{\nu \mu} \).

This brings us to the definition of a scalar. The quantity that remains invariant under Lorentz transformation. An example of such scalar is \( x_\mu x^\mu = -c^2 t^2 + x^2 + y^2 + z^2 \). Note that this quantity is zero for propagation of a photon from the origin, defining a light cone. Negative values of the above quantity is a time-like four vector while positive values are space-like four vector. As it is scalar, Lorentz transformation does not change space-like vector to time-like and vice versa.

In matrix notation (1.33) is written as \( \Lambda g \Lambda^T = g \). Taking determinant on both sides, we have \( \det \Lambda = 1 \). Hence \( \det \Lambda = \pm 1 \). Also it follows from (1.33),
\[ \Lambda^\mu_0 g_{\mu \nu} \Lambda^\nu_0 = -1. \]  
\((1.35)\)

Or,
\[ (\Lambda^0_0)^2 - (\Lambda^i_i)^2 = 1. \]  
\((1.36)\)

Therefore, \( \Lambda^0_0 \geq 1 \) or \( \Lambda^0_0 \leq -1 \). The signs of \( \det \Lambda \) and \( \Lambda^0_0 \) are normally used to classify Lorentz transformation. The case which incorporates boosts and rotations is \( \Lambda^0_0 \geq 1 \) and \( \det \Lambda = 1 \). This forms proper orthochronous Lorentz group. Unless otherwise mentioned, in subsequent discussion, we will only consider proper orthochronous Lorentz transformations.

In the following, we give expression of various quantities in four vector notation. These will be used in the future.

**Gradient:** Four dimensional generalization of the gradient is
\[
\partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial (ct)}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) = \left( \frac{\partial}{\partial \theta^i}, \nabla \right).
\]  
\((1.37)\)

A scalar can be immediately formed as
\[ \partial_\mu A^\mu = \partial_0 A^0 + \partial_i A^i = \frac{\partial A^0}{\partial \theta^i} + \nabla \cdot \mathbf{A}. \]  
\((1.38)\)
Similarly another scalar is the operator

\[ \partial_\mu \partial^\mu = -\frac{\partial^2}{c^2 \partial t^2} + \nabla^2, \quad (\nabla^2 = \nabla_i \nabla_i = \nabla^i \nabla_i). \]  

(1.39)

**Momentum:** Define \( p^\mu = (E/c, \mathbf{p}) \). Hence

\[ p_\mu p^\mu = -\frac{E^2}{c^2} + \mathbf{p}^2 = -m_0^2 c^2 \]  

(1.40)

(design \( m_0 \) that way). Now since \( c \) is frame independent, so is \( m_0 \) as \( p_\mu p^\mu \) is frame independent. Note that \( m_0 \) has the dimension of [mass] and we will call the rest mass. From here, it follows that a particle with zero momentum, \( E = \pm m_0 c^2 \). Taking the plus sign, we get Einstein mass-energy relation. We will discuss later the one with the minus sign. In quantum mechanics we write

\[ p^i \rightarrow -i \frac{\partial}{\partial x^i} \]  

(1.41)

Similarly, in four dimensional space,

\[ p^\mu \rightarrow -i \frac{\partial}{\partial x_\mu} \]

\[ = \left( -\frac{i}{c} \frac{\partial}{\partial x_0}, -i \frac{\partial}{\partial x_1}, -i \frac{\partial}{\partial x_2}, -i \frac{\partial}{\partial x_3} \right) \]

\[ = \left( \frac{i}{c} \frac{\partial}{\partial x^0}, -i \frac{\partial}{\partial x^1}, -i \frac{\partial}{\partial x^2}, -i \frac{\partial}{\partial x^3} \right) \]

\[ = i \left( \frac{\partial}{\partial c \partial t}, \frac{\partial}{\partial x^i} \right). \]  

(1.42)

So

\[ p_\mu p^\mu = \left( -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right), \quad (\nabla^2 = \nabla_i \nabla_i). \]  

(1.43)

### 1.2 Poincare Group

Space-time translation plus Lorentz transformation constitutes Poincare group (inhomogeneous):

\[ x'^\mu = a^\mu + A^\mu_\nu x^\nu. \]  

(1.44)

