DYON SPECTRUM IN GENERIC $\mathcal{N}=4$ SUPERSYMMETRIC $\mathbb{Z}_N$ ORBIFOLDS

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hep-th/0609109

- DYONS FROM D1-D5 SYSTEM
- SUMMARY OF RESULTS
- (4,4) SCFT & MODULAR FORMS
- DYON SPECTRUM FROM PARTON COUNTING

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DYONS FROM D1-D5 SYSTEM

- TYPE IIB STRING THEORY COMPACTIFIED ON $\mathcal{M} \times S^1 \times \tilde{S}^1$ ($\mathcal{M}$ = K3 or T^4)
- MOD OUT BY Z_N SYMMETRY GENERATED BY g.

  g: $1/N$ UNIT OF TRANSLATION ALONG S^1
  & ORDER N TRANSFORMATION $\tilde{g}$ ON $\mathcal{M}$.

- $\tilde{g}$ PRESERVES $\mathcal{N}=4$ SUSY (1/2 SUSY BROKEN FOR T^4)

- **CONFIGURATION:**
  - SINGLE D5-BRANE WRAPPING $(\mathcal{M} \times S^1)/Z_N$
  - Q_1 D1-BRANES WRAPPING S^1/Z_N
  - SINGLE KK MONOPOLE ASSOCIATED WITH $\tilde{S}^1$
  - $-n/N$ UNITS OF MOMENTUM ALONG S^1/Z_N
  - J UNITS OF MOMENTUM ALONG $\tilde{S}^1$

SINGLE NS5 WRAPPING $\frac{(M \times S^1)}{Z_N}$

Q_1 F1 WRAPPING S^1/Z_N

SINGLE KK MONOPOLE ($\tilde{S}^1$)

$-n/N$ UNITS OF MOMENTUM S^1/Z_N

J UNITS OF MOMENTUM $\tilde{S}^1$

- S-DUALITY TRANSFORMS THIS CONFIGURATION TO

- S-DUALITY FOLLOWED BY T-DUALITY ALONG $\tilde{S}^1$ GIVES US

SINGLE KK MONOPOLE ALONG $\hat{S}^1$

Q_1 F1 BRANES WRAPPING S^1/Z_N

SINGLE NS5 WRAPPING $(M \times S^1)/Z_N$

$-n/N$ UNITS OF MOMENTUM S^1/Z_N

J F1 BRANES WRAPPING $\hat{S}^1$

$\hat{S}^1$: CIRCLE DUAL TO $\tilde{S}^1$

SINGLE KK MONOPOLE ALONG $\hat{S}^1$

Q_1 NS5 WRAPPING $(\mathcal{M} \times S^1)/\mathbb{Z}_N$

SINGLE F1 WRAPPING $S^1/\mathbb{Z}_N$

$-n/N$ UNITS OF MOMENTUM $S^1/\mathbb{Z}_N$

J NS5 WRAPPING $\mathcal{M}/\mathbb{Z}_N \times \hat{S}^1$

- S-DUALITY + T-DUALITY ALONG $\tilde{S}^1$ + 6D STRING-STRING DUALITY $\left[Q_e^2 = \frac{2n}{N},\ Q_m^2 = 2(Q_1 - \beta),\ Q_e \cdot Q_m = J \right]$

GIVES US • HETEROTIC STRING ON AN ASYMMETRIC $\mathbb{Z}_N$ ORBIFOLD $(T^4 \times S^1)/\mathbb{Z}_N \times \hat{S}^1$ FOR $\mathcal{M} = K3$ OR • TYPE IIA STRING ON AN ASYMMETRIC $\mathbb{Z}_N$ ORBIFOLD $(T^4 \times S^1)/\mathbb{Z}_N \times \hat{S}^1$ FOR $\mathcal{M} = T^4$

- 1/4 BPS SUPERMULTIPLET IN $\mathcal{N}=4$ THEORY IS 64 DIMENSIONAL.
- IT CONTAINS EQUAL NO. OF BOSONIC & FERMIONIC COMPONENTS.
- WE WILL CALL A MULTIPLET BOSONIC (FERMIONIC) IF ITS FIRST COMPONENT IS BOSONIC (FERMIONIC).
- S-DUALITY IN ASYMMETRIC ORBIFOLD DESCRIPTION $(\Gamma_1(N))$

$$\begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Q_e \\ Q_m \end{pmatrix} \qquad \begin{array}{l} ad-bc=1 \\ a,d = 1 \bmod N \quad c = 0 \bmod N \end{array}$$

