# Chern Simons term in the entropy function formalism

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# Introduction

- In the presence of the gravitational Chern simons term the Lagrangian density cannot be written in a manifestly covariant form and as a result Wald's formalism cannot be applied in a straightforward fashion.
- As a result one cannot apply Sen's entropy function method directly to action containing Chern simons term since we have seen tha Sen's entropy function is just walds formalism applied to extremal black holes
- But one can see that after dimensional reduction and throwing away certain total derivative terms one gets a covariant piece from the Chern simons term (worked out by Jackiew et-al in hep-th/0305117) and then use the entropy function methods to compute the entropy

# Plan of the Talk

- In this talk I wish to present a way to handle Chern simons term using walds Noether Charge method or alternatively Ashoke's entropy function formalism. I wish to present it in two different contexts.
  - BTZ black holes in the presence of higher derivatives and Chern Simons terms. (Work done with Ashoke Sen hep-th/0601228)
  - $\alpha'$-Corrections to Extremal Dyonic Black Holes in Heterotic String Theory(Work done with Ashoke Sen hep-th/0608182)

- BTZ Black holes with higher derivatives and Chern Simons terms
  - BTZ solution describes a black hole in three dimensional theory of gravity with negative cosmological constant and often appears as a factor in the near horizon geometry of higher dimensional black holes in string theory
  - In three dimensions one can also add to the action the gravitational Chern-Simons terms
  - In this case the Lagrangian density cannot be written in a manifestly covariant form and as a result Wald's formalism cannot be applied in a straightforward fashion

- But one can use Walds Noether charge method to compute the entropy of BTZ black holes in the presence of Chern-Simons and higher derivative terms
- In order to do this we regard the BTZ black hole as a two dimensional configuration by treating the angular coordinate as a compact direction
- The black hole entropy is then calculated using the dimensionally reduced two dimensional theory.
- This has the advantage that the Chern-Simons term, which was not manifestly covariant in three dimensions, becomes manifestly covariant in two dimensions

# The two dimensional view

- Let us consider a three dimensional theory of gravity with metric $G_{MN}$ $(0 \leq M, N \leq 2)$ and a general action of the form:

$$S = \int d^3x \sqrt{-\det G} \left[ \mathcal{L}_0^{(3)} + \mathcal{L}_1^{(3)} \right] . \qquad (1)$$

  - $\mathcal{L}_0^{(3)}$ denotes an arbitrary scalar constructed out of the metric, the Riemann tensor and covariant derivatives of the Riemann tensor
  - $\sqrt{-\det G}\, \mathcal{L}_1^{(3)}$ denotes the gravitational Chern-Simons term:

$$\sqrt{-\det G}\, \mathcal{L}_1^{(3)} = K\, \Omega_3(\widehat{\Gamma}) , \qquad (2)$$

  where $K$ is a constant

- We consider dimensional reduction along one of the direction say y
- In this case we can define two dimensional fields through the relation:

$$G_{MN}dx^M dx^N = \phi\left[g_{\mu\nu}dx^\mu dx^\nu + (dy + A_\mu dx^\mu)^2\right]\,. \qquad (3)$$

- Here $g_{\mu\nu}$ ($0 \le \mu, \nu \le 1$) denotes a two dimensional metric, $A_\mu$ denotes a two dimensional gauge field and $\phi$ denotes a two dimensional scalar field.
- In terms of these two dimensional fields the action takes the form:

$$S = \int d^2x \sqrt{-\det g}\left[\mathcal{L}_0^{(2)} + \mathcal{L}_1^{(2)}\right] \qquad (4)$$

