From Space Time to World sheet : Four point correlators

Based on

hep-th/0606078
with R. Gopakumar

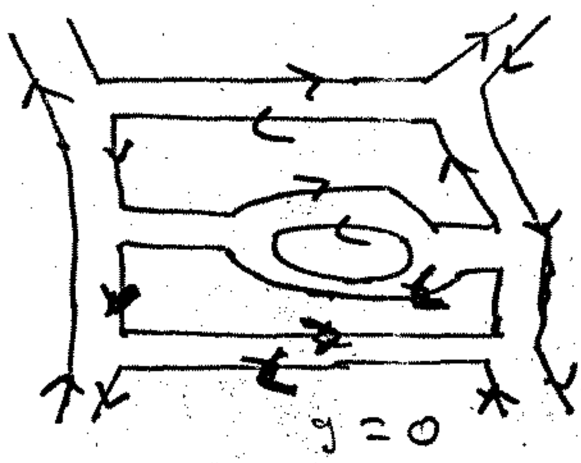
& work with

O. Aharony, R. Gopakumar,
Z. Komargodski,
R. Sholomo.

Introduction

- There is considerable evidence that gauge theories are dual to string Theories

→ Recall 't Hooft's organization of $U(N)$ theories



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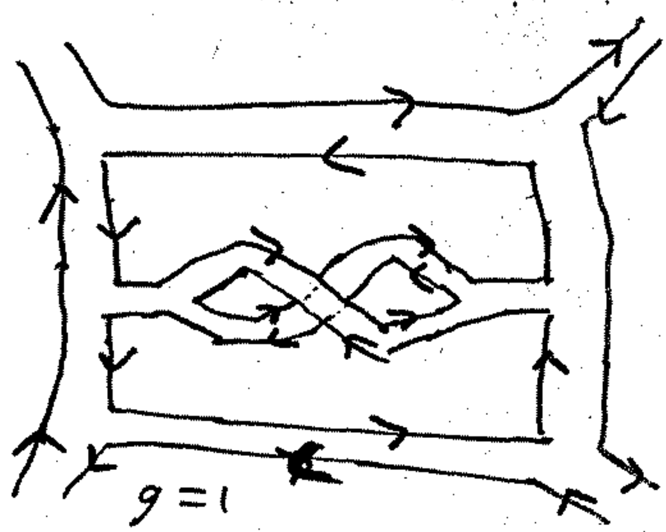


Suppressed by $1/N^2$.

$$\sum_n \lambda^{n} \sum_g N^{2-2g}$$

$$\lambda = g_{YM}^2 N .$$

The 2-dimensional topology of the Feynman graphs play important role in the organization of the expansion

Resembles : string perturbation theory

$$\lambda \sim \alpha' , \quad \frac{1}{N^2} \sim g_s^2$$

$\Rightarrow$ A concrete realization of gauge theories dual to string theories

AdS/CFT

e.g $\mathcal{N}=4$ SYM with gauge group $U(N)$ dual to Type IIB on $AdS_5 \times S^5$

$\Rightarrow$ In units of Radius of AdS

$$\frac{1}{\alpha'} = \sqrt{\lambda} \quad : \quad g_s = \frac{1}{N^2}$$

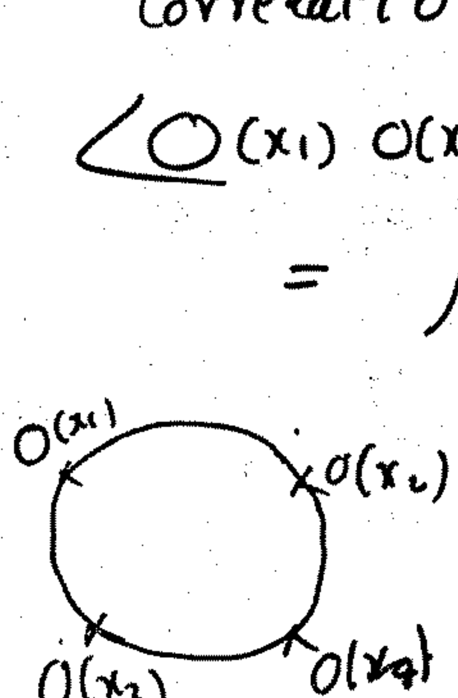
$\rightarrow$ Gauge Invariant operators $\longleftrightarrow$ physical states of string in $AdS_5 \times S^5$

