Split States, Hole Halos and Entropy Enigmas

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with G. Moore, hep-th/yymmnnn $< T_{PR}$

Outline

Realizations of BPS states

Attractor flow trees and BPS state counting

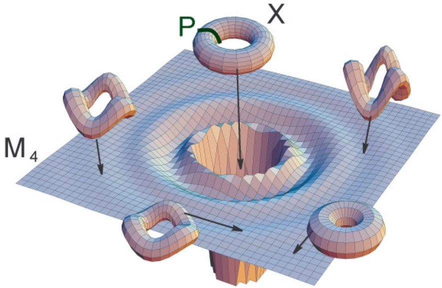
Application to the OSV conjecture

Motivation

- $AdS_2 \times S^2$ "simplest" yet most poorly understood case of gauge-gravity correspondence.
- Structure of black hole microstates.
- Mathematics: geometrical invariants (GW, GV, DT, J, ...), notion of stability.
- Phenomenology and landscape statistics (from OSV to LHC?):
  - BPS states $\Leftrightarrow$ susy brane configurations $\Leftrightarrow$ string vacua
  - So: counting black hole microstates $\Leftrightarrow$ landscape statistics.
  - May hugely affect estimates of relative fraction of vacua with discrete symmetries, low energy susy, split susy, etc...

Realizations of BPS states

Setting



- IIA on Calabi-Yau X $\rightsquigarrow$ 4d $\mathcal{N} = 2$ supergravity $+(h^{1,1} + 1)$ gauge fields
- D6-D4-D2-D0 BPS bound st. (D-branes + gauge flux) $\rightsquigarrow$ BPS black holes with magn. and el. charges (p^0, p^A, q_A, q_0)

BPS states and stability

- BPS bound for mass of particle with charge $\Gamma = (p^0, p, q, q_0)$ in vacuum with complexified Kähler moduli $t \equiv B + iJ$:

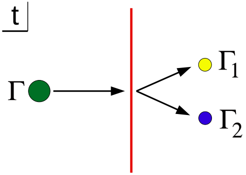
$$M \geq M_{BPS} = |Z|\, M_p$$

where

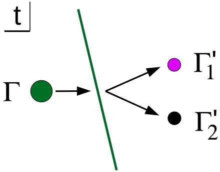
$$Z = \left(\frac{(\mathrm{Im}\, t)^3}{6} \right)^{-1/2} \left(p^0 \frac{t^3}{6} - p \cdot \frac{t^2}{2} + q \cdot t - q_0 \right) + \text{inst. corr.}$$

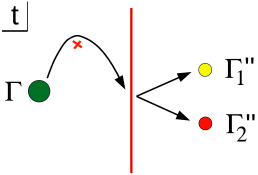
- For generic t: $|Z(\Gamma_1 + \Gamma_2, t)| < |Z(\Gamma_1, t)| + |Z(\Gamma_2, t)|$
  $\Rightarrow$ BPS states absolutely stable.
- Exception: when t such that $\arg Z(\Gamma_1, t) = \arg Z(\Gamma_2, t)$: $|Z(1+2)| = |Z(1)| + |Z(2)|$: marginal stability.
- $\Rightarrow$ BPS states can disappear from spectrum when crossing walls of marginal stability.

Decay at marginal stability



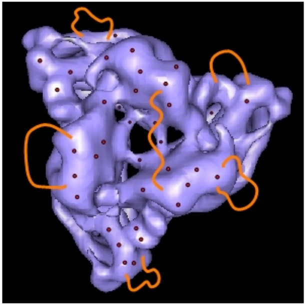
BPS particle splits in two BPS particles conserving different susies.
Even index of BPS states jumps!





For same total charge, different MS lines and decay products possible, depending on initial state and path.

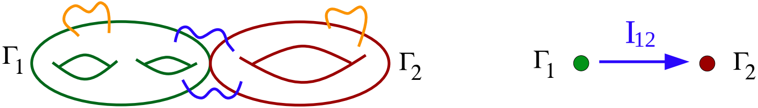
BPS states at $g_s \to 0$ and $V_{CY} \to \infty$



- Localized at single point in noncompact space.
- "Pure" D($2k$)-branes: infinitely thin, holomorphically wrapped.
- Bound states with lower dim branes:
  - gauge flux: μ-stable holomorphic vector bundles
  - brane "gas": Π-stable ideal sheaves