- IN TYPE IIB DESCRIPTION, S-DUALITY BECOMES DIFFEOMORPHISM SYMMETRY OF $S^1 \times \tilde{S}^1$ WHICH PRESERVES $2\pi/N$ TRANSLATION ALONG S^1.
- THE DEGENERACY FORMULA WILL BE WRITTEN IN TERMS OF T-DUALITY INVARIANTS Q_e^2, Q_m^2 & $Q_e \cdot Q_m$

SUMMARY OF RESULTS

- LET $d(Q_e, Q_m)$ BE No. OF BOSONIC MINUS FERMIONIC 1/4 BPS SUPERMULTIPLETS WITH CHARGE (Q_e, Q_m), THEN

$$d(Q_e,Q_m) = \frac{1}{N}\int_{C} d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v}\; \frac{\exp\left[-i\pi\left(N\tilde{\rho}\,Q_e^2 + \frac{\tilde{\sigma}}{N}Q_m^2 + 2\tilde{v}\,Q_e\cdot Q_m\right)\right]}{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}$$

WHERE INTEGRATION CONTOUR IS $0\le \mathrm{Re}(\tilde{\rho})\le 1$, $0\le \mathrm{Re}(\tilde{\sigma})\le N$, $0\le \mathrm{Re}(\tilde{v})\le 1$. & $\mathrm{Im}\,\tilde{\rho}$, $\mathrm{Im}\,\tilde{\sigma}$ & $\mathrm{Im}\,\tilde{v}$ ARE LARGE POSITIVE NUMBERS.

$\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})$ IS GENUS 2 MODULAR FORM

$$\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v}) = e^{2\pi i(\tilde{\alpha}\tilde{\rho}+\tilde{\gamma}\tilde{\sigma}+\tilde{v})} \times \prod_{b=0}\prod_{r=0}^{N-1}\prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z},\, j\in 2\mathbb{Z}+b \\ k',l\ge 0,\ j<0 \text{ for } k'=l=0}} \left(1-\exp\left[2\pi i(k'\tilde{\sigma}+l\tilde{\rho}+j\tilde{v})\right]\right)^{\sum_{s=0}^{N-1} e^{-\frac{2\pi i s l}{N}} c_b^{(r,s)}(4k'l-j^2)}$$

$\tilde{\alpha}$, $\tilde{\gamma}$ AND $c_b^{(r,s)}(u)$ ARE DEFINED AS FOLLOWS

• CONSIDER 2-DIM. (4,4) SUSY σ-MODEL WITH TARGET SPACE $\mathcal{M}/\tilde{\mathbb{Z}}_N$

• DEFINE TWISTED ELLIPTIC GENUS (BEFORE $\tilde{g}$ PROJECTION)

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{N} Tr_{RR,\tilde{g}^r}\left(\tilde{g}^s (-1)^{F_L+F_R} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} e^{2\pi i F_L z}\right).$$

F_L, F_R ARE WORLD SHEET FERMION NO. CORRESPONDING TO LEFT, RIGHT CHIRAL FERMIONS.

L_0 & $\bar{L}_0$ ARE NORMALIZED SO THAT RR GROUND STATES HAVE $L_0 = \bar{L}_0 = 0$.

$F^{(r,s)}(\tau,z)$ CAN BE WRITTEN AS

$$F^{(r,s)}(\tau,z) = \sum_{b=0}^{1} \sum_{\substack{j\in 2\mathbb{Z}+b \\ n\in \mathbb{Z}/N \\ 4n-j^2\geq -b^2}} C_b^{(r,s)}(4n-j^2)\, e^{2\pi i\tau n+2\pi i z j}$$