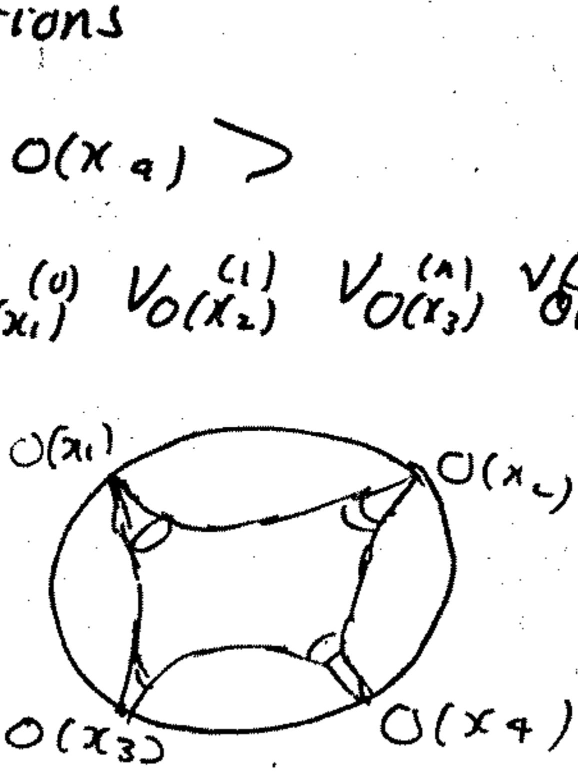
$$\mathcal{O}(x_1) \longleftrightarrow V_{\mathcal{O}(x_1)}$$

$\Rightarrow$ Correlation functions

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\, \mathcal{O}(x_3)\, \mathcal{O}(x_4) \rangle$$

$$= \int d^2\eta \, \langle V^{(0)}_{\mathcal{O}(x_1)}\, V^{(1)}_{\mathcal{O}(x_2)}\, V^{(\eta)}_{\mathcal{O}(x_3)}\, V^{(\infty)}_{\mathcal{O}(x_4)} \rangle$$





. Tests of AdS/CFT has been restricted to supergravity limit $\boxed{\begin{array}{l} \alpha' \to 0 \\ \lambda \to \infty \end{array}}$

or semi classical string configurations

- Reason String theory on $AdS_5 \times S^5$ has not been quantized. Yet !

- But $\lambda = 0$: free YM ! string Theory is strongly coupled.

- Pushing the AdS/CFT correspondence: One should be able to write the free field correlators as a string Amplitude.

Free field correlators are simple & we have a hint from t' Hooft s observation.

But how to Recast a field Theory amplitude into a "string" Amplitude" ??

- Gopakumar has put forward a procedure to rewrite any given field Theory amplitude as a "string amplitude"

- We will Review Gopakumar's proposal focussing on the 4 pt function.

- We will implement it explictly for a class of 4 pt functions and show the resulting "string amplitude" has the required properties of ~~a~~ a 2-D CFT

Topics

- Review Gopakumar's proposal
  (4 pt function)

- Strebel differentials for the 4-punctured sphere

- A world sheet 4-pt function
  Y– diagram

- The π – & the spade diagrams

- Conclusions

Gopakumar's proposal (4 pt fn)

- Consider a free field Theory of scalars in the adjoint representation of $U(N)$

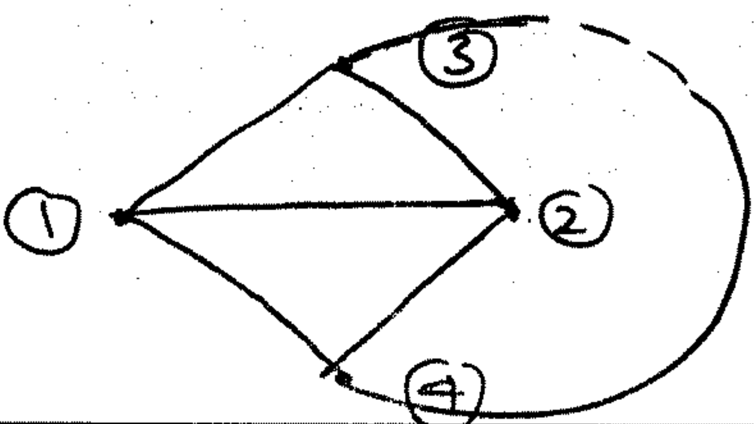
$$g \langle \mathrm{Tr}\, \varphi^3(x_1)\, \mathrm{Tr}(\varphi^3(x_2))\, \mathrm{Tr}(\varphi^3(x_3))\, \mathrm{Tr}(\varphi^3(x_4)) \rangle$$

correlation fn obtained by Wick contraction

$$\sim \frac{1}{(x_{12})^2 (x_{14})^2 (x_{13})^2 (x_{24}^2)(x_{23}^2)(x_{34})^2}$$

① Organize the diagram in the genus expansion ($1/N^2$ expansion)

- lets focus on the planar diagram.