- where
  - $$\sqrt{-\det g}\,\mathcal{L}_0^{(2)} = \int dy \sqrt{-\det G}\,\mathcal{L}_0^{(3)} = 2\pi \sqrt{-\det G}\,\mathcal{L}_0^{(3)}\,, \tag{5}$$
- and
  - $$\sqrt{-\det g}\,\mathcal{L}_1^{(2)} = K\,\pi\,\left[\frac{1}{2} R \varepsilon^{\mu\nu} F_{\mu\nu} + \frac{1}{2}\varepsilon^{\mu\nu} F_{\mu\tau} F^{\tau\sigma} F_{\sigma\nu}\right]\,. \tag{6}$$
- Here $R$ is the scalar curvature of the two dimensional metric $g_{\mu\nu}$:

- A general BTZ black hole in the three dimensional theory is described by the metric:

$$G_{MN}dx^M dx^N = -\frac{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}{l^2\rho^2} d\tau^2 \qquad (7)$$

$$+ \frac{l^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)} d\rho^2 + \rho^2 \left(dy - \frac{\rho_+\rho_-}{l\rho^2} d\tau\right)^2$$

  - where $l$, $\rho_+$ and $\rho_-$ are parameters labelling the solution.
- Extremal Black holes are defined by taking $\rho_- = \pm\rho_+$ and the near horizon limit is obtained by taking $\rho$ close to $\rho_+$.

- Defining
  - $$r = \rho - \rho_+, \qquad t = \frac{4}{l^2}\,\tau\,, \tag{8}$$
- we can reexpress the metric for $\rho_- = \pm\rho_+$ and small $r$ as
  - $$G_{MN}dx^M dx^N = \frac{l^2}{4}\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) \tag{9}$$

$$+\rho_+^2\,\left(dy \pm \left(-\tfrac{l}{4} + \tfrac{l}{2\rho_+}\,r\right)\,dt\right)^2$$

- $$\phi = \rho_+^2\,, \qquad A_\mu dx^\mu = \pm\left(-\frac{l}{4} + \frac{l}{2\rho_+}\,r\right)\,dt, \tag{10}$$

  $g_{\mu\nu}dx^\mu dx^\nu = \frac{l^2}{4\rho_+^2}\,\left(-r^2dt^2 + \frac{dr^2}{r^2}\right)$ . are the scalar fields, gauge fields and metric from the two dimensional point of view

- Let

  - $$u = \rho_+^2, \qquad v = \frac{l^2}{4\rho_+^2}, \qquad e = \pm\frac{l}{2\rho_+}\,. \tag{11}$$

  be the near horizon values of the scalar fields, size of the Ads space and the near horizon values of the electric fields respectively

- we now have two independent parameters $l$ and $\rho^+$ labelling the near horizon geometry.
- In particular $v$ and $e$ satisfy the relation
  - $$v = e^2 \, . \tag{12}$$
- We shall choose $e$ and

$$l = 2\sqrt{ue^2} \, , \tag{13}$$

as independent variables

- Since the BTZ black hole is locally the maximally symmetric $AdS_3$ space, $\mathcal{L}_0^{(3)}$, being a scalar constructed out of the Riemann tensor and its covariant derivatives, must be a constant. Furthermore since locally BTZ metrics for different values of $\rho_\pm$ are related by a coordinate transformation, $\mathcal{L}_0^{(3)}$ must be independent of $\rho_\pm$ and hence is a function of $l$ only.
- Let us define
  -

$$h(l) = \mathcal{L}_0^{(3)} \tag{14}$$

evaluated in the BTZ black hole geometry.

- Since
  - $$\sqrt{-\det G} = \frac{l^2 \rho_+}{4} = \frac{l^3}{8\,|e|}\,, \tag{15}$$

  we get for the entropy function
- - $$\mathcal{E} = 2\pi \left( q\, e - \frac{1}{|e|} g(l) - \frac{\pi\, K}{e} \right)\,. \tag{16}$$
- where
  - $$g(l) = \frac{\pi\, l^3\, h(l)}{4}\,. \tag{17}$$