BPS states at $g_s \to 0$ near marginal stability

- Decay $\Gamma \to \Gamma_1 + \Gamma_2$ at marginal stability often invisible in IIA large volume geometrical D-brane picture.
- Stringy microscopic description [Kachru-McGreevy]:



Light $1 \to 2$ open string modes ϕ_i, $i = 1, \ldots, I_{12}$ have D-term potential:

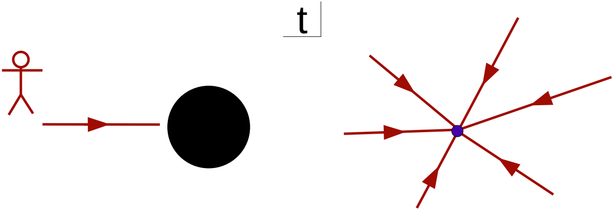
$$V_D \sim \sum_i (|\phi_i|^2 - \xi)^2$$

- FI term ξ changes sign when crossing MS wall $\Rightarrow$ susy config. exists on one side, not on other: $\exists$ "tachyon glue" iff $\xi > 0$.

BPS states in 4d supergravity ($g_s|\Gamma| \gg 1$)

Simplest possibility: spherically symmetric BPS black hole of charge $\Gamma \equiv (p^0, p, q, q_0)$:

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}d\vec{x}^2$$



Solutions $\Leftrightarrow$ attractors [Ferrara-Kallosh-Strominger]:

Radial inward flow of moduli $t(r)$ is gradient flow of $\log|Z(\Gamma, t)|$.

Existence of spherically symmetric BPS black holes

Three possibilities [Moore]:

1. Gradient flow ends in minimum $t = t_*(\Gamma)$ with $Z(\Gamma, t_*) \neq 0$.
   $\Rightarrow$ Regular black hole with horizon area $A = 4\pi|Z(\Gamma, t_*)|^2$.

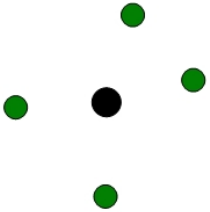
2. Flow ends in boundary point $t = t_0$ with $Z(\Gamma, t_0) = 0$.
   $\Rightarrow$ Zero area black hole, but still BPS solution (e.g. pure D6, D2-D0; note: regular after uplifting to 5d).

3. Flow ends in interior point $t = t_0$ with $Z(\Gamma, t_0) = 0$.
   $\Rightarrow$ No BPS black hole solution.

BPS black hole molecules

More general BPS solutions exist: multi-centered bound states:

$$ds^2 = -e^{2U(\vec{x})}\left(dt - \omega(\vec{x})\right)^2 + e^{-2U(\vec{x})} d\vec{x}^2.$$



- Centers have nonparallel charges.
- Bound in the sense that positions are constrained by balancing gravitational, scalar and electromagnetic attraction and electromagnetic repulsion.
- Stationary but with intrinsic spin from e.m. field

Explicit multicentered BPS solutions

- N-centered solutions characterized by harmonic function $H(\vec{x})$ from 3d space into charge space:

$$H(\vec{x}) = \sum_{i=1}^{N} \frac{\Gamma_i}{|\vec{x} - \vec{x}_i|} + H_\infty$$

  with H_∞ determined by $t_{|\vec{x}|=\infty}$ and total charge Γ.
- Positions constrained by

$$\sum_{j=1}^{N} \frac{\langle \Gamma_i, \Gamma_j \rangle}{|\vec{x}_i - \vec{x}_j|} = 2\,\mathrm{Im}\left(e^{-i\alpha} Z(\Gamma_i)\right)_{|\vec{x}|=\infty}$$

  where $\langle \Gamma_1, \Gamma_2 \rangle = \Gamma_1^{\mathrm{m}} \cdot \Gamma_2^{\mathrm{e}} - \Gamma_1^{\mathrm{e}} \cdot \Gamma_2^{\mathrm{m}}$ and $\alpha = \arg Z(\Gamma)$.
- All fields can be extracted completely explicitly from the entropy function $S(\Gamma)$ on charge space, e.g.

$$e^{2U(\vec{x})} = \frac{\pi}{S(H(\vec{x}))}$$

Decay at marginal stability

2-centered case:

- Equilibrium distance from position constraint:

$$|\vec{x}_1 - \vec{x}_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{|Z_1 + Z_2|}{\mathrm{Im}(Z_1 \overline{Z_2})}\bigg|_{|\vec{x}|=\infty}$$

- When MS wall is crossed: RHS $\to \infty$ and then becomes negative: decay

Example: pure $D4 = D6 - \overline{D6}$ molecule

- Pure (ample) D4 with D4-charge P has

$$Z \sim -P \cdot \frac{t^2}{2} - \frac{P^3 + c_2 P}{24}.$$

  $Z(t) = 0$ at $t \sim i\,P \Rightarrow$ No single centered solution.

