The Heat Kernel on $AdS_5$ and its Applications

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6 Summary
The one-loop partition function contains the leading quantum corrections to a classical background.

It is also potentially tractable.

It decomposes into separate contributions from every particle in the spectrum of the theory, and is sensitive only to that part of the Lagrangian which is quadratic in the fluctuations.

If Gravity is the only background field, then the contribution of a particle to the one-loop partition function is $\ln \det (\Delta_{(s)} + m^2)$ where $\Delta_{(s)}$ is the Laplacian for a spin-$s$ particle.

We will define this determinant through the heat kernel and evaluate it for the thermal quotient of $AdS_5$. 
Motivations

- This program has already yielded insights into 3-d gravity.
- The heat kernel on $AdS_3$ has been calculated in this manner and the one-loop partition function of ($\mathcal{N} = 1$ super-)gravity has been shown to be the vacuum character of the ($\mathcal{N} = 1$ super-)Virasoro algebra.
  
  [Giombi, Maloney, Yin arXiv:0804.1773]
  [David, Gaberdiel, Gopakumar arXiv:0911.5085]
- This has also provided evidence that topologically massive gravity at its chiral point is dual to a $log$-CFT.
  
- The one-loop partition function of higher spin theories on $AdS_3$ has been shown to be determined in terms of the vacuum characters of $\mathcal{W}_n$ algebras.
  
- Further, this has lead to a proposal for the CFT dual of these higher spin theories.
  
The Heat Kernel

- We will now define the determinant of the laplacian using the property that $\ln \det (-\Delta_{(s)}) = \text{Tr} \ln (-\Delta_{(s)})$, and the relation that $\ln (-\Delta_{(s)}) = -\int_0^\infty \frac{dt}{t} e^{t\Delta_{(s)}}$.

- This leads us to define

$$\ln \det (-\Delta_{(s)}) = -\text{Tr} \int_0^\infty \frac{dt}{t} e^{t\Delta_{(s)}} = -\int_0^\infty \frac{dt}{t} \text{Tr} e^{t\Delta_{(s)}}.$$ 

Now the trace is given by

$$\text{Tr} e^{t\Delta} \equiv \int \sqrt{g} d^n x \sum_a \langle x, a | e^{t\Delta} | x, a \rangle$$

$$= \int \sqrt{g} d^n x \sum_a \sum_n \psi_{n,a}(x) \psi_{n,a}(x)^* e^{tE_n},$$

where $\psi_{n,a}(x)$ are eigenfunctions of the Laplacian, and $a$ is a local Lorentz index.
The Heat Kernel

- The expression above is just the trace of the heat kernel defined as
  \[ K_{a,b}(x, y, t) = \sum_n \psi_{n,a}(x) \psi_{n,b}(y)^* e^{tE_n}, \]
  where the trace was done over both the local Lorentz index \( a \), as well as the continuous space-time index \( x \).
- To calculate the one-loop partition function we therefore need to know
  - the eigenvalues \( E_n \) of the Laplacian
  - its eigenfunctions \( \psi_{n,a}(x) \)
  - how to do the sum over \( n \) and the trace over \( a \) and \( x \).
- We will see that because \( AdS_5 \) is a symmetric space, there is a group theoretic structure underlying all three of these questions, which we can exploit to evaluate the traced heat kernel for particles with arbitrary spin.
A toy model: The scalar on $S^2$

- As an illustration of the techniques that are involved, we evaluate the heat kernel for a scalar on $S^2$.
- The eigenvalues of the Laplacian are $\ell (\ell + 1)$, which is the quadratic Casimir of the spin-$\ell$ representation of $SU(2)$.
- The corresponding eigenfunctions are the $Y_{\ell m}(\theta, \phi)$, which are elements of the spin-$\ell$ Wigner matrices.

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} D_{0,m}^{\ell} \left( R(\phi, \theta, 0)^{-1} \right)$$

- The sum over the degenerate eigenvalues can be performed by using the addition theorem

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{x}) Y_{\ell m}(\hat{x})^* = \frac{2\ell + 1}{4\pi},$$

which is a consequence of the group multiplication law of Wigner Matrices.
• The heat kernel is then given by

\[ K(\hat{x}_1, \hat{x}_2, t) = \sum_{\ell, m} \frac{2\ell + 1}{4\pi} D_{0,m}^\ell (R_1^{-1}) D_{m,0}^\ell (R_2) e^{-t E_\ell} \]

\[ = \sum_{\ell} \frac{2\ell + 1}{4\pi} D_{0,0}^\ell (R_1^{-1} R_2) e^{-t E_\ell} \]

• We recognize the prefactor \( \frac{2\ell + 1}{4\pi} \) as \( \frac{d_\ell}{d_0} \frac{1}{V_{S^2}} \), where \( d_\ell \) is the dimension of the spin-\( \ell \) representation of \( SU(2) \), \( d_0 = 1 \) is the dimension of the singlet representation of \( U(1) \).