• DEFINE

$$Q_{r,s} = N\left(C_0^{(r,s)}(0) + 2\,C_1^{(r,s)}(-1)\right)$$

THEN

$$\tilde{\alpha} = \frac{1}{24N} Q_{0,0} - \frac{1}{2N}\sum_{s=1}^{N-1} Q_{0,s}\frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2}$$

$$\tilde{\gamma} = \frac{1}{24N} Q_{0,0}$$

$$d(Q_e, Q_m) = \frac{1}{N}\int_C d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v}\; e^{-i\pi(N\tilde{\rho}Q_e^2 + \frac{\tilde{\sigma}}{N}Q_m^2 + 2\tilde{v}Q_e\cdot Q_m)} \times \frac{1}{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}$$

BEHAVIOUR OF $d(Q_e, Q_m)$ FOR LARGE CHARGES CAN BE DETERMINED BY INTEGRATING OVER $\tilde{v}$ BY PICKING UP RESIDUES AT POLES OF THE INTEGRAND, AND INTEGRATING OVER $\tilde{\rho}$ & $\tilde{\sigma}$ IN THE SADDLE POINT APPROXIMATION.

THE STATISTICAL ENTROPY FUNCTION DEFINED USING $d(Q_e, Q_m)$ IS

$$-\tilde{\Gamma}_B(\tau) = \frac{\pi}{2\tau_2}|Q_e + \tau Q_m|^2 - \ln g(\tau) - \ln g(-\bar{\tau}) - (k+2)\ln 2\tau_2 + \text{const.} + O(Q^{-2})$$

ENTROPY OBTAINED BY EXTREMISING $\tilde{\Gamma}_B$ AGREES WITH THE BLACK HOLE ENTROPY UP TO FIRST SUBLEADING ORDER.

(4,4) SCFT AND MODULAR FORMS

- CONSIDER A MANIFOLD $\mathcal{M}$ (K3 OR T^4) AND AN ORDER N DISCRETE SYMMETRY $\tilde{\mathbb{Z}}_N$ GENERATED BY $\tilde{g}$, SATISFYING FOLLOWING PROPERTIES
  - CHOOSE COMPLEX STRUCTURE ON $\mathcal{M}$ SUCH THAT $\tilde{g}$ PRESERVES (2,0) AND (0,2) FORMS & $\mathcal{M}/\tilde{\mathbb{Z}}_N$ HAS SU(2) HOLONOMY.
  - (4,4) SCFT WITH $\mathcal{M}$ AS ITS TARGET SPACE HAS $SU(2)_L \times SU(2)_R$ R-SYMMETRY & $\tilde{g}$ PRESERVES BOTH R-SYMMETRY & (4,4) SC.
- $\mathcal{M}$ HAS 3 SELF-DUAL AND P ANTI-SELFDUAL 2-FORMS. ($P=3: T^4$; $P=19: K3$)
- DEFINE

$$F^{(r,s)}(\tau,z) \equiv \frac{1}{N} \mathrm{Tr}_{RR,\tilde{g}^r}\left(\tilde{g}^s (-1)^{F_L+F_R} e^{2\pi i \tau L_0} \cdot e^{-2\pi i \bar{\tau} \bar{L}_0} e^{2\pi i z F_L}\right) \quad 0 \leq r,s \leq N-1$$

$F_{L(R)}$ ARE WORLDSHEET LEFT (RIGHT) FERMION NO. OR $2\times$ GENERATORS OF $U(1)_L$ ($U(1)_R$) SUBGROUPS OF $SU(2)_L \times SU(2)_R$ R-SYMMETRY.

- DUE TO $(-1)^{F_R}$ INSERTION, $F^{(r,s)}$ IS INDEPENDENT OF $\bar{\tau}$.

• $SU(2)_L$ CURRENT ALGEBRA SYMMETRY AND $e^{2\pi i F_L z}$ INSERTION IMPLIES $F^{(r,s)}$ CAN BE WRITTEN IN TERMS OF CHARACTERS OF $SU(2)_L$ CURRENT ALGEBRA AT LEVEL l, i.e. IN TERMS OF $\theta_3(2\tau, 2z)$ & $\theta_2(2\tau, 2z)$.

$$F^{(r,s)}(\tau, z) = h_0^{(r,s)}(\tau)\theta_3(2\tau, 2z) + h_1^{(r,s)}(\tau)\,\theta_2(2\tau, 2z)$$