- Instead: realized as bound state of single D6 with $U(1)$ flux $F = P/2$ and anti-(single D6 with flux $F = -P/2$):

  Stable for $\mathrm{Im}\, t > \mathcal{O}(P)$.

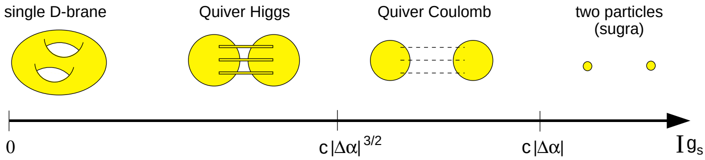
- Total charge ok ✓
- If P not even, flux must be turned on on D4 as well (✓[Freed-Witten,Minasian-Moore])
- M-theory uplift: smooth "bubbling" geometry.

Transition between $g_s|\Gamma| \gg 1$ and $g_s|\Gamma| \ll 1$ pictures

- Mass squared lightest bosonic modes of open strings between Γ_1 and Γ_2:

$$
\begin{aligned}
M^2/M_s^2 \quad &\sim \quad \frac{|\vec{x}_1 - \vec{x}_2|^2}{\ell_s^2} + \Delta\alpha \\
&= \quad c(t)\, g_s^2 + \Delta\alpha
\end{aligned}
$$

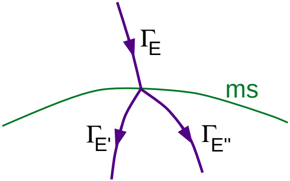
- On stable side of MS wall $\Delta\alpha < 0$, so if g_s gets sufficiently small, open strings become tachyonic and branes condense into single centered D-brane. [FD qqhh]

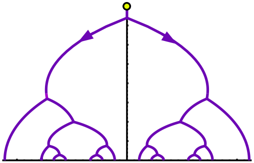


Attractor flow trees and BPS state counting

The flow tree - BPS state correspondence

- Establishing existence of multicentered BPS configurations not easy: position constraints, $S(H(\vec{x})) \in \mathbb{R}^+ \; \forall \vec{x}$, ... However:
- Theorem/conjecture: Branches of multicentered configuration moduli spaces in 1-1 correspondence with attractor flow trees:



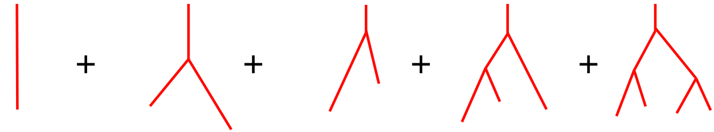


  - Initial point = background $t_{r=\infty}$.
  - Each edge E is attractor flow for some charge Γ_E
  - Charge and energy "conserved" at vertices $E \to E' + E''$:
    1. $\Gamma_E = \Gamma'_E + \Gamma_{E''}$
    2. $|Z(\Gamma_E)| = |Z(\Gamma_{E'})| + |Z(\Gamma_{E''})|$, i.e. splits on MS walls.
  - Terminal points = attractor points $t_*(\Gamma_i)$.
- Much simpler to check!

Flow tree decomposition of BPS Hilbert space

- Flow trees can also be given microscopic interpretations (decay sequences / tachyon gluing).
- $\Rightarrow$ Hilbert space of BPS states of charge Γ in background t can be decomposed in attractor flow tree sectors:

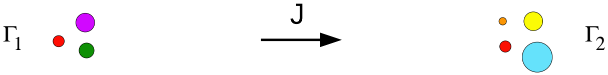
$$\mathcal{H}(\Gamma, t) =$$



Side note: scaling solutions

- There exists multicentered configurations asymptotically connected to single centered black holes, e.g. $\mathrm{D6}$ + flux, $\overline{\mathrm{D6}}$ + flux, N D0 for N sufficiently large.
- Associated to quivers with closed loops.
- No walls of marginal stability.
- Prime candidates for horizonless supergravity microstates resolving black holes $\rightsquigarrow$ superconformal quantum mechanics (but: probably need new d.o.f.)
- Represented by ordinary single attractor flow $\rightsquigarrow$ $\exists$ refinement of flow tree picture to represent different such multicentered configurations?