• This is (almost) the answer for an arbitrary-spin particle on a generic symmetric space!
• We can express these results in a more “group theoretic” manner which anticipates the more general theory that we’ll use.

• The scalar transforms under the $s = 0$ representation of $U(1)$ generated by $J_3$.

• The eigenfunctions of the Laplacian are unitary representation matrices of $SU(2)$ that transform in any spin-$\ell$ that contains the representation $s = 0$ of the $U(1)$.

• These representation matrices are functions of an $SU(2)$ element which depend on the coordinates on $S^2 \simeq SU(2)/U(1)$.

• The corresponding eigenvalue is the difference between the quadratic Casimir of the spin-$\ell$ representation of $SU(2)$ and the quadratic Casimir of the $s = 0$ representation of the $U(1)$.
Strategy

- $AdS_5$ is realised as $SO(5,1)/SO(5)$, which is a symmetric space. Harmonic analysis on such spaces is well understood, and in principle one can carry out the above program directly for $AdS_5$.

- It is still simpler to consider $S^5 \equiv SO(6)/SO(5)$, and analytically continue our answers to the $AdS_5$ case.

- We will carry out this procedure for the thermal quotient of $AdS_5$.

- Our analysis is quite general and should be extendable to other quotients and symmetric spaces. It extends directly to $AdS_{2n+1}$. 
Review: Harmonic Analysis on Homogeneous Spaces

- Fields are sections over $G/H$ and are in one-to-one correspondence with irreps of $H$.
- Consider a particle transforming in some representation $S$ of $H$, or a “spin-$S$” particle.
- If $R$ is a representation of $G$ such that it contains $S$ when restricted to $H$, then the eigenvalue of the Laplacian is given by
  \[-E_R^{(S)} = C_2(R) - C_2(S).\]
- Further, if $\sigma(x) \in G$ is a section over $G/H$, then the corresponding eigenfunctions are
  \[U^{(R)}(\sigma(x)^{-1}) \equiv \langle a, S | U^{(R)}(\sigma(x)^{-1}) | l \rangle.\]
The Heat Kernel for a spin-$S$ particle is then

$$K_{a,b}^S(x, y, t) = \sum_R \frac{dR}{dS} \frac{1}{V_{G/H}} U^{(R)} \left( \sigma(x)^{-1} \sigma(y) \right)^b_a e^{tE_R^{(S)}},$$

which is analogous to the expression on $S^2$. 

- The group multiplication used here is the “addition theorem” for harmonics on generic symmetric spaces.

- We can also trace over the $H$ indices to define

$$K^S(x, y, t) = \sum_R \frac{dR}{dS} \frac{1}{V_{G/H}} \text{Tr}_S U^{(R)} \left( \sigma(x)^{-1} \sigma(y) \right) e^{tE_R^{(S)}}$$
Harmonic Analysis on Quotients of $G/H$

- Consider an abelian discrete group $\Gamma$, generated by $\gamma$.
- We will consider the quotient space $\Gamma \backslash G/H$.
- If $G/H$ is compact, $\Gamma \simeq \mathbb{Z}_N$.
- The trace over the heat kernel can be implemented when the choice of section is compatible with the quotienting.
- *i.e.* we must choose a section $\sigma$ such that
  \[ \sigma (\gamma (x)) = \gamma \cdot \sigma (x). \]
- The other ingredient we need is the method of images.
The method of Images
Giombi, Maloney, Yin arXiv:0804.4589

- We can use the method of images to evaluate the heat kernel over quotients of $G/H$, e.g. $AdS$.
- The heat kernel on the quotient space $\Gamma \backslash G/H$ is given by

$$K_{\Gamma}^{(S)}(x, y, t) = \sum_{\gamma \in \Gamma} K^{(S)}(x, \gamma(y), t)$$

- The heat kernel on $G/H$ satisfies the heat equation and

$$K_{ab}^{(S)}(x, y, 0) = \delta(x - y) \delta_{ab}$$

- On $\Gamma \backslash G/H$, this boundary condition gets modified to

$$K_{ab}^{(S)}(x, y, 0) = \sum_{\gamma \in \Gamma} \delta(x - \gamma(y)) \delta_{ab}$$

- The solution on $\Gamma \backslash G/H$ then follows.
The thermal quotient of $S^5$

- Let us consider the unit $S^5$, which can be realised as the surface $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, with $z_i \in \mathbb{C}$, and define the phases of the $z_i$ to be $\phi_i$.