WHERE

$$h_b^{(r,s)}(\tau) = \sum_{n \in \frac{\mathbb{Z}}{N} - \frac{b^2}{4}} c_b^{(r,s)}(4n)\, e^{2\pi i n \tau}$$

$$\theta_3(2\tau, 2z) = \sum_{j \in 2\mathbb{Z}} e^{2\pi i j z} e^{i\pi\tau j^2/2}, \quad \theta_2(2\tau, 2z) = \sum_{j \in 2\mathbb{Z}+1} e^{2\pi i j z} e^{i\pi\tau j^2/2}$$

• $$F^{(r,s)}(\tau, z) = \sum_{b=0}^{1} \sum_{\substack{j \in 2\mathbb{Z}+b \\ n \in \mathbb{Z}/N}} c_b^{(r,s)}(4n - j^2)\, e^{2\pi i n \tau + 2\pi i j z}$$

• SINCE $L_0 \geqslant 0$ IN RR SECTOR $\Rightarrow$ $c_0^{(r,s)}(u) = 0$ for $u < 0$ $\quad c_1^{(r,s)}(u) = 0$ for $u < -1$.

• THINK OF $F^{(r,s)}(\tau, z)$ AS THE TORUS PARTITION FUNCTION WITH MODULAR PARAMETER τ, $\tilde{g}^s e^{2\pi i F_L z}$ TWIST ALONG b-CYCLE & $\tilde{g}^r$ TWIST ALONG a-CYCLE

- $\tilde{g}^s e^{2\pi i F_L z}$ TWIST ALONG b-CYCLE & $\tilde{g}^r$ TWIST ALONG a-CYCLE.

- SUPPOSE (σ_1, σ_2) DENOTE TORUS COORDINATES WITH UNIT PERIOD,

THEN UNDER $(\sigma_1, \sigma_2) \to (-\sigma_1, -\sigma_2)$: $(r,s) \to (-r,-s)$ & $z \to -z$

$\Rightarrow F^{(r,s)}(\tau, z) = F^{(-r,-s)}(\tau, -z)$

- UNDER $(\sigma_1, \sigma_2) \to (\sigma_1 + \sigma_2, \sigma_2)$: $\tau \to \tau + 1$, $(r,s) \to (r, s+r)$

$\Rightarrow F^{(r,s)}(\tau, z) = F^{(r,s+r)}(\tau+1, z)$

- SINCE r, s ARE DEFINED MODULO $N \Rightarrow F^{(r,s)}(\tau, z) = F^{(r,s)}(\tau+N, z)$

- THIS IMPLIES $n \in \mathbb{Z}/N$

- IN THE UNTWISTED SECTOR, $n=0$ CONTAINS GEOMETRIC DATA

WEIGHTED BY $(-1)^p \tilde{g}^s$ FOR HARMONIC p-FORMS.

- TO CAPTURE THIS INFORMATION, LET US DEFINE

$$Q_{r,s} = Tr_{RR,\tilde{g}^r}\left(\tilde{g}^s (-1)^{F_L+F_R} e^{2\pi i \tau L_0} e^{-2\pi i \bar{\tau} \bar{L}_0}\right)$$

$$= N\left(c_0^{(r,s)}(0) + 2\, c_1^{(r,s)}(-1)\right)$$

$\because (-1)^{F_L+F_R} \Rightarrow Q_{r,s}$ IS INDEPENDENT OF τ & $\bar{\tau}$.

- WE CAN USE THIS INFORMATION & EVALUATE THE THRESHOLD INTEGRAL

$$\tilde{I}(\rho,\sigma,v) = \sum_{r,s=0}^{N-1} \sum_{b=0}^{1} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z} \\ n_1\in\mathbb{Z}+\frac{r}{N},\, j\in 2\mathbb{Z}+b}} q^{p_L^2/2}\, \bar{q}^{p_R^2/2}\, e^{2\pi i m_1 s/N}\, h_b^{(r,s)}(\tau)$$

$$q = e^{2\pi i\tau}, \quad \tfrac{1}{2}p_L^2 = \tfrac{1}{2}p_R^2 + m_1n_1 + m_2n_2 + \tfrac{1}{4}j^2$$

$$\tfrac{1}{2}p_R^2 = \frac{1}{4\det \mathrm{Im}\,\Omega}\left|-m_1\rho + m_2 + n_1\sigma + n_2(\rho\sigma - v^2) + jv\right|^2 \qquad \Omega = \begin{pmatrix}\rho & v \\ v & \sigma\end{pmatrix}$$

