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Integrability and the AdS/CFT Correspondence in the Limit of Large R-Charge

hep-th/0604175

and H. Chen, ND, K. Okamura
hep-th/0605155
hep-th/0608047
hep-th/0610295

AdS/CFT equates:

- Spectrum of operator dimensions in planar $\mathcal{N}=4$ SUSY Yang-Mills
- Spectrum of free strings on $AdS_5 \times S^5$

dual theories weakly coupled in different limits

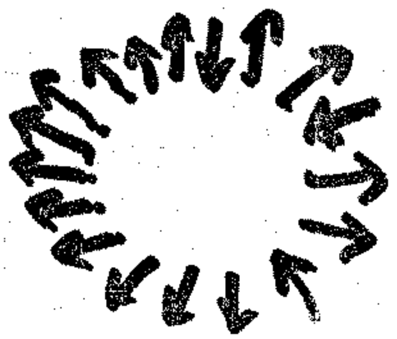
i) $\lambda \ll 1 \Rightarrow \mathcal{N}=4$ SYM weakly coupled

ii) $\lambda \gg 1 \Rightarrow$ string σ-model weakly coupled

't Hooft coupling: $\lambda = g^2 N$

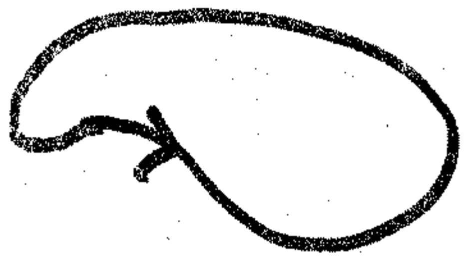
emergence of <u>integrability</u> - on both sides of correspondence,

<u>Gauge Theory</u> - Minahan + Zarembo
Beisert + Staudacher



One-loop planar dilatation operator
⟺ integrable spin chain

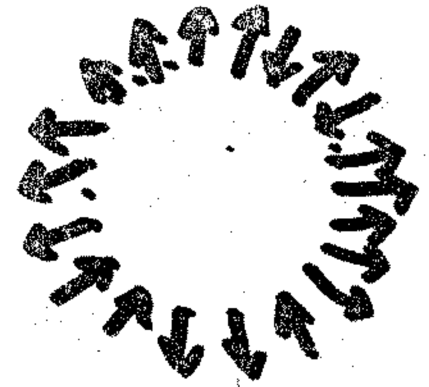
<u>String Theory</u> - Bena, Polchinski + Roiban

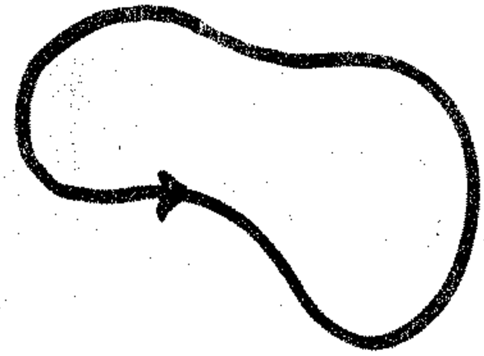


string σ-model is classically integrable

∞ tower of conserved charges

Gauge Theory	String Theory
Minahan + Zarembo	Bena, Polchinski + Roiban





Periodic BC,

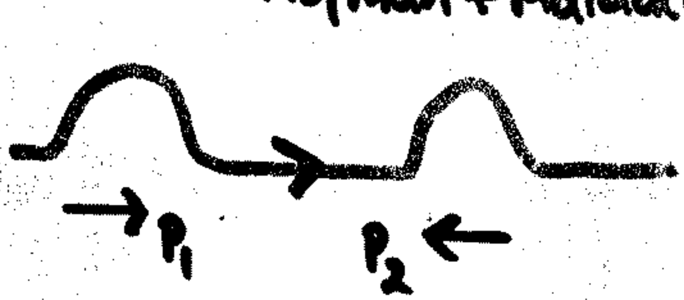
Integrability $\overset{?}{\Longrightarrow}$ Bethe Ansatz

- $J \to \infty$ limit $\Rightarrow$ long chain/string

Staudacher
Beisert
Hofman + Maldacena

$\rightarrow p_1 \quad p_2 \leftarrow$

$\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\uparrow\downarrow\uparrow\uparrow\ldots$