- A choice of the coset representative in $SO(6)$ for points in $S^5$ is $x = e^{i\phi_1}Q_{12}e^{i\phi_2}Q_{34}e^{i\phi_3}Q_{56}e^{i\psi}Q_{35}e^{i\theta}Q_{13}$, where $Q$’s are the generators of $SO(6)$.

- We consider the quotient $\phi_1 \mapsto \phi_1 + \beta$, which we call the thermal quotient.

- The action of the thermal quotient on this coset representative is given by $\gamma : x \mapsto e^{i\beta}Q_{12} \cdot x$.

- The thermal section is then given by

$$\sigma (x) = x.$$
The Traced Heat Kernel on Quotients of Symmetric Spaces

- We shall calculate the expression

\[ \sum_{m \in \mathbb{Z}_N} \int_{\Gamma \backslash G/H} d\mu(x) K^{(S)}(x, \gamma^m(x); t), \]

where \( d\mu(x) \) is the invariant measure on \( G/H \).

- The integral over \( \Gamma \backslash G/H \) can be replaced by the same integral over \( G/H \) with the multiplication of an appropriate volume factor. Therefore we shall instead calculate

\[ \sum_{m \in \mathbb{Z}_N} \int_{G/H} d\mu(x) K^{(S)}(x, \gamma^m(x); t) \]
• With our choice of section, this implies that we shall need to evaluate the integral

\[ \mathcal{I}_m = \int_{G/H} d\mu(x) \text{Tr}_S \left( (x^{-1} \gamma^m x)^{(R)} \right), \]

in order to evaluate the heat kernel. Here \( g^{(R)} \) is shorthand for \( U^{(R)}(g), g \in G \).

• This integral over \( G/H \) can be lifted to an integral over \( G \), and the trace over the representation \( S \) of \( H \), which is a subspace of \( R \), can be lifted to a trace over \( R \).

• Using these simplifications, we can write

\[ \mathcal{I}_m = \frac{dS}{dR} V_{G/H} \chi_R (\gamma^m). \]
A Sketch of the Proof

- As outlined above,

\[ \mathcal{I}_m = \int_{G/H} d\mu(x) \text{Tr}_S \left( (x^{-1}\gamma^mx)^{(R)} \right) \]

\[ \sim \int_G dg \text{Tr}_R \left( (g^{-1}\gamma^mg)^{(R)} \right) \sim \text{Tr}_R (\gamma^m). \]

- Now, \( \gamma \sim e^{\theta_i H_i} \), where the \( H_i \)s are the Cartans of \( SO(6) \). This trace is then the character in the representation \( R \).

- We therefore have

\[ \mathcal{I}_m = \frac{dS}{dR} V_{G/H} \chi_R (\gamma^m), \]

where \( \chi_R (\gamma^m) \) is the character in the representation \( R \).
The traced Heat Kernel over Thermal $S^5$

- For the thermal quotient, the volume factor is just $\frac{\beta}{2\pi}$. Carrying through the above procedure gives us the expression for the traced heat kernel over thermal $S^5$

$$K^{(S)} \equiv \sum_{m \in \mathbb{Z}_N} \int_{\Gamma \backslash G/H} d\mu(x) K^{(S)}(x, \gamma^m(x); t)$$

$$= \frac{\beta}{2\pi} \sum_{m \in \mathbb{Z}_N} \sum_R \chi_R(\gamma^m) e^{tE_R^{(S)}}.$$

- This is an expression purely in terms of the characters of $\Gamma$ embedded in $SO(6)$ and the quadratic Casimirs of $SO(6)$ and $SO(5)$, both of which are known.

- We shall analytically continue this expression to obtain the traced heat kernel over $AdS_5$. 
The Analytic Continuation to $AdS_5$

- The analytic continuation of the traced heat kernel is implemented by
  - replacing the $SO(6)$ character by the $SO(5,1)$ character, and
  - analytically continuing the sum over UIRs of $SO(6)$ to a sum over UIRs of $SO(5,1)$.