FINAL RESULT IS

$$\tilde{I}(\rho,\sigma,v) = -2\ln\left[(\det \mathrm{Im}\,\Omega)^k\right] - 2\ln\tilde{\Phi}(\rho,\sigma,v) - 2\ln\bar{\tilde{\Phi}}(\rho,\sigma,v) + \text{CONST.}$$

WHERE

$$\tilde{\Phi}(\rho,\sigma,v) = e^{2\pi i(\tilde{\alpha}\rho + \tilde{\gamma}\sigma + v)}$$

$$\times \prod_{b=0}^{1} \prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z} \\ j\in 2\mathbb{Z}+b \\ k',l\geq 0,\, j<0 \text{ for } k'=l=0}} \left(1 - e^{2\pi i(k'\sigma + l\rho + jv)}\right)^{\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4k'l - j^2)}$$

$$\tilde{\alpha} = \frac{1}{24N} Q_{0,0} - \frac{1}{2N}\sum_{s=1}^{N-1} Q_{0,s} \frac{e^{-2\pi i s/N}}{(1-e^{-2\pi i s/N})^2}, \quad \tilde{\gamma} = \frac{1}{24N} Q_{0,0}$$

DYON SPECTRUM FROM PARTON COUNTING

- WE WILL WORK IN TYPE IIB DESCRIPTION.
- IN THIS FRAME, THE CHARGES (Q_e, Q_m) ARE LABELLED BY Q_1, n, & J.
- OTHER CHARGES CORRESPONDING TO D5 & KK MONOPOLE ARE SET TO 1
- LET $h(Q_1, n, J)$ DENOTE THE NO. BOSONIC MINUS FERMIONIC MULTIPLETS WITH QUANTUM NUMBERS (Q_1, n, J).
- IN THE TYPE IIB FRAME, $h(Q_1, n, J)$ CAN BE COMPUTED IN THE WEAK COUPLING LIMIT.
- IN THE WEAK COUPLING LIMIT, WE CAN TREAT THIS SYSTEM TO BE MADE UP OF 3 INDEPENDENT SUBUNITS.

1) EXCITATIONS OF KK MONOPOLE WITH MOMENTUM $-l_0'$ ALONG S^1
2) OVERALL MOTION OF D1-D5 WITH MOMENTUM $-l_0$ ALONG S^1 & j_0 ALONG $\tilde{S}^1$
3) MOTION OF D1 IN THE PLANE OF D5 WITH MOMENTUM $-L$ ALONG S^1 AND MOMENTUM J' ALONG $\tilde{S}^1$. $\left[l_0' + l_0 + L = n\,,\ j_0 + J' = J \right]$

• DENOTE

$$f(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \sum_{Q_1,n,J} h(Q_1,n,J)\, e^{2\pi i(\tilde{\rho} n + \tilde{\sigma} Q_1/N + \tilde{v} J)}$$

• USING 3 SUBSYSTEMS, WE CAN WRITE

$$f(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \frac{1}{64}\left(\sum_{Q_1,L,J'} d_{D1}(Q_1,L,J')\, e^{2\pi i(\tilde{\sigma} Q_1/N + \tilde{\rho} L + \tilde{v} J')}\right)$$

$$\times\left(\sum_{l_0,j_0} d_{CM}(l_0,j_0)\, e^{2\pi i l_0 \tilde{\rho} + 2\pi i j_0 \tilde{v}}\right)\left(\sum_{l_0'} d_{KK}(l_0')\, e^{2\pi i l_0' \tilde{\rho}}\right)$$

• RECALL THE DISCRETE SYMMETRY GENERATED BY $\tilde{g}$ PRESERVES ALL SUPERSYMMETRIES OF IIB COMPACTIFIED ON K3.

• KK MONOPOLE IN IIB STRING THEORY COMPACTIFIED ON $K3\times S^1\times\tilde{S}^1$ BREAKS 8 OF THE 16 SUPERSYMMETRIES.

• QUANTIZATION OF FERMION ZERO MODE COMING FROM BROKEN SUSY GIVES 16-FOLD DEGENERACY.