Vacuum BC,

Integrability $\Rightarrow$ Factorised scattering

Outline

I) The spin-chain: $\lambda \ll 1$
SU(2) sector at one-loop

II) Exact Analysis all λ

III) Dual string theory $\lambda \gg 1$

The Spin-Chain Minahan + Zarembo

SU(2) sector contains $\mathcal{N}=4$ operators formed from two adjoint complex scalars X, Y

$$\hat{O} \sim \mathrm{Tr}_N\left[X^{J_1} Y^{J_2}\right]$$

- operators $\longleftrightarrow$ configurations of periodic spin chain

$$\mathrm{Tr}_N[\ldots XXY\,XX\ldots XXXYXXY]$$
$$\ldots\uparrow\uparrow\downarrow\,\uparrow\uparrow\ldots\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow$$

length, $L = J_1 + J_2$

impurity #, $M = J_2$

trace $\Rightarrow$ cyclicity constraint

Scaling dimensions Δ $\equiv$ Eigenvalues of dilatation operator $\hat{D}$

to one-loop,

$$\hat{D} = L\,\mathbb{I} + \frac{\lambda}{8\pi^2}\hat{H} + O(\lambda^2)$$

eigenvalues,

$$\Delta = L + \frac{\lambda}{8\pi^2}\mathcal{E} + O(\lambda^2)$$

where,

permutation operator

$$\hat{H} = \sum_{\ell=1}^{L}\left(I - P_{\ell,\ell+1}\right)$$

$$I|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle$$

$$P|\uparrow\downarrow\rangle = |\downarrow\uparrow\rangle$$

is Hamiltonian of Heisenberg $XXX_{1/2}$ spin chain which is Integrable infinite chain: $L \to \infty$, M fixed

<u>$M=0$</u> ferromagnetic ground state

$\varepsilon = 0$

$\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\ldots$

<u>$M=1$</u> one impurity

$|\ell\rangle = \ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots$

ℓ'th site

Magnon:

$$|p\rangle = \sum_{\ell} e^{ip\ell}|\ell\rangle \qquad \psi_p(\ell) = e^{ip\ell}$$

$\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots$

$\longrightarrow p$

Dispersion Relation:

$$\varepsilon(p) = 4\sin^2(p/2)$$

$\underline{M=2}$ two magnons

$$\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots\ldots\ldots\uparrow\uparrow\downarrow\uparrow\uparrow\ldots$$
$$\longrightarrow p_1 \qquad\qquad p_2 \longleftarrow$$

S-matrix

- <u>scattering states</u>

$$\psi_{p_1,p_2}(\ell_1,\ell_2) = e^{ip_1\ell_1+ip_2\ell_2} + S(p_1,p_2)\, e^{ip_1\ell_2+ip_2\ell_1}$$

$$S(p_1,p_2) = \frac{\varphi(p_1) - \varphi(p_2) + i}{\varphi(p_1) - \varphi(p_2) - i}$$

$$\varphi(p) = \tfrac{1}{2}\cot\left(\tfrac{p}{2}\right)$$

- <u>boundstate</u>

S-matrix pole: $\varphi(p_1) - \varphi(p_2) = i$

$$\psi_{p_1,p_2}(\ell_1,\ell_2) = \left[\cos\left(P/2\right)\right]^{|\ell_1-\ell_2|} e^{iP/2(\ell_1+\ell_2)}$$