- A UIR of $SO(6)$ is labelled by (half-)integers $(m_1, m_2, m_3)$ with $m_1 \geq m_2 \geq |m_3|$.

- The analytic continuation is:
  \[
  m_1 \mapsto i\lambda - 2, \quad \lambda \in \mathbb{R}_+.
  \]

- This is an analytic continuation to the principal series of $SO(5,1)$ representations. For odd dimensional spacetimes, these are the only ones that matter for us.
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Summary

The Traced Heat Kernel on Thermal $AdS_5$

- The traced heat kernel may be obtained from analytically continuing the expression for the traced heat kernel over $S^5$.

$$K(S) = \frac{\beta}{2\pi} \sum_{m \in \mathbb{Z}_N} \sum_R \chi_R(\gamma^m) e^{tE_R(S)}.$$  

- The sum over $R$ above is an implicit sum over $m_1, m_2, m_3$. The sum over $m_1$ gets analytically continued to an integral over $\lambda$ and the other sums stay.

- We therefore obtain

$$K(S)(\gamma, t) = \frac{\beta}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{m_2, m_3} \int_0^\infty d\lambda \chi_R(\gamma^m) e^{tE_R(S)}$$

as the expression for the traced heat kernel over thermal $AdS_5$. 
The eigenvalue of the Laplacian is fixed by the analytic continuation to be

$$E_R^{(S)} = - (\lambda^2 + \zeta),$$

where $\zeta$ is a constant independent of $\lambda$ and is a function of the representation $S$ of $H$ and of $m_2, m_3$.

The thermal quotient acts by

$$t \mapsto t + \beta$$

The character $\chi_R (\gamma^m)$ then has a very simple form

$$\chi_R (\gamma^m) = \frac{1}{8} \frac{\cos m\beta\lambda}{(\sinh \frac{\beta}{2})^4} d_{m_2,m_3},$$

where $d_{m_2,m_3}$ is the dimension of the UIR of $SO(4)$ labelled by the highest weight $(m_2, m_3)$.
The integral over $\lambda$ is therefore a gaussian integral and can be performed to obtain

$$K^{(S)}(\beta, t) = \frac{\beta}{32\pi} \sum_{m \in \mathbb{Z}} \sum_{m_2, m_3} d_{m_2, m_3} \frac{1}{(\sinh \frac{m\beta}{2})^4} \sqrt{\frac{\pi}{t}} e^{-\frac{m^2 \beta^2}{4t} - t\zeta}$$

This expression for the heat kernel also yields the expression for the one-loop partition function

$$\log Z^{1-loop}_{(S)} = \frac{1}{16} \sum_{m \in \mathbb{Z}_+} \sum_{m_2, m_3} \frac{d_{m_2, m_3}}{m} \frac{1}{(\sinh \frac{m\beta}{2})^4} e^{-m\beta \sqrt{\zeta + m_R^2}}$$

where we have subtracted out the divergence at one-loop by dropping the $m = 0$ term.
Example: Partition Function of a Scalar on $AdS_5$

- For a scalar, $m_2 = m_3 = 0$ and $d_{m_2,m_3} = 1$, and
  \[ \sqrt{\zeta + m_R^2} = \Delta - 2, \] where $\Delta$ is the conformal dimension of the scalar.
- The one-loop determinant is, therefore
  \[ \log Z^{1\text{-loop}} = \sum_{m \in \mathbb{Z}_+} \frac{1}{m(1 - e^{-m \beta})^4} e^{-m \beta \Delta} \]
- This sum can be evaluated to obtain
  \[ \log Z = -\sum_{n=0}^{\infty} \frac{(n + 1)(n + 2)(n + 3)}{6} \log \left(1 - e^{\beta(\Delta+n)}\right). \]
- This matches with the result obtained using existing methods.

Denef, Hartnoll, Sachdev, arxiv:0908.2657
The heat kernel on a general manifold is hard to evaluate.

The group theoretic properties of symmetric spaces are of great use in carrying out this evaluation.

We have calculated the heat kernel for an arbitrary-spin particle on thermal $AdS_5$.

Our method is directly extendable to thermal $AdS_{2n+1}$.

We are now in a position to go forward with the program outlined in the beginning.
Thank You