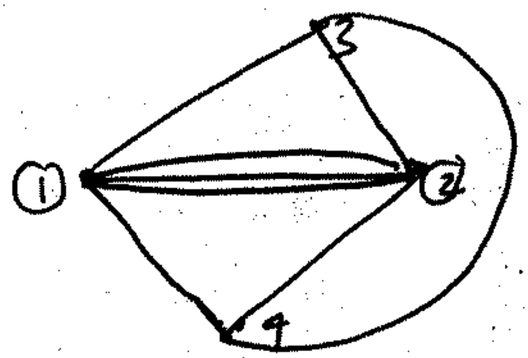
(supressing double lines)

(2) Schwinger parametrize the Feynman diagrams

$$\sim \int d\sigma_1 d\sigma_2 d\sigma_3 d\sigma_4 d\sigma_5 d\sigma_6 \, e^{-[\sigma_1 x_{12}^2 + \sigma_2 x_{14}^2 + \sigma_3 x_{13}^2 + \sigma_4 x_{24}^2 + \sigma_5 x_{23}^2 + \sigma_6 x_{34}^2]}$$

$$= \frac{1}{x_{12}^2 x_{14}^2 \; x_{13}^2 x_{24}^2 \; x_{23}^2 x_{34}^2}$$

(3) Glue Homotopic edges to obtain a skeleton diagram



$$\int d\sigma_1 d\sigma_1' d\sigma_1'' \, e^{-(\sigma_1 x_{12}^2 + \sigma_1' x_{12}^2 + \sigma_1'' x_{12}^2)}$$

$$\sim \int d\sigma_1 \, \sigma_1^2 \, e^{-\sigma_1 x_{12}^2}.$$

so one has for the generic planar 4 pt fn

$$\int \prod_{i=1}^{6} d\sigma_i \prod_i \sigma_i^{J_i - 1} \, e^{-(\sigma_1 x_{12}^2 + \sigma_2 x_{14}^2 + \sigma_3 x_{13}^2 + \sigma_4 x_{24}^2 + \sigma_5 x_{23}^2 + \sigma_6 x_{34}^2)}$$

Need to write this as

$$\sim \int d^2\eta \cdot F(\eta, \bar{\eta}).$$

Strebel differentials for the 4-punctured sphere (7)

Thus we need to find a map from schwinger parameter space to moduli space of 4 punctured sphere.

This map is through a special holomorphic quadriatic differential $\phi(z)\, d^2z$ on the (sphere).

For the case of sphere with 4-marked
→ ~~The map~~ pts.

8 4 Residues.

4 marked pts on sphere.
→ characterized by a complex No:

4 Real Residues →

→ Total No. of parameters = 6.

→ No: of Schwinger lengths = 6.

Corresponding "Strebel differential"

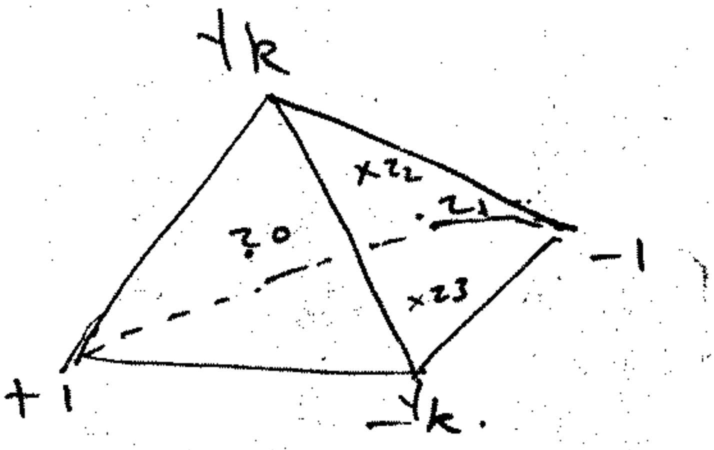
$$\phi(z)\, dz^2 = -c \frac{(z^2-1)(z^2k^2-1)}{(z-z_0)^2(z-z_1)^2(z-z_2)^2(z-z_3)^2}\, dz^2$$

$$\varphi(z)\,dz^2 = -c\,\frac{(z^2-1)(z^2k^2-1)}{(z-z_0)^2(z-z_1)^2(z-z_2)^2(z-z_3)^2}\,dz^2$$

q

- Characterized by double poles at the 4 marked points z_0, z_1, z_2, z_3.
- $\mathrm{Res}\left(\sqrt{\varphi(z)}\right)\Big|_{z=z_i} = r_i$

- There is a critical graph with zeros as the vertices enclosing the double poles



Order of zero $= (V-2)$

Trivalent $V=3$

$=$ simple zero

Edges of the graph are trajectories along which $\sqrt{\varphi(z)}\,dz$ is real.

$$\ell_{e_i} = \int_{\text{Edge}} \sqrt{\varphi(z)}\, dz.$$

(one zero to another)

= strebel lengths.