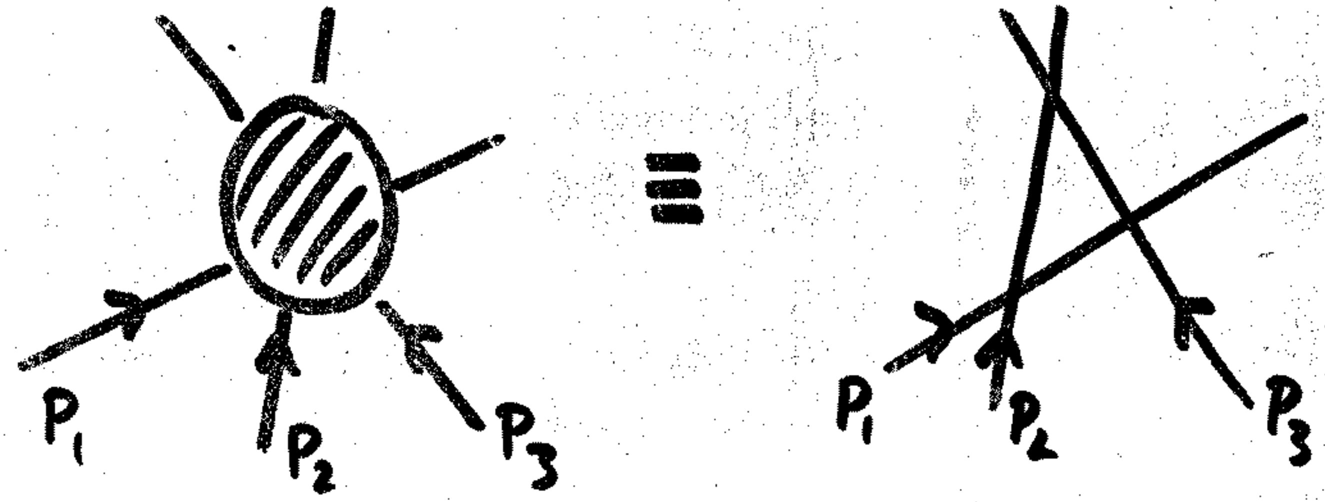
normalisable

$P = p_1 + p_2$

$\underline{M>2}$ multi-magnon

scattering + boundstates

<u>integrability</u> $\Rightarrow$ factorization of multi-particle S-matrix,



$$S(P_1, P_2, P_3) = S(P_1, P_2)\, S(P_1, P_3)\, S(P_2, P_3)$$

3-particle S-matrix

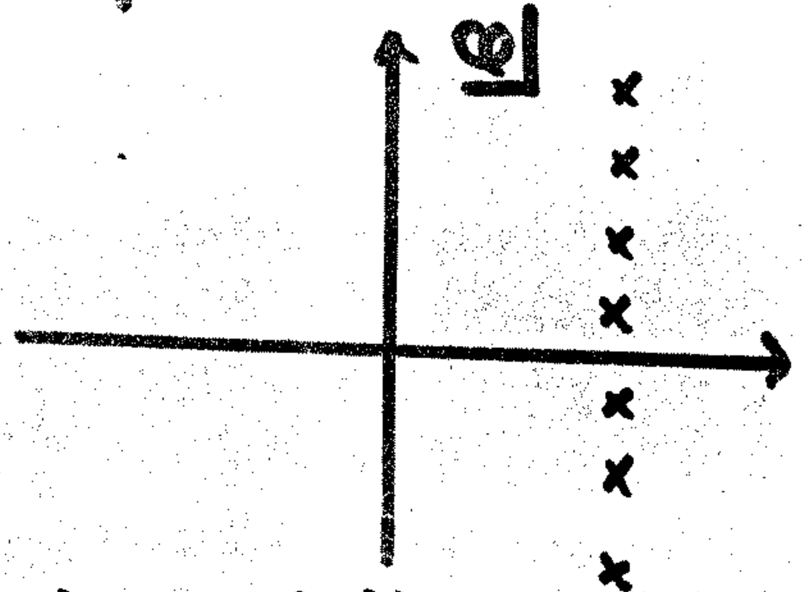
2-particle S-matrix

- conserved charges $\equiv$ individual particle momenta $\{P_1, P_2, \ldots, P_M\}$

pole at,

$$\varphi_1 - \varphi_2 = \varphi_2 - \varphi_3 = i \Rightarrow \text{3-magnon boundstate}$$

Q-magnon boundstates correspond to "Bethe strings"



$u_j - u_{j+1} = i$

$j = 1, \dots, Q-1$

dispersion relation,

$$\underline{\underline{\varepsilon_Q = 4/Q \, \sin^2(p/2)}} \qquad Q = 1, 2, 3, \dots$$

correspond to operators in $\mathcal{N}=4$ SYM of form,

$$\hat{O} \sim \mathrm{Tr}[\dots\dots XXY^{Q}XX\dots]$$

Q impurities bound together

full spectrum in $L \to \infty$ limit
M fixed

$\equiv$ Free multi-particle Fock space

$$\hat{O} \sim \mathrm{Tr}\Big[\ldots X X Y \overset{Q_1}{X} \ldots X X Y \overset{Q_2}{X} X \ldots X Y \overset{Q_3}{X} \ldots\Big]$$

$$\overrightarrow{p_1} \qquad \overleftarrow{p_2} \qquad \overrightarrow{p_3}$$

$$\Delta = L + \frac{\lambda}{8\pi^2} \sum_j \frac{4}{Q_j} \sin^2\left(\frac{p_j}{2}\right) + \ldots$$

What happens,

- Beyond SU(2) sector ?
- Beyond one-loop ?