- Extremization wrt e and l gives
  - $$e = \sqrt{\frac{\pi(C-K)}{q}} \quad \text{for } q > 0\,, \qquad = \sqrt{\frac{\pi(C+K)}{|q|}} \quad \text{for } q < 0\,. \tag{18}$$
- Furthermore, at the extremum,
  - $$\mathcal{E} = 2\pi\sqrt{\frac{c_R\, q}{6}} \quad \text{for } q > 0\,, \qquad = 2\pi\sqrt{\frac{c_L\, |q|}{6}} \quad \text{for } q < 0\,, \tag{19}$$

- where we have defined
  - $$c_L = 24\,\pi\,(C + K)\,, \qquad c_R = 24\,\pi\,(C - K)\,. \tag{20}$$
- where
  - $$C = -\frac{1}{\pi} g(l) \tag{21}$$

    at the extremum of $g$
- Note that $c_L - c_R = 48\,\pi\,K$ , is determined completely by the coeefieciernts of the chern simons term

- $\alpha'$-Corrections to Extremal Dyonic Black Holes in Heterotic String Theory
- String theory at low energy describes Einstein gravity coupled to certain matter fields, together with infinite number of higher derivative corrections.
- Thus study of black holes in string theory involves study of black holes in higher derivative theories of gravity.
- most of the analysis so far has been done by taking into account only a subset of these corrections, *e.g.* by including only the terms in the action proportional to Gauss-Bonnet term , or by including the set of all terms which are related to the curvature squared terms by supersymmetry transformation.

- Later Krauss and Larsen in hepth-0506176,hepth-0508218 proved certain non-renormalization theorems establishing that for a certain class of supersymmetric black holes the results of the above works are in fact exact
- The underlying assumption behind this proof is the existence of an $AdS_3$ component of the near horizon geometry of the black hole solution when embedded in the full ten dimensional space-time, and supersymmetry of the resulting two dimensional theory that lives on the boundary of this $AdS_3$.
- Notwithstanding these non-renormalization theorems, it is important to verify the result by a direct calculation that takes into account all the higher derivative corrections in a given order

- An attempt in this direction was made by G. Exirifard in hep-th 0607094 where the author tried to include all the tree level four derivative corrections to the action of heterotic string theory compactified on a six dimensional torus $T^6$, and used this to compute correction to the entropy of an extremal dyonic black hole
- The apparent conclusion of this paper was that the entropy computed this way disagrees with the earlier results thus contradicting Krauss and Larsens non-renormalization theorems
- A closer look however reveals that the analysis of hep-th 0607094 left out one important term, – the coupling of the gravitational Chern-Simons term to the 3-form field strength.

- We recalculate the entropy of a dyonic black hole in tree level heterotic string theory by including the complete set of tree level four derivative terms in the heterotic string effective action. We find that after the effect of gravitational Chern-Simons term is included, the resulting entropy agrees perfectly with the results of earlier analysis, in accordance with the non-renormalization theorems of Krauss and Larsen in hep-th0506176,0508218

- We begin with the low energy effective field theory of ten dimensional heterotic string theory compactified on $T^4$ or $K3$
- we ignore all the ten dimensional gauge fields and the massless fields associated with the components of the metric and the anti-symmetric tensor fields along the compact space $T^4$ or $K3$
- So the remaining massless fields consist of the string metric $G^{(6)}_{MN}$, the anti-symmetric tensor field $B^{(6)}_{MN}$ and the dilaton field $\Phi^{(6)}$ with $0 \leq M, N \leq 5$

- The gauge invariant field strength associated with the anti-symmetric tensor field is given by:
  - $$H^{(6)}_{MNP} = \partial_M B^{(6)}_{NP} + \partial_N B^{(6)}_{PM} + \partial_P B^{(6)}_{MN} + \lambda\, \Omega^{(6)}_{MNP}\,. \quad (22)$$
- where
  - $\lambda$ is a coefficient to be specified later and
  - $\Omega^{(6)}_{MNP}$ denotes the gravitational Chern-Simons 3-form
- We shall denote the action of this theory as
  - $$S = \int d^6x\, \sqrt{-\det G^{(6)}}\, \mathcal{L}^{(6)} \quad (23)$$