- D1-D5 SYSTEM IN THE BACKGROUND OF KK MONOPOLE BREAKS

4 OF THE REMAINING 8 SUSYs. THIS GIVES 4-FOLD DEGENERACY.

- COUNTING STATES OF KK MONOPOLE:
  - WORLD VOLUME OF KK MONOPOLE IS COMPACTIFIED ON $\mathcal{M} \times S^1$
  - TAKING $\mathcal{M}$ SMALL GIVES 1+1 D THEORY WITH CHIRAL SUSY ($\because$ IIB ON K3 IS CHIRAL)
  - THUS 1/2 BPS STATES OF KK MONOPOLE $\Rightarrow$ RIGHT MOVING OSCILLATORS OF 2D THEORY ARE IN THE GROUND STATE.
  - LOW ENERGY DEGREES OF FREEDOM OF 2D THEORY:
    - 3 NON-CHIRAL MASSLESS BOSONS CORRESPONDING TO 3 TRANSVERSE DIR$^{\underline{NS}}$
    - 2 NON-CHIRAL BOSONS FROM $B^{NS}_{\mu\nu}$, $B^{R}_{\mu\nu}$ REDUCED ALONG THE HARMONIC 2-FORM OF TAUB-NUT.
    - SELF-DUAL 4-FORM OF IIB REDUCED ALONG THE HARMONIC 2-FORMS OF TAUB-NUT & OF $\mathcal{M}$ GIVE RISE TO CHIRAL SCALARS. 3 RIGHT MOVING AND P LEFT MOVING.

      ($P=3$ FOR T^4, $P=19$ FOR K3)

- IIB ON K3 HAS 16 CHIRAL SUSYs. OF THESE 8 ARE BROKEN BY TAUB-NUT.
- 8 FERMIONIC ZERO MODES DUE TO BROKEN SUSY ARE CHIRAL & RIGHT-MOVERS.
- IIB ON T^4 HAS 32 NON-CHIRAL SUSYs. TAUB-NUT BREAKS 16 OF THEM.
- 2D THEORY THUS HAS 8 LEFT MOVING & 8 RIGHT-MOVING FERMION ZERO-MODES.
- SINCE $\tilde{g}$ PRESERVES SUSY OF IIB ON K3, ALL RIGHT-MOVING FERMIONS ARE NEUTRAL UNDER $\tilde{g}$. SAME IS TRUE FOR 8 RIGHT-MOVING BOSONS.
- ACTION OF $\tilde{g}$ ON P+5 LEFT-MOVING BOSONS IS DEDUCED FROM ACTION OF $\tilde{g}$ ON P+5 EVEN DEGREE HARMONIC FORMS ON $\mathcal{M}$.
- ACTION OF $\tilde{g}$ ON LEFT-MOVING FERMIONS IS GIVEN BY ACTION OF $\tilde{g}$ ON HARMONIC 1 AND 3 FORMS ON $\mathcal{M}$.
- ALTERNATIVELY, IN (4,4) SCFT DEFINE

$$Q_{0,s} = \mathrm{Tr}_{RR}\left((-1)^{F_L+F_R}\, \tilde{g}^s\, e^{2\pi i \tau L_0}\, e^{-2\pi i \bar{\tau} \bar{L}_0}\right)$$

= # OF LEFT MOVING BOSONS WEIGHTED BY $\tilde{g}^s$
− # OF —"— "— FERMIONS —"— "—

• DEFINE n_ℓ = # OF LEFT-MOVING BOSONS − FERMIONS WITH $\tilde{g}$ QUANTUM # $e^{2\pi i \ell/N}$

THEN $$n_\ell = \frac{1}{N}\sum_{s=0}^{N-1} e^{-2\pi i s\ell/N} Q_{0,s}$$

• DISTRIBUTION OF MOMENTUM $-\ell_0'$ AMONG THESE MODES IS GIVEN BY

$$\sum_{\ell_0'} d_{KK}(\ell_0')\, e^{2\pi i \tilde{\rho} \ell_0'} = 16\, e^{2\pi i C \tilde{\rho}} \prod_{\ell=1}^{\infty} \left(1 - e^{2\pi i \ell \tilde{\rho}}\right)^{-n_\ell}$$

• CONSTANT C CAN BE DETERMINED BY NOTING THAT KK MONOPOLE BECOMES TWISTED FUNDAMENTAL (HET/IIA) STRING IN ASYM. ORB.