Beisert - Staudacher approach

- **assume** integrability $\forall \lambda$
- **derive** constraints on exact spectrum and S-matrix
  $\Rightarrow$ exact Bethe ansatz equations
- **compare** with explicit calculations

$\lambda \ll 1$ Perturbative gauge theory

$\lambda \gg 1$ semiclassical string theory

Asymptotic States Beisert

$L \to \infty$, λ fixed

- ferromagnetic groundstate,

$$\ldots XXXXXXXX \ldots$$

has unbroken susy,

$$SU(2|2) \times SU(2|2)$$

linearly realised on individual impurities,

$$\ldots XXXXYXXX \ldots$$

with central charge,

$$\{Q, Q\} \sim \mathbb{P}$$

$$\delta_{\mathbb{P}} Y \sim [X, Y]$$ ← gauge transformation

($\mathbb{P}$ vanishes on physical states) asymptotic state, $\quad$ l'th site

$$\hat{O} \sim \sum_{l} e^{ipl} \ldots\ldots XXX\,I\,XXX\ldots$$

$$I \in \{Y, Z, D_\mu X, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}\}$$

- impurities form short multiplet 1b of $SU(2|2)^2$

$\Rightarrow$ exact dispersion relation,

$$\Delta - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$$

- $\lambda \ll 1$ $\quad \Delta - J = 1 + \frac{\lambda}{2\pi^2}\sin^2(p/2) + O(\lambda^2)$

Heisenberg spin chain

- $\lambda \gg 1$

$$\Delta - J = \frac{\sqrt{\lambda}}{\pi}\left|\sin(p/2)\right| + O(1)$$

giant magnon = stringy soliton

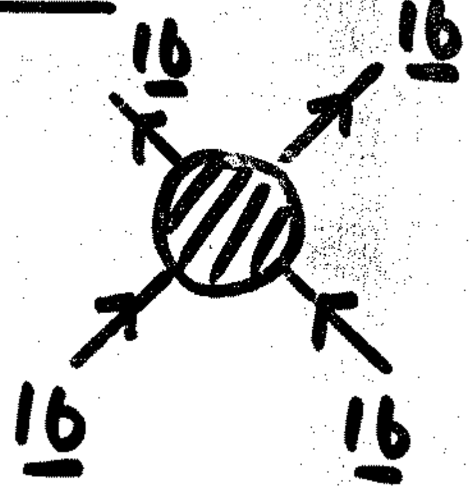
The $\mathcal{S}$-matrix

integrability ⇒ factorized scattering

two-particle $\mathcal{S}$-matrix

$$\mathcal{S}_{IJ}(p_1, p_2; \lambda)$$

256×256



determined up to an overall phase, Beisert

$$S_0(p_1, p_2; \lambda)$$

by unbroken SUSY, $SU(2|2)^2$

recent progress:

crossing (Janik)

transcendentality (Beisert, Eden + Staudacher)

Boundstate spectrum

ND

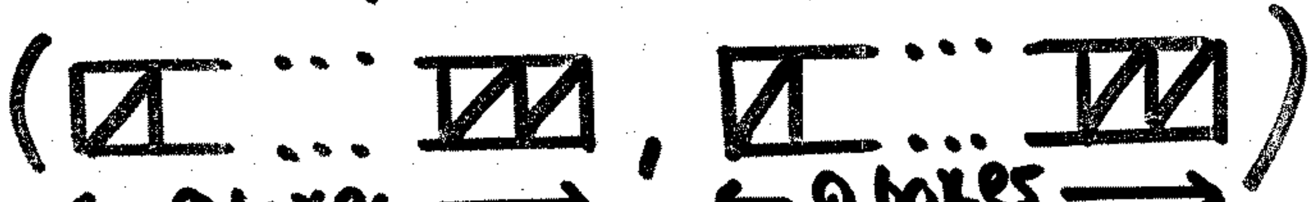
Chen, ND, Okamura

pole structure of exact S-matrix

$\Rightarrow \exists$ BPS Q-magnon boundstate

$Q = 1, 2, 3 \ldots$

- lies in, short rep



← Q boxes → , ← Q boxes →

of $SU(2|2) \times SU(2|2)$ dimension $= 16Q^2$

- exact dispersion relation,

$$\Delta - J_1 = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)} \quad - (*)$$