- where the Lagrangian density $\mathcal{L}^{(6)}$ is a function of $G^{(6)}_{MN}$, the Riemann tensor $R^{(6)}_{MNPQ}$, $H^{(6)}_{MNP}$, $\Phi^{(6)}$ and covariant derivatives of these fields
- In order to bring the Chern Simons term into a form well known to handle we work in the dual field strength $K^{(6)}_{MNP}$ instead of $H^{(6)}_{MNP}$
- The Bianchi identity of $H^{(6)}_{MNP}$ becomes the equation of motion of $K^{(6)}_{MNP}$ and vice versa

- The algorithm to achieve the above purpose is to introduce $C^{(6)}_{MN}$ and its field strength
  - $$K^{(6)}_{MNP} = \partial_M C^{(6)}_{NP} + \partial_N C^{(6)}_{PM} + \partial_P C^{(6)}_{MN} \,. \qquad (24)$$
- and consider Lagrangian density
  - $$\sqrt{-\det G^{(6)}}\,\tilde{\mathcal{L}}^{(6)} \equiv \sqrt{-\det G^{(6)}}\,\mathcal{L}^{(6)} \qquad (25)$$

$$+\frac{1}{16\pi^2}\frac{1}{(3!)^2}\,\epsilon^{MNPQRS}K^{(6)}_{MNP}H^{(6)}_{QRS}$$
$$-\frac{1}{16\pi^2}\frac{1}{(3!)^2}\,\lambda\,\epsilon^{MNPQRS}K^{(6)}_{MNP}\Omega^{(6)}_{QRS}$$

where we treat $H^{(6)}_{MNP}$ and $C^{(6)}_{MN}$ as independent variables

- We now dimensionally reduce this theory to four dimensions by introducing the fields $G_{\mu\nu}$, $C_{\mu\nu}$, $\Phi$, $\widehat{G}_{mn}$, $\widehat{C}_{mn}$ and $\mathcal{A}_\mu^{(i)}$ ($0 \leq \mu \leq 3$, $4 \leq m, n \leq 5$, $1 \leq i \leq 4$) via the relations
  - $$\widehat{G}_{mn} = G^{(6)}_{mn}, \quad \widehat{C}_{mn} = C^{(6)}_{mn},$$
    $$\mathcal{A}_\mu^{(m-3)} = \frac{1}{2}\widehat{G}^{mn}G^{(6)}_{m\mu}, \quad \mathcal{A}_\mu^{(m-1)} = \frac{1}{2}C^{(6)}_{m\mu} - \widehat{C}_{mn}\mathcal{A}_\mu^{(n-3)},$$
    $$G_{\mu\nu} = G^{(6)}_{\mu\nu} - \widehat{G}^{mn}G^{(6)}_{m\mu}G^{(6)}_{n\nu},$$
    $$C_{\mu\nu} = C^{(6)}_{\mu\nu} - 4\widehat{C}_{mn}\mathcal{A}_\mu^{(m-3)}\mathcal{A}_\nu^{(n-3)} \qquad (26)$$
    $$-2(\mathcal{A}_\mu^{(m-3)}\mathcal{A}_\nu^{(m-1)} - \mathcal{A}_\nu^{(m-3)}\mathcal{A}_\mu^{(m-1)}),$$
    $$\Phi = \Phi^{(6)} - \frac{1}{2}\ln V_M, \qquad (27)$$

- where $x^4$ and $x^5$ are the coordinates labelling the torus and $V_M$ is the volume of $T^2$ measured in the string metric. We shall normalize $x^4$ and $x^5$ so that they have coordinate radius 1.
- Then
  - $$V_M = 4\pi^2 \sqrt{\det \widehat{G}}\,. \tag{28}$$