Wall crossing formula for primitive splits



- Near marginal stability wall $\Gamma \to \Gamma_1 + \Gamma_2$ (with Γ_1 and Γ_2 primitive), the decaying part of $\mathcal{H}(\Gamma, t)$ has following factorized form:

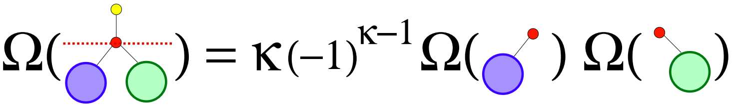
$$(J) \otimes \mathcal{H}'(\Gamma_1, t) \otimes \mathcal{H}'(\Gamma_2, t)$$

  with $J = \frac{1}{2}(\langle \Gamma_1, \Gamma_2 \rangle - 1)$.
- Spin J factor comes from intrinsic angular momentum monopole-electron system ($-1/2$ from relative position d.o.f.)
- Implies index jump

$$\Delta\Omega = -(-)^{2J}(2J+1)\,\Omega(\Gamma_1, t_{\rm ms})\,\Omega(\Gamma_2, t_{\rm ms}).$$

Recursive index factorization formula for primitive splits

From wall crossing formula:



where $\kappa = |\langle \Gamma_1, \Gamma_2 \rangle|$.

Total index formula

- From recursion formula (ignoring nonprimitive splits):

$$\Omega(\Gamma,t) = \Omega(\Gamma,t_*(\Gamma)) + \sum_{\Gamma\to\Gamma_1+\Gamma_2} (-)^{\langle\Gamma_1,\Gamma_2\rangle-1}|\langle\Gamma_1,\Gamma_2\rangle|\,\Omega(\Gamma_1,t_{\rm ms})\,\Omega(\Gamma_2,t_{\rm ms})$$

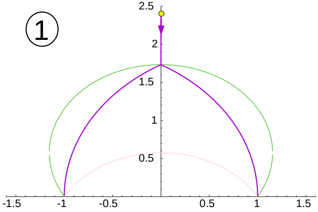
  where sum is over all allowed splits.

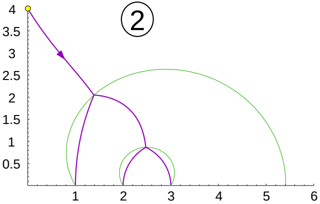
- Iteration results in sum over flow trees, of terms involving products of intersection products and terminal $\Omega_*(\Gamma_i) := \Omega(\Gamma_i, t_*(\Gamma_i))$ factors:

$$\Omega(\Gamma,t) = \sum_{\text{trees}\,T} P(T)\prod_i \Omega_*(\Gamma_i)$$

- Note: reminiscent of Joyce's formulas.

Simple (single tree) examples





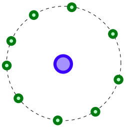
1. Pure D4 on $P > 0$ with pulled back flux S : $\Gamma_1 = \eta e^{P/2+S}$, $\Gamma_2 = -\eta e^{-P/2+S}$, $\eta := 1 + \frac{c_2}{24}$, [up to signs for simplicity]:

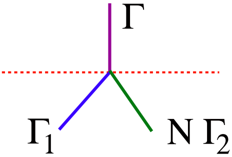
$$\Omega = \langle \Gamma_1, \Gamma_2 \rangle = \frac{P^3}{6} + \frac{c_2 P}{12} = I_P = \chi(\mathcal{M}_P). \quad \checkmark$$

2. D6-D2-D0 ideal sheaf $\mathcal{I}$ with D2 $= U \cap V$; $U, V, V - U > 0$: $\Gamma_1 = \eta e^U$, $\Gamma_2 = \eta e^V$, $\Gamma_3 = -\eta e^{U+V}$:

$$\Omega = |I_V - I_{V-U}| I_U = \chi(\mathcal{M}_{\mathcal{I}}) = N_{DT}(\mathcal{I}). \quad \checkmark$$

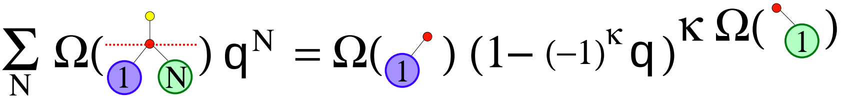
Nonprimitive splits 1: halos





Halo = bound state of one Γ_1 particle ("core") with N Γ_2 particles.