$$\simeq Q + \frac{1}{2}\frac{\lambda}{Q\pi^2}\sin^2(p/2) + \ldots$$

matches spectrum of $XXX_{1/2}$ chain

- scaling limit,

$\lambda \to \infty$, $Q \sim \sqrt{\lambda}$

$\Rightarrow (*)$ should hold in classical string

Result

Exact spectrum of planar operator dimensions in $J_1 \to \infty$ limit,

$$\hat{O} \sim \mathrm{Tr}_N[\dots XX]_1 XX \dots XX]_2 XX \dots XX]_3 XX \dots]$$

$$\xrightarrow{\;\;} p_1 \qquad p_2 \xleftarrow{\;\;} \qquad \xrightarrow[p_3]{\;\;}$$

Each impurity has, $\sum_i p_i = 0$

- conserved momentum p
- boundstate # $Q \in \mathbb{Z}^+$
- choice of $16Q^2$ polarisations

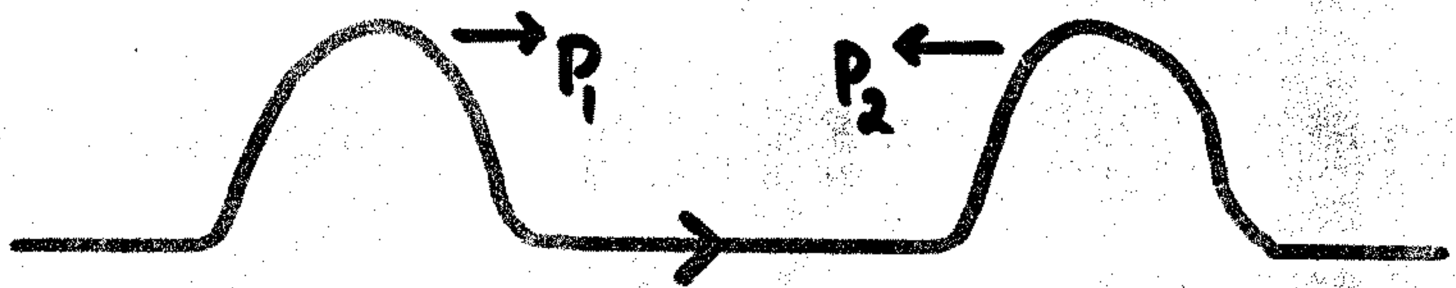
total dimension,

$$\Delta - J_1 = \sum_i \sqrt{Q_i^2 + \lambda/\pi^2 \sin^2(p_i/2)}$$

Extra states ?

String Theory

$J_1 \to \infty$ limit yields infinitely long string Hofman + Maldacena



magnon boundstates $\equiv$ classical solitons
$Q \sim \sqrt{\lambda}$

SU(2) sector: $Q = J_2$

S^3 σ-model + Virasoro $\xrightarrow{\text{reduction}}$ complex sine-Gordon equation