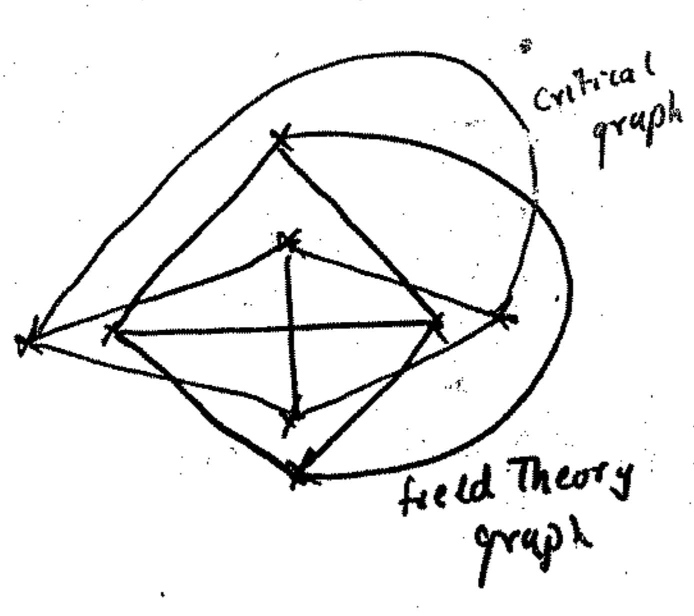
- Thus given 4 marked pts. & the residues. (r_i)
  $\exists$ a one to one mapping to the 6 - strebel lengths.

- Gopakumar proposed

→ Indantify poles of Strebel differential with pts of closed string Insertions

→ Critical graph of strebel differential with dual of field Theory Skeleton graph

$$\left.\begin{array}{l} V \longleftrightarrow F \\ F \longleftrightarrow V \\ E \longleftrightarrow E \end{array}\right\} \text{dual}$$



→ The schwinger parameters are identified with the strebel lengths

Since the strebel differential is uniquely specified by the 4-residues and the $SL(2,C)$ invariant location of the poles $z_1\ z_2\ z_3\ z_4$. (cross ratio)

$$\eta = \frac{(z_3 - z_2)(z_1 - z_0)}{(z_1 - z_2)\ (z_3 - z_0)}$$

⇒ There exists a unique map from the strebel lengths (schwinger parameter) to the Residues r_i & η.

Amplitude

$$\therefore \int \prod_{i=1}^{6} d\sigma_i \ f(\sigma_i) = \int \prod_{i=1}^{4} dr_i \, d\eta \, d\bar{\eta} \ f(r_i, \eta, \bar{\eta})$$

. Perform the Integral over r_i → left with

→ $\int d\eta \, d\bar{\eta} \ f(\eta, \bar{\eta})$

'world sheet 4 pt fn'

Let us see the <u>change of variables</u>
for the <u>most general 4 pt fn.</u>

The general strebel differential for the 4 punctured sphere is given by

$$\varphi(z)\,d^2z = -c\,\frac{(z^2-1)(z^2k^2-1)}{(z-z_0)(z-z_1)^2(z-z_2)^2(z-z_3)^2}\,dz^2$$

used $SL(2,C)$ to fix. zeros at
± 1 & $\pm 1/k$

poles at z_0, z_1, z_2, z_3.

Introduce the variable

$$u = \int_1^z \frac{dz}{\sqrt{(z^2-1)(z^2k^2-1)}}$$

$$z = \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}$$ Jacobi Elliptic functions
of periodicity
$(2\omega_1, 2\omega_2)$ modulus k

Then we can obtain the following relations

$$\sum_{i=0}^{3} \frac{r_i}{\mathrm{sn}\, u_i} = 0$$

$$\sum_{i=0}^{3} r_i \, \mathrm{sn}\, u_i = 0$$

$$\sum_{i=0}^{3} r_i \frac{\mathrm{cn}\, u_i \, \mathrm{dn}\, u_i}{\mathrm{sn}\, u_i} = 0$$

$$a = \sum r_i \left[\pi - 2i\left(\zeta(u_i)\, \omega_1 - \zeta(\omega_2)\, u_i \right) \right]$$

$$b = \sum r_i \left[\pi + 2i\left(\zeta(u_i)\, \omega_1 - \zeta(\omega_2)\, u_i \right) \right]$$

$$\zeta'(u) = -\wp(u)$$

Wierstrass $\wp$-fn

Note: Given a, b, r_i (strebel lengths)

can find u_i → Thus cross ratio of poles.

→ In principle one can perform the change variables.

→ In practice hard.