# Computation of the Entropy

- we shall be interested in the correction due to the four derivative terms in the action. For this let us split the original action $\mathcal{L}^{(6)}$ as
  - $$\mathcal{L}^{(6)} = \mathcal{L}_0^{(6)} + \mathcal{L}_1^{(6)} \,, \qquad (29)$$
- where $\mathcal{L}_0^{(6)}$ denotes the supergravity action and $\mathcal{L}_1^{(6)}$ denotes four derivative corrections.
- The entropy function obtained from this Lagrangian density has the form:
  - $$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 \,, \qquad (30)$$

- with $\mathcal{E}_0$ and $\mathcal{E}_1$ reflecting the contribution from the two and four derivative terms respectively:
- 

$$\begin{aligned} \mathcal{E}_0 &= 2\pi\Bigg(\sum_{i=1}^{4}\widetilde{q}_i\widetilde{e}_i - \int d\theta d\phi dx^4 dx^5 \,\Big(\sqrt{-\det G^{(6)}}\,\mathcal{L}_0^{(6)} \\ &\quad + \frac{1}{16\pi^2}\frac{1}{(3!)^2}\,\epsilon^{MNPQRS}K^{(6)}_{MNP}H^{(6)}_{QRS}\Big)\Bigg) \end{aligned} \quad (32)$$

- $$\mathcal{E}_1 = 2\pi \Bigg( - \int d\theta d\phi dx^4 dx^5 \sqrt{-\det G^{(6)}}\, \mathcal{L}_1^{(6)}$$

  - $\int d\theta\, d\phi \sqrt{-\det G}\, \widetilde{\mathcal{L}}'' \Bigg)$ .(33)

- Since the entropy is given by the value of $\mathcal{E}$ at its *extremum*, a first order error in the determination of the near horizon background will give a second order error in the value of the entropy. Thus we can find the near horizon background, including the auxiliary field $H^{(6)}_{MNP}$, by extremizing $\mathcal{E}_0$ and then evaluate $\mathcal{E}_0 + \mathcal{E}_1$ in this background. This gives the value of the entropy correctly up to first order.

- We consider an extremal black hole solution in this theory with near horizon configuration:
  - $$ds^2 \equiv G_{\mu\nu}dx^\mu dx^\nu = v_1\left(-r^2dt^2 + \frac{dr^2}{r^2}\right) + v_2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$
    $$\widehat{G} = diag\left(u_1^2, u_2^2\right)\,, \qquad \widehat{C} = 0, \qquad e^{-2\Phi} = u_S\,,$$
    $$\mathcal{F}^{(1)}_{rt} = \widetilde{e}_1, \qquad \mathcal{F}^{(3)}_{rt} = \widetilde{e}_3,$$
    $$\mathcal{F}^{(2)}_{\theta\phi} = \frac{\widetilde{p}_2}{4\pi}\sin\theta\,, \qquad \mathcal{F}^{(4)}_{\theta\phi} = \frac{\widetilde{p}_4}{4\pi}\sin\theta\,. \qquad (34)$$

- One can then put this background in the action and evaluate the entropy function. After eliminating the near horizon electric fields by their equation of motion one gets for the electric fields and the entropy function
  - $$\widetilde{e}_1 = \frac{2v_1\widetilde{q}_1}{v_2 u_S u_1^2}\,, \qquad \widetilde{e}_3 = \frac{v_1 u_S \widetilde{q}_3}{32\pi^2 v_2 u_2^2}\,. \tag{35}$$

    and

    $$\mathcal{E}_0 = \frac{\pi}{4} v_1 v_2 u_S \left[\frac{2}{v_1} - \frac{2}{v_2} + \frac{8\widetilde{q}_1^2}{v_2^2 u_S^2 u_1^2} + \frac{\widetilde{q}_3^2}{8\pi^2 v_2^2 u_2^2} + \frac{u_2^2 \widetilde{p}_2^2}{8\pi^2 v_2^2} + \frac{8u_1^2 \widetilde{p}_4^2}{v_2^2 u_S^2}\right.\,.$$