Pohlmeyer

CSG is classically integrable

$\Rightarrow$ multi-soliton solutions exhibit factorised scattering

Complex sine-Gordon Eqn

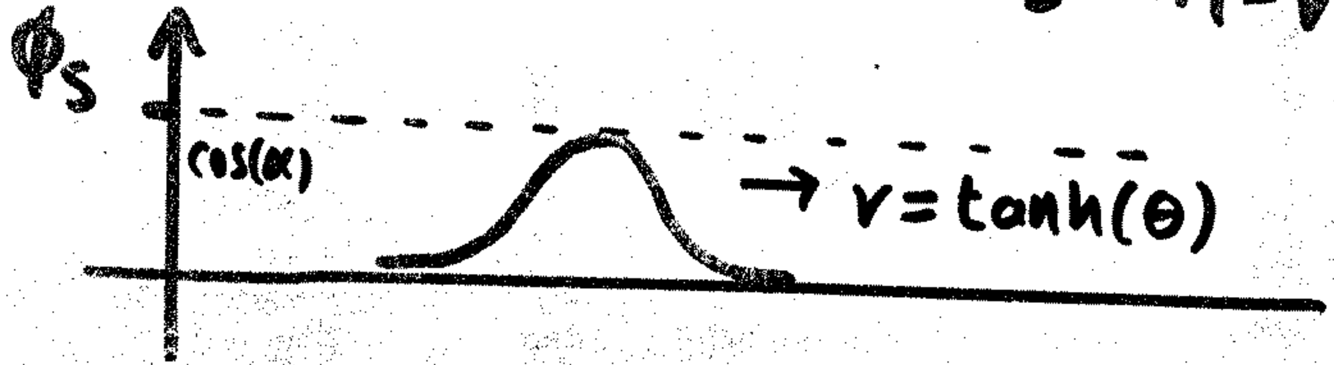
integrable PDE for $\psi(\sigma,\tau) \in \mathbb{C}$,

$$\partial_+ \partial_- \psi + \psi^* \frac{\partial_+ \psi \partial_- \psi}{1-|\psi|^2} + \psi(1-|\psi|^2) = 0$$

Getmanov
Lund + Regge
......

Soliton solution,

$$\psi(\sigma,\tau) = e^{i \sin(\alpha) \left[\frac{\tau - v\sigma}{\sqrt{1-v^2}}\right]} \phi_S\left[\frac{\sigma - v\tau}{\sqrt{1-v^2}}\right]$$



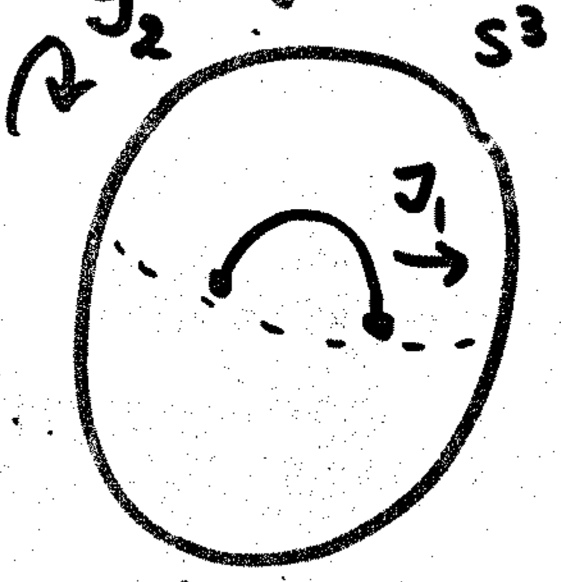
two parameters,

$\theta \sim$ rapidity

$\alpha \sim$ rotation parameter

.. reconstruct corresponding string motion,

"Dyonic giant magnon"



Chen, ND, Okamura
Arutyonov, Frolov, Zamaklar
Minahan, Tirziu, Tseytlin
Spradlin, Volovich

dictionary,

$$J_2 = \frac{\sqrt{\lambda}}{\pi}\,\frac{s(\alpha)c(\alpha)}{c^2(\alpha)+sh^2(\theta)}$$

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\,\frac{sh(\theta)ch(\theta)}{c^2(\alpha)+sh^2(\theta)}$$

$$\cot(p/2) = 2\,sh(\theta)/c(\alpha)$$

$$\Rightarrow \Delta - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$$

in $SU(2)$ sector $J_2 = Q$ = # of magnons

$Q = J_2 \to 0$ case

$$\Delta - J_1 = \frac{\sqrt{\lambda}}{\pi}\left|\sin\left(\frac{p}{2}\right)\right|$$

$\Rightarrow$ single magnon $\equiv$ classical soliton ???

original giant magnon of HM

outstanding puzzles.....

Planar $\mathcal{N}=4$ SYM $J \to \infty$ limit

Exact results valid $\forall \lambda = g^2 N$

Spectrum

- magnons
- ∞ tower of boundstates

universal dispersion relation, ND

$$\Delta - J = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2(p/2)}$$

$Q = \#$ of constituents

$Q=1$: Beisert, Dippel + Staudacher
Beisert

S-matrix Beisert

Almost uniquely determined.....