→ Have to look for simpler

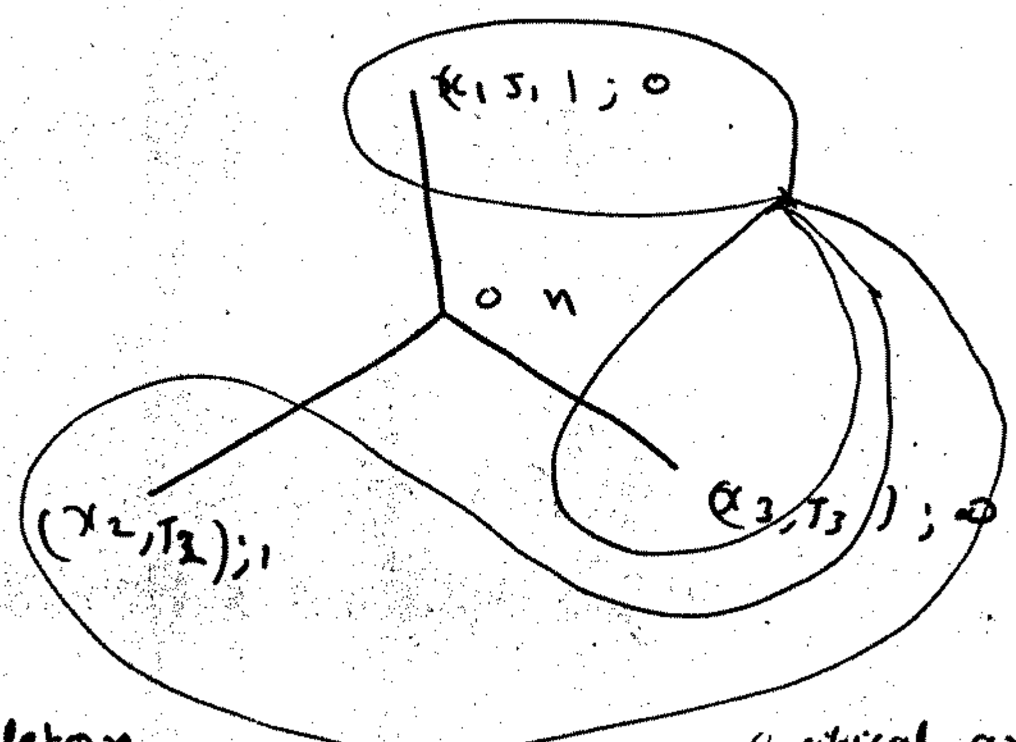
4 pt fn

In hep-th/060226 O. Aharony, Z. Komargodski & S. Razamat observed that in certain 4 point functions the change of variables can be explicity carried out.

The Y- diagram

Consider the following correlator

$$\langle \mathrm{Tr}\, X^{J}(x_1)\ \mathrm{Tr}\, Y^{J_2}(x_2)\ \mathrm{Tr}\, Z^{J_3}(x_3)\ \mathrm{Tr}(\bar{X}^{J_1} \bar{Y}^{J_2} \bar{Z}^{J_3})(\infty) \rangle$$



Skeleton

$E = 3$

$F = 1$

$V = 4$

Critical graph

$E = 3$

$F = 4$

$V = 1$

One vertex of valency 6

order of zero = 4

→ The Relations involving Elliptic functions reduce to algebraic relations.

$$r_1 + r_2 + r_3 = r_0 \quad \text{(seen from diagram)}$$

$$r_1 z_1 + r_2 z_2 + r_3 z_3 = r_0 z_0$$

$$r_1 z_1^2 + r_2 z_2^2 + r_3 z_3^2 = r_0 z_0^2$$

(z_1, z_2, z_3, z_0) locations of poles

One can solve for the cross-ratio

$$\eta = \frac{(z_3 - z_2)(z_1 - z_0)}{(z_1 - z_2)(z_3 - z_0)}$$

$$= \left(\frac{\sqrt{r_0 r_2} \pm i \sqrt{r_1 r_3}}{r_1 + r_2} \right)^2$$

We now invert the relations.