• C = ZERO POINT ENERGY OF TWISTED FUNDAMENTAL STRING.

$$C = -\frac{N}{24}\sum_{\ell=0}^{N-1} n_\ell + \frac{N}{4}\sum_{\ell=0}^{N-1} n_\ell \frac{\ell}{N}\left(1-\frac{\ell}{N}\right) = -\frac{Q_{0,0}}{24N} + \frac{1}{2N}\sum_{s=1}^{N-1} Q_{0,s}\frac{e^{-2\pi i s/N}}{\left(1-e^{-2\pi i s/N}\right)^2} = -\tilde{\alpha}$$

• THUS

$$\sum_{\ell_0'} d_{KK}(\ell_0')\, e^{2\pi i \tilde{\rho} \ell_0'} = 16\, e^{-2\pi i \tilde{\alpha} \tilde{\rho}} \prod_{\ell=1}^{\infty} \left(1 - e^{2\pi i \ell \tilde{\rho}}\right)^{-\sum_{s=0}^{N-1} e^{-2\pi i s \ell/N}\left(c_0^{(0,s)}(0) + 2 c_1^{(0,s)}(-1)\right)}$$

- LET US LOOK AT OVERALL MOTION OF D1-D5 SYSTEM.
- THIS COMPUTATION WAS CARRIED OUT BY DAVID & SEN FOR K3.
- SINCE MOTION IN TAUB-NUT SPACE IS BLIND TO THE CHOICE OF $\mathcal{M}$, THEIR RESULTS HOLD EVEN FOR $\mathcal{M} = T^4$.

$$\sum_{l_0, j_0} d_{tr}(l_0, j_0)\, e^{2\pi i l_0 \tilde{\rho} + 2\pi i j_0 \tilde{v}} = 4\, e^{-2\pi i \tilde{v}} (1 - e^{-2\pi i \tilde{v}})^2$$

$$\times \prod_{n=1}^{\infty} \Big\{ (1 - e^{2\pi i n N \tilde{\rho}})^4 (1 - e^{2\pi i n N \tilde{\rho} + 2\pi i \tilde{v}})^{-2} (1 - e^{2\pi i n N \tilde{\rho} - 2\pi i \tilde{v}})$$

- AN ADDITIONAL CONTRIBUTION COMES FROM WILSON LINES. (ON T^4)
- WE HAVE 4 WILSON LINES ON T^4.
- UNDER $\tilde{g}$, 2 OF THEM TAKE PHASE $e^{2\pi i/N}$ & OTHER 2 TAKE $e^{-2\pi i/N}$
- Z_N INVARIANCE $\Rightarrow$ THEY CARRY $Nk-1$ & $Nk+1$ UNITS OF S^1 MOMENTUM.
- FERMIONS TRANSFORM IN THE SAME WAY (SUSY)
- BOSONS BEING NEUTRAL UNDER TN ISOMETRY, DO NOT CARRY MOMENTUM ALONG $\tilde{S}^1$.
- FERMIONS TRANSFORM AS SPINORS WITH $\pm 1/2$ UNITS OF ANGULAR MOMENTUM

• COMBINING THESE THINGS TOGETHER GIVES

$$\sum_{l_0,j_0} d_W(l_0,j_0)\, e^{2\pi i l_0 \tilde{\rho} + 2\pi i j_0 \tilde{\vartheta}}$$

$$= \prod_{\substack{l \in N\mathbb{Z}+1 \\ l>0}} \frac{(1-e^{2\pi i l\tilde{\rho}-2\pi i\tilde{\vartheta}})(1-e^{2\pi i l\tilde{\rho}+2\pi i\tilde{\vartheta}})}{(1-e^{2\pi i l\tilde{\rho}})^2} \prod_{\substack{l \in N\mathbb{Z}-1 \\ l>0}} \frac{(1-e^{2\pi i l\tilde{\rho}-2\pi i\tilde{\vartheta}})(1-e^{2\pi i l\tilde{\rho}+2\pi i\tilde{\vartheta}})}{(1-e^{2\pi i l\tilde{\rho}})^2}$$