- In order to compare our charges and the charges used in hep-th/050842 we need to write the original field H in terms of these charges and then by using the relation between near horizon fields and charges in both description, one then gets
  - $$q_1 = \widetilde{q}_1, \quad p_2 = \widetilde{p}_2, \quad q_3 = -\widetilde{p}_4, \quad p_4 = -\widetilde{q}_3 \,. \qquad (38)$$
- The entropy function may now be rewritten as
  - $$\mathcal{E}_0 = \frac{\pi}{4} v_1 v_2 u_S \left[ \frac{2}{v_1} - \frac{2}{v_2} + \frac{8q_1^2}{v_2^2 u_S^2 u_1^2} + \frac{p_4^2}{8\pi^2 v_2^2 u_2^2} + \frac{u_2^2 p_2^2}{8\pi^2 v_2^2} + \frac{8u_1^2 q_3^2}{v_2^2 u_S^2} \right.$$

- Extremizing the entropy function with respect to $v_1$, $v_2$, $u_1$, $u_2$ and $u_S$ we get
  -

$$v_1 = v_2 = \frac{1}{4\pi^2}\,|p_2 p_4|\,, \quad u_S = 8\pi\sqrt{\left|\frac{q_1 q_3}{p_2 p_4}\right|},$$

$$u_1 = \sqrt{\left|\frac{q_1}{q_3}\right|}, \quad u_2 = \sqrt{\left|\frac{p_4}{p_2}\right|}$$

$$\widetilde{e}_1 = \frac{1}{4\pi q_1}\sqrt{|p_2 p_4 q_1 q_3|}, \qquad \widetilde{e}_3 = -\frac{1}{4\pi p_4}\sqrt{|p_2 p_4 q_1 q_3|}. \quad (41)$$

- And the leading order contribution to the black hole entropy:

$$\mathcal{E}_0 = \sqrt{|p_2 p_4 q_1 q_3|}\,. \quad (42)$$

- We now turn to the evaluation of $\mathcal{E}_1$. We shall divide the contribution into two parts:
  - $$\mathcal{E}_1 = \mathcal{E}_1' + \mathcal{E}_1'' \,, \tag{43}$$
- where
  - $$\mathcal{E}_1' = -2\pi \int d\theta d\phi dx^4 dx^5 \sqrt{-\det G^{(6)}}\, \mathcal{L}_1^{(6)} \tag{44}$$

    and

    $$\mathcal{E}_1'' = -\int d\theta d\phi \sqrt{-\det G} \widetilde{\mathcal{L}}'' \,. \tag{45}$$

- In oder to obtain the contribution for $\mathcal{E}_1'$ we just substitute the leading order solution of the near horizon fields into the expression for $\mathcal{E}_1'$. By doing so we obtain
  - $$\mathcal{E}_1' = 16\pi^2 \sqrt{\left|\frac{q_1 q_3}{p_2 p_4}\right|}\,. \tag{46}$$
- In order to compute $\mathcal{E}''$ we need to first dimensionally reduce the Chern simons term to bring it to manifest covariant form. This analysis can be simplified by considering the sphere as a compact direction and doing the dimensional reduction all the way to 2 dimensions.

- The resulting two dimensional lagrangian density has the form
  - $$\begin{aligned}&\sqrt{-\det G^{(2)}}\,\widetilde{\mathcal{L}}^{(2)\prime\prime}\\ &= -\frac{1}{16\pi^2}\frac{1}{(3!)^2}\lambda\int dx^4 dx^5\, d\theta\, d\phi\,\epsilon^{MNPQRS} K^{(6)}_{MNP}\Omega^{(6)}_{MNP}\\ &+\text{total derivative terms}\,, \end{aligned} \tag{47}$$
- The contribution $\mathcal{E}_1''$ to the entropy function is then given by
  $$\mathcal{E}_1'' = -2\pi\sqrt{-\det G^{(2)}}\,\widetilde{\mathcal{L}}^{(2)\prime\prime}\,, \tag{48}$$
  evaluated in the near horizon background of the black hole.