As η depends only on the ratio. Define

$$S_1 = \frac{r_1}{r_3}, \quad S_2 = \frac{r_2}{r_3} \Big\} \text{independent}$$

$$S_0 = \frac{r_0}{r_3} = 1 + S_1 + S_2.$$

$$S_1 = \frac{1 - |n| + |1-n|}{-1 + |n| + |1-n|}$$

$$S_2 = \frac{1 + |n| + |1-n|}{-1 + |n| + |1-n|}.$$

$$S_0 = \frac{1 + |n| + |1-n|}{-1 + |n| + |1-n|}$$

Rewriting the 4 pt function.

$$\langle \mathrm{Tr}\, X^{J_1}(x_1)\ \mathrm{Tr}\, X^{J_2}(x_2)\ \mathrm{Tr}\, X^{J_3}(x_3)\ \mathrm{Tr}(\bar{X}^{J_1} \bar{Y}^{J_2} \bar{Z}^{J_3})(0) \rangle$$

$$= \frac{C(J_i)}{x_1^{2J_1}\, x_2^{2J_2}\, x_3^{2J_3}} = \Gamma^4.$$

$$= C \int_0^\infty d\sigma_1\, d\sigma_2\, d\sigma_3\ \sigma_1^{J_1-1} \sigma_2^{J_2-1} \sigma_3^{J_3-1}\, e^{-(\sigma_1 x_1^2 + \sigma_2 x_2^2 + \sigma_3 x_3^2)}$$

Identify schwinger parameter with the strebel lengths

$$\sigma_1 = r_1, \quad \sigma_2 = r_2, \quad \sigma_3 = r_3$$

$\rightarrow$ Change variables to $(n, \bar{n}, r_3.)$

$$\Gamma^{a} = c\int_0^\infty dr_3\, r_3^{J-1}\int ds_1\, ds_2\, s_1^{J_1-1}\, s_2^{J_2-1}\, e^{-\sigma_3(s_1 x_1^2+s_2 x_2^2+x_3^2)}$$

$$= \tilde{c}\int ds_1\, ds_2\, \frac{s_1^{J_1-1}\, s_2^{J_2-1}}{(s_1 x_1^2+s_2 x_2^2+x_3^2\ )}$$

$$ds_1\, ds_2 = dn\, d\bar{n}\ \frac{|n-\bar{n}|}{(-1+|n|+|1-n|)^3\ |n|\,|1-n|}$$

$$\Gamma^{a}_{J_i}(x_i) = \int d^2\eta\ G^{J_i}_{x_i}(n,\bar{n})$$

$$= \int d^2\eta\ \frac{(1+|n|+|1-n|)^{1/2}}{|n|\ \ |1-n|}$$

$$\times\ \frac{(1-|n|+|1-n|)^{J_1-1/2}(1+|n|-|1-n|)^{J_2-1/2}(-1+|n|+|1-n|)^{J_3-1/2}}{\left[x_1^2(1-|n|+|1-n|)+x_2^2(1+|n|-|1-n|)+x_3^2(-1+|n|+|1-n|)\right]}$$

$G^{J_i}_{x_i}(n,\bar{n}) \longrightarrow$ Interpretable as a world sheet correlator

Satisfies the following requirements

① $G^{J_2\, J_1\, J_3\, J}_{\underline{x_2\, x_1}\, x_3}(1-\eta, 1-\bar{\eta})$

$$= G^{\underline{J_1\, J_2}\, J_3\, J}_{\underline{x_1\, x_2}\, x_3}(\eta, \bar{\eta})$$

$1 \Leftrightarrow 2$ Exchange.

② $G^{J_3\, J_2\, J_1\, J}_{x_3\, x_2\, x_1}(1/\eta, 1/\bar{\eta}) = |\eta|^4\, G^{J_1\, J_2\, J_3\, J}_{x_1\, x_2\, x_3}(\eta, \bar{\eta})$

$1 \Leftrightarrow 3$ Exchange

(world sheet operators are (1,1))

Change of variables has the permutation symmetry inbuilt.

③ OPE : expansion in powers of $\eta^h\, \bar{\eta}^{\bar{h}}$ is consistent with locality. i.e. $h - \bar{h}$ Integer.

Correlators have a similarity with Ising model spin fields.

$$\langle \sigma(1)\,\sigma(2)\,\sigma(3)\,\sigma(4)\rangle = \sqrt{\frac{1}{2(z_{13}z_{14})^{1/2}} \cdot \frac{1}{\sqrt{|\eta|\,|1-\eta|}}}\;(1+|\eta|+|1-\eta|)^{1/2}.$$

2 order 2 dis order

$$\langle \sigma(1)\,\mu(2)\,\sigma(3)\,\mu(4)\rangle = \sqrt{\frac{1}{2(z_{13}z_{14})^{1/2}}} \times \frac{1}{\sqrt{|\eta|\,|1-\eta|}} \times (-1+|\eta|+|1-\eta|)^{1/2}.$$