It has six plus four that is 10 independent parameters and hence ten generators.
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We can realize Poincare group on the space of functions. Let us consider infinitesimal Poincare transformation

\[ f(x') = f(x + a + \omega x) = f(x) + a^\mu \partial_\mu f(x) + \omega^{\mu \nu} x^\nu \partial_\mu f(x) = \left(1 + a_\mu \partial^\mu + \frac{1}{2} \omega_{\mu \nu} (x^\mu \partial^\nu - x^\nu \partial^\mu)\right) f(x). \]  

(1.45)

In the last line we used

\[ \omega_{\mu \nu} x^\mu x^\nu \partial^\mu = \omega_{\mu \nu} x^\mu \partial^\nu - \omega_{\mu \nu} x^\nu \partial^\mu = 2 \omega_{\mu \nu} x^\mu \partial^\nu. \]  

(1.46)

We now define

\[ M_{\mu \nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu), \quad \text{and} \quad P^\mu = -i \partial^\mu, \]  

(1.47)

and write

\[ f(x') = \left(1 + ia_\mu P^\mu + \frac{i}{2} \omega_{\mu \nu} M^{\mu \nu}\right) f(x). \]  

(1.48)

\[ M^{\mu \nu} \] and \( P^\mu \) are the ten generators of Poincare transformation.

We note that the above generators satisfy following commutation relations among themselves.

\[ [x^\mu, P^\nu] = -i\left[x^\mu, \partial^\nu\right] f = -i x^\mu \partial^\nu f + i(\partial^\nu x^\mu) f \]  

(1.49)

However, using \( \partial^\nu x^\mu = g^{\mu \nu} \), we see

\[ [x^\mu, P^\nu] = ig^{\mu \nu}. \]  

(1.50)

Similarly, the one gets

\[ [P^\mu, M^{\rho \sigma}] = [\partial^\mu, x^\rho \partial^\sigma - x^\sigma \partial^\rho] = -i(g^{\mu \rho} P^\sigma - g^{\mu \sigma} P^\rho), \]  

(1.51)

and finally

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i(g^{\mu \rho} M^{\nu \sigma} - g^{\mu \sigma} M^{\nu \rho} - g^{\mu \sigma} M^{\nu \rho} + g^{\mu \sigma} M^{\nu \rho}). \]  

(1.52)

Let us now define

\[ J_x = M^{21}, J_y = M^{31}, J_z = M^{12}. \]  

(1.53)
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Then it follows,

\[
[J_x, J_y] = [M^{23}, M^{31}] = ig^{33}M^{12} = iJ_z.
\] (1.54)

Or in general, writing \( J_i = \epsilon_{ijk}M^{jk}/2 \), we get the algebra of generators of the angular momentum.

\[
[J_i, J_j] = i\epsilon_{ijk}J_k.
\] (1.55)

Similarly, we define boost generators \( K_i = M_{i0} \) to get

\[
[K_i, K_j] = -i\epsilon_{ijk}J_k.
\] (1.56)

That is, commutator of two boosts generates rotation. Finally, it is easy to show

\[
[J_i, K_j] = i\epsilon_{ijk}K_k.
\] (1.57)

**Problem 5**: Consider an unitary operator \( U(\Lambda) \) for a proper orthochronous transformation \( \Lambda \) which satisfies

\[
U(\Lambda')U(\Lambda) = U(\Lambda'\Lambda).
\] (1.58)

Now writing \( U(\Lambda) = U(1 + \delta \omega) = 1 + (i/2)\omega_{\mu\nu}M^{\mu\nu} \), show that

\[
U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^\nu_\rho \Lambda^\mu_\sigma M^{\rho\sigma}.
\] (1.59)

Further, find \([M^{\mu\nu}, M^{\rho\sigma}]\) form here.

**Problem 6**: \( A \) is a hermitian matrix with \( \det A = -1 \) and is given by

\[
A = \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix}.
\] (1.60)

Under Lorentz transformation, if it transforms as

\[
A' = BAB^\dagger,
\] (1.61)

(a) Show that \( \det B = 1 \) and it has six independent parameters.

(b) Further, find explicitly the form of \( B \) representing a boost along \( z \) by \( \beta \).