• CONTRIBUTION FROM OVERALL MOTION IS OBTAINED BY ADDING

d_{tr} AND d_W

$$\sum_{l_0,j_0} d_{CM}(l_0,j_0)\, e^{2\pi i l_0\tilde{\rho}+2\pi i j_0\tilde{\vartheta}} = 4\, e^{-2\pi i\tilde{\vartheta}} \prod_{l=1}^{\infty} (1-e^{2\pi i l\tilde{\rho}})^{2\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_1^{(0,s)}(-1)}$$

$$\times \prod_{l=1}^{\infty} (1-e^{2\pi i l\tilde{\rho}+2\pi i\tilde{\vartheta}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_1^{(0,s)}(-1)}$$

$$\times \prod_{l=0}^{\infty} (1-e^{2\pi i l\tilde{\rho}-2\pi i\tilde{\vartheta}})^{-\sum_{s=0}^{N-1} e^{-2\pi i l s/N} c_1^{(0,s)}(-1)}$$

- RELATIVE MOTION OF D1-D5 SYSTEM

- THIS INVOLVES GENERALIZATION OF A PROCEDURE GIVEN BY DIJKGRAAF ET AL.

$$\sum_{Q_1, L, J'} d_{D1}(Q_1, L, J')\, e^{2\pi i(\tilde{\sigma} Q_1/N + \tilde{\rho} L + \tilde{v} J')}$$

$$= \prod_{\substack{\omega, l, j \in \mathbb{Z} \\ \omega > 0,\, l \geq 0}} \left(1 - e^{2\pi i(\tilde{\sigma}\omega/N + \tilde{\rho} l + \tilde{v} j)}\right)^{-n(\omega, l, j)}$$

WHERE

$$n(\omega, l, j) = \sum_{s=0}^{N-1} e^{-2\pi i l s/N}\, c_b^{(r,s)}(4 l\omega/N - j^2) \qquad \begin{array}{l} r = \omega \bmod N \\ b = 0, 1 \end{array}$$

- FULL PARTITION FUNCTION

- RECALL TOTAL PARTITION FUNCTION FOR 3 SUBSYSTEMS IS

$$f(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \frac{1}{64}\left(\sum_{Q,L,J'} d_{D_1}(Q,L,J')\, e^{2\pi i(\tilde{\sigma}Q_1/N + \tilde{\rho}L + \tilde{v}J')}\right)$$

$$\times\left(\sum_{l_0,j_0} d_{CM}(l_0,j_0)\, e^{2\pi i l_0\tilde{\rho} + 2\pi i j_0\tilde{v}}\right)\times\left(\sum_{l_0'} d_{KK}(l_0')\, e^{2\pi i l_0'\tilde{\rho}}\right)$$

- COMBINING THESE 3 CONTRIBUTIONS TOGETHER GIVES

$$f(\tilde{\rho},\tilde{\sigma},\tilde{v}) = e^{-2\pi i(\tilde{\alpha}\tilde{\rho}+\tilde{v})}\prod_{b=0}^{1}\prod_{r=0}^{N-1}$$

$$\times \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{N},\, l\in\mathbb{Z},\, j\in 2\mathbb{Z}+b \\ k',l\geq 0,\ j<0 \text{ for } k'=l=0}} \left(1 - e^{2\pi i(\tilde{\sigma}k' + \tilde{\rho}l + \tilde{v}j)}\right)^{-\sum_{s=0}^{N-1} e^{-2\pi i s l/N} c_b^{(r,s)}(4lk'-j^2)}$$

- COMPARING WITH $\tilde{\Phi}$ WE GET

$$f(\tilde{\rho},\tilde{\sigma},\tilde{v}) = \frac{e^{2\pi i\tilde{\gamma}\tilde{\sigma}}}{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}$$

THUS

$$h(Q_1,n,J) = \frac{1}{N}\int_C d\tilde{\rho}\, d\tilde{\sigma}\, d\tilde{v}\, e^{-2\pi i(\tilde{\rho}n + \tilde{\sigma}(Q_1-\tilde{\gamma}N)/N + \tilde{v}J)} \times \frac{1}{\tilde{\Phi}(\tilde{\rho},\tilde{\sigma},\tilde{v})}$$