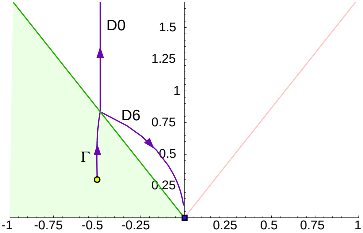
Recursive index formula in terms of generating function:



Nonprimitive splits 2: $(\Gamma_1, N\Gamma_2) \rightarrow (M\Gamma_1, N\Gamma_2) \rightsquigarrow$??

Example: D6-D0 bound states in large B-field

- For $-B \sim -\mathrm{Re}\, t$ sufficiently large: D6 - N D0 BPS bound states exist as halos:



$$\mathcal{Z}_{D6-D0}(u;t) := \sum_n \Omega(D6+nD0,t)\, u^n = \prod_k (1-(-u)^k)^{-k\chi(X)} \quad \checkmark$$

(product is over halos of D0-particles of D0-charge k.)

- Similarly: D6-D2-D0 halos $\rightsquigarrow$ Gopakumar-Vafa type product formulas counting D6-D2-D0 bound states $\rightsquigarrow \mathcal{Z}_{D6} \sim \mathcal{Z}_{\mathrm{top}}$.

Application to the OSV conjecture

The OSV conjecture

Defining

$$\mathcal{Z}_{osv}(\phi) \equiv \sum_q \Omega(p,q)\, e^{\phi\cdot q}$$

[Ooguri-Strominger-Vafa] conjectured:

$$\mathcal{Z}_{osv}(\phi) \sim \mathcal{Z}_{top}(g_{top}, t)\, \overline{\mathcal{Z}_{top}(g_{top}, t)}$$

with identifications:

$$g_{\rm top} = \frac{1}{\phi^0 + i\, p^0}, \qquad t^A = \frac{\phi^A + i\, p^A}{\phi^0 + i\, p^0}.$$

Inverting:

$$\Omega(p,q) \sim \int d\phi\, e^{-\phi\cdot q}\, |\mathcal{Z}_{top}|^2(p,\phi).$$

RHS in leading saddle point approx. $= e^{S_{\rm BH}(p,q)}$.

Deriving OSV for $p^0 = 0$: rough outline

1. Identify $\mathcal{Z}_{osv} = \lim_{\beta\to 0} \mathcal{Z}_{D4}(\beta, C_1, C_3)$.
2. Use $SL(2,\mathbb{Z})$ TST-duality to rewrite $\mathcal{Z}_{D4}$ at $t = i\infty$ as Fareytail/Rademacher series built on polar part $\mathcal{Z}_{D4}^-$:

$$\mathcal{Z}_{osv} = \sum_{A\in SL(2,\mathbb{Z})} f(A,\phi^0)\, \mathcal{Z}_{D4}^-(A\cdot(\phi^0,\phi))$$

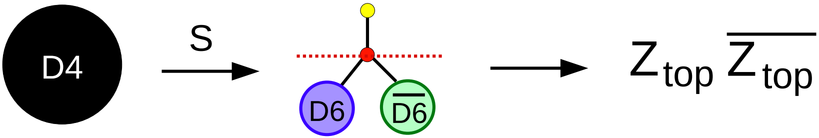
3. Polar BPS states split: no attractor point, $\Omega_*(\Gamma^-) = 0$, so

$$\Omega(\Gamma^-,t) = \sum_{\Gamma^-\to\Gamma_1+\Gamma_2} (-)^{\langle\Gamma_1,\Gamma_2\rangle-1} |\langle\Gamma_1,\Gamma_2\rangle|\, \Omega(\Gamma_1,t_{\rm ms})\, \Omega(\Gamma_2,t_{\rm ms})$$

4. At large P, $SL(2,\mathbb{Z})$ element $A = S : \phi^0 \mapsto 1/\phi^0$, and splits into Γ_1 (Γ_2) = single (anti-)D6 with dilute D2-D0 gas dominate nonpolar part of $\mathcal{Z}_{D4}$ in Fareytail sum, provided $\phi^0 (\sim 1/g_{\rm top})$ not too large (and P large).
5. Dilute gasses fully factorize $\Rightarrow$ in suitable regime:

$$\mathcal{Z}_{osv} \sim \mathcal{Z}_{D6}\, \mathcal{Z}_{\overline{D6}} \sim \mathcal{Z}_{DT}\, \overline{\mathcal{Z}_{DT}} = |\mathcal{Z}_{\rm top}|^2.$$