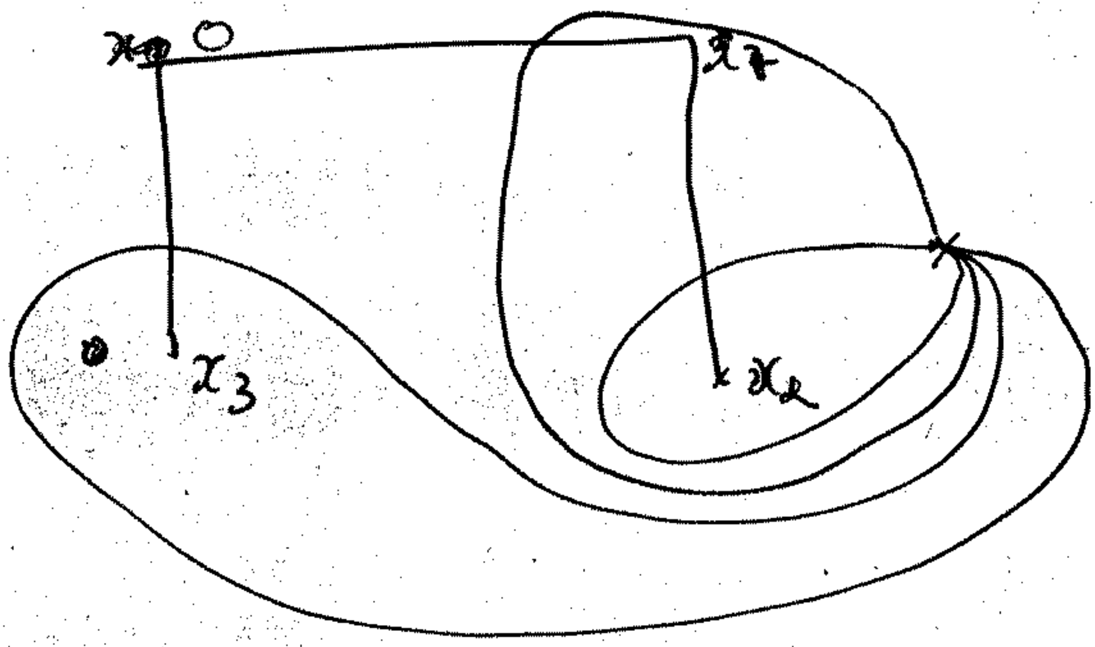
can get $(1-|\eta|+|1-\eta|)^{1/2}$

& $(1+|\eta|-|1-\eta|)^{1/2}$

by permuting the order of operators

The π diagram & the spade diagram

with O. Aharony, Z. Komargodsky
& S. Razamat.
& R. Gopakumar



The Equations determining the locations of the poles

$$r_0 = r_3 - r_2 + r_1 \rightarrow (\text{from diagram})$$

$$r_0 z_0 = r_1 z_1 - r_2 z_2 + r_3 z_3$$

$$r_0 z_0^2 = r_1 z_1^2 - r_2 z_2^2 + r_3 z_3^2$$

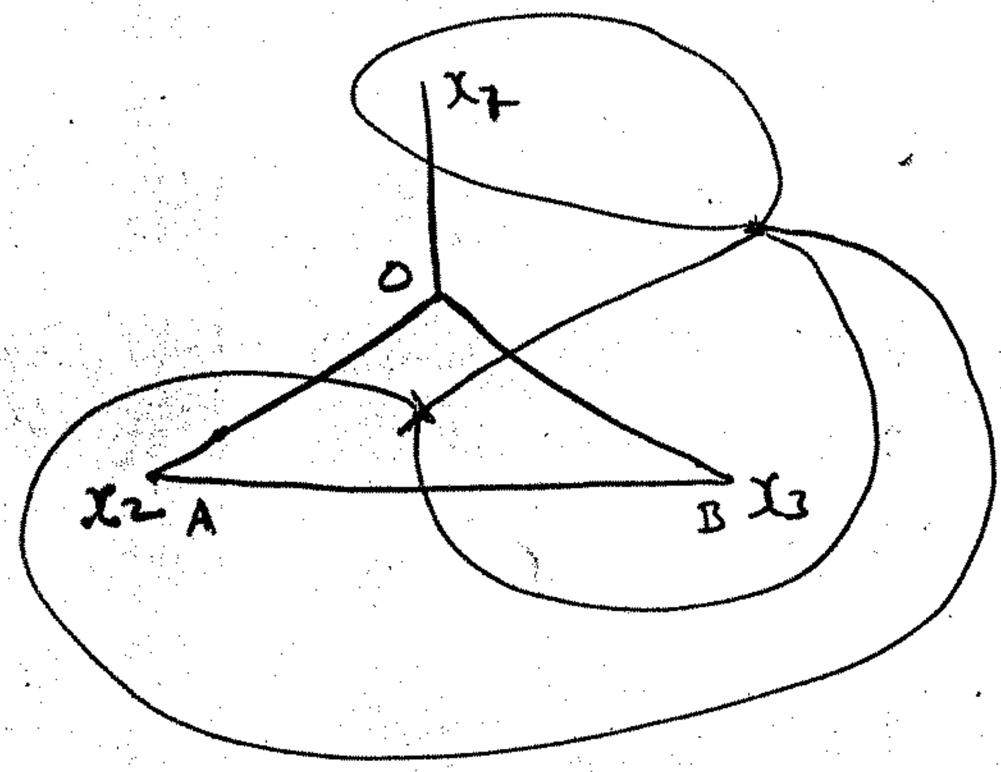
($r_2 \rightarrow -r_2$ In the Y-diagram)