Pictorial summary

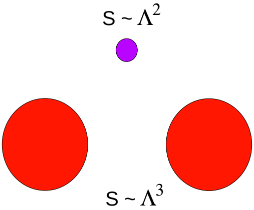


The Entropy Enigma

For $\Gamma = \Lambda(0, P, Q, Q_0)$ in large Λ limit, and in background with $t \gg O(\Lambda)$, there always exists two centered D6-anti-D6 type black hole configuration such that

$$S_{\mathrm{BH},2} := S_{\mathrm{BH},1}(\Gamma_1) + S_{\mathrm{BH},1}(\Gamma_2) \sim \Lambda^3$$

while leading order OSV prediction is $\log \Omega \sim S_{\mathrm{BH},1}(\Gamma) \sim \Lambda^2$.



$\Rightarrow$ OSV terribly wrong??

Possible resolutions

1. Can't trust solutions? ✗
2. $\Omega(\Gamma)$ is index, receives many contributions, from many different attractor flow trees, with different signs $\Rightarrow$ there may be miraculous cancelations.

   $\leadsto$ Would highly (and mysteriously) constrain various indices of subsystems.
3. OSV only valid at attractor point:

$$\Omega(p, q; t_*(p, q)) \sim \int d\phi\, e^{-\phi \cdot q} |\mathcal{Z}_{top}|^2(p, \phi).$$

   [No troubling Λ^3 solutions there.]

   $\leadsto$ But then OSV loses interpretation of counting large volume D-brane ground states.

So what about those OSV derivations?

- Derivations [Gaoitto-Strominger-Yin, Cheng-deBoer-Dijkgraaf-Manschot-Verlinde, we] are supposed to be valid in large volume limit (needed to make $\mathcal{Z}_{D4}$ Jacobi form). $\rightsquigarrow$???
- Upon closer examination: derivations fail at weak g_{top} ($\sim 1/\Lambda$) if indeed $\log \Omega(\Lambda\Gamma) \sim \Lambda^3$ (non-dilute gas becomes dominant).
- However, even with Λ^3 growth, there is regime in which conjecture can be derived at large volume:

$$g_{top} > g_{top}^{\text{crit}}, \quad \text{or equivalently} \quad \hat{q}_0/p^3 > c_{\text{crit}}.$$

$\rightsquigarrow$ phase transition.

OSV: bottom line

$$
\begin{aligned}
\mathcal{Z}^{+}_{\rm osv}(\phi) &= \frac{1}{2\pi}\,\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}\sum_k i\phi^0 e^{\mathcal{F}^\epsilon(p,\phi+ik,\alpha)-2\pi i\Delta q\cdot k}\\
\mathcal{F}^\epsilon(p,\phi,\alpha) &:= F^\epsilon_{\rm top}(g,t)+\overline{F^\epsilon_{\rm top}(g,t)}+\Delta\mathcal{F}\\
F^\epsilon_{\rm top} &:= \log\mathcal{Z}^\epsilon_{\rm top}=\log\mathcal{Z}_{\rm pol}+\log\mathcal{Z}'^{\epsilon}_{\rm DT}+\frac{1}{2}\log\mathcal{Z}^{0,\epsilon}_{\rm DT}\\
\mathcal{Z}^\epsilon_{\rm DT}(g,t) &:= \sum_{|n|<\epsilon P^3,\,P\cdot\beta<\epsilon P^3} N_{DT}(n,\beta)\,e^{-gn}\,e^{i\beta\cdot t}
\end{aligned}
$$

with substitutions

$$
g\equiv\frac{2\pi}{\phi^0}+2\pi\alpha,\qquad t\equiv\frac{1}{\phi^0}(\Phi+i\frac{P}{2})+i\alpha P.
$$

and error

$$
\Delta\mathcal{F}=\mathcal{O}\big(e^{(\Sigma(P,\epsilon)-\frac{2\pi}{\phi^0}\epsilon)P^3}\big)+\mathcal{O}(e^{-\frac{2\pi}{\phi^0}\epsilon^{-1}|P|}).
$$

where $\Sigma(P,\epsilon)\sim$ independent of P if indeed Λ^3 index growth, $\sim 1/|P|$ if miraculous cancelation $\Lambda^3\to\Lambda^2$.