Evaluating $\eta = \left(\frac{\sqrt{r_0 r_2} + \sqrt{r_1 r_3}}{r_1 - r_2} \right)^2$

→ Cross Ratio becomes real.

Under change of variables one does not have an Integral over the complex plane.

To understand this feature look at the spade diagram.



Skeleton

$E = 4$

$F = 2$

$V = 4$

Dual

$E = 4$

$F = 4$

$V = 2$

Remove AB ⟶ Y diagram.

Remove OB, or OA → λ → diagram.

The Equations determining the position of poles is <u>algebraic</u>

$$r_1 \sqrt{z_1} + r_2 \sqrt{z_2} + r_3 \sqrt{z_3} + r_0 \sqrt{z_0} = 0$$

$$\frac{r_1}{\sqrt{z_1}} + \frac{r_2}{\sqrt{z_2}} + \frac{r_3}{\sqrt{z_3}} + \frac{r_0}{\sqrt{z_0}} = 0$$

$$r_1 z_1^{3/2} + r_2 z_2^{3/2} + r_3 z_3^{3/2} + r_0 z_0^{3/2} = 0$$

One can Think of the <u>π diagram</u> as a <u>limit</u> of the <u>spade diagram</u> in which <u>No: of contractions</u> on the OB edge is <u>small</u> compared to the remaining edges

We can perform the perturbation expansion of the above equations around this limit to obtain

$$\eta = \left(\frac{\sqrt{r_0 r_2} + \sqrt{r_1 r_3}}{r_1 - r_2}\right)^2 + \boxed{\frac{3}{46}}\boxed{\omega^2}\, 2^{8/3} \underline{\underline{(r_3 + r_1 - r_2 - r_0)}}^{2/3} \times$$

$$\times \Big[\frac{1}{(r_1 r_2 r_3 r_0)^{1/6} (r_1 - r_2)^{8/3}} \times \sqrt{r_1 r_2}\,(r_0 - r_3) + \sqrt{r_3 r_0}\,(r_2 + r_1) \Big]^{4/3}$$

$(r_3 + r_1 - r_2 - r_0) \rightarrow$ length of Edge OB

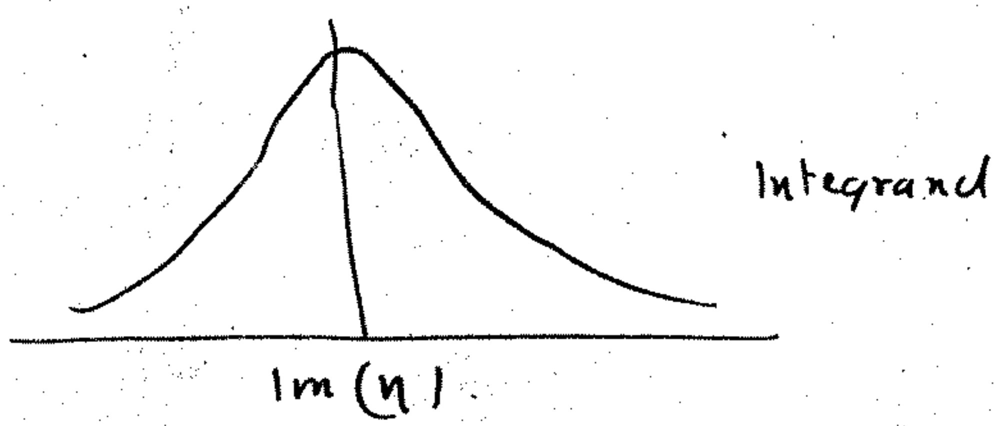
$\rightarrow$ perturbation parameter.

ω : Cube root of unity.

Thus η develops an Imaginary part.

$\rightarrow$ One can show

after performing the change of variable in the field Theory Integrand



Integrand localizes at Im(η) = 0

Thus the π diagram can be thought of as a delta function distribution in the moduli space.

Conclusions.

- Provided the change of variables for the most general 4 pt function

- Performed the change of variables explicitly fo the Y - diagram and extracted a world sheet correlator

  This satisfies.

- crossing symmetry
  locality of OPE.

- The confusing Π diagram can be thought of as a δ-fn distribution in Moduli Space.