- First of all we note that the six dimensional field configuration has the structure of a product of two three dimensional spaces, the first one labelled by $(\theta, \phi, x^5)$ and the second one labelled by $(t, r, x^4)$. Thus we can make a consistent truncation where we consider only those field configurations which respect this product structure. In this case the two dimension lagrangian simplifies to

$$\sqrt{-\det G^{(2)}}\,\widetilde{\mathcal{L}}^{(2)\prime\prime} = -\frac{1}{16\pi^2}\frac{1}{(3!)^2}\lambda \int dx^4\, dx^5\, d\theta\, d\phi\, \epsilon^{\check{m}\check{n}\check{p}}\epsilon^{\check{\alpha}\check{\beta}\check{\gamma}} \left(K^{(6)}_{\check{m}\check{n}\check{p}}\Omega^{(6)}_{\check{\alpha}\check{\beta}\check{\gamma}} - \Omega^{(6)}_{\check{m}\check{n}\check{p}}K^{(6)}_{\check{\alpha}\check{\beta}\check{\gamma}}\right) \quad (49)$$

where the indices $\check{m}$, $\check{n}$, $\check{p}$ run over $(\theta, \phi, x^5)$ and the indices $\check{\alpha}$, $\check{\beta}$, $\check{\gamma}$ run over $(t, r, x^4)$.

- Let us now label the components of the six dimensional metric as

$$G^{(6)}_{\check{m}\check{n}}dx^{\check{m}}dx^{\check{n}} = G^{(6)}_{55}\left(h_{mn}dx^m dx^n + (dx^5 + 2\mathcal{A}^{(2)}_m dx^m)^2\right) \quad (50)$$

and

$$G^{(6)}_{\check{\alpha}\check{\beta}}dx^{\check{\alpha}}dx^{\check{\beta}} = G^{(6)}_{44}\left(g_{\alpha\beta}dx^\alpha dx^\beta + (dx^4 + 2\mathcal{A}^{(1)}_\alpha dx^\alpha)^2\right) \quad (51)$$

- where the indices $m, n$ run over $(\theta, \phi)$ and the indices $\alpha, \beta$ run over $(t, r)$.

- Then it follows from our previous analysis of BTZ case that

$$\int dx^5\, d\theta\, d\phi\, \epsilon^{\check{m}\check{n}\check{p}}\Omega^{(6)}_{\check{m}\check{n}\check{p}}$$
$$= \pi \int d\theta\, d\phi\, \epsilon^{mn} \left[R_h\, \mathcal{F}^{(2)}_{mn} + 4\, h^{m'p'}\, h^{q'q}\, \mathcal{F}^{(2)}_{mm'}\, \mathcal{F}^{(2)}_{p'q'}\, \mathcal{F}^{(2)}_{qn}\right] \quad (52)$$

and

$$\int dx^4 \epsilon^{\check{\alpha}\check{\beta}\check{\gamma}}\Omega^{(6)}_{\check{\alpha}\check{\beta}\check{\gamma}}$$
$$= \pi\, \epsilon^{\alpha\beta} \left[R_g \mathcal{F}^{(1)}_{\alpha\beta} + 4\, g^{\alpha'\gamma'}\, g^{\delta'\delta}\, \mathcal{F}^{(1)}_{\alpha\alpha'} \mathcal{F}^{(1)}_{\gamma'\delta'}\, \mathcal{F}^{(1)}_{\delta\beta}\right]$$
$$+\text{total derivative terms} \quad (53)$$

- Thus we get

$$\sqrt{-\det G^{(2)}}\,\widetilde{\mathcal{L}}^{(2)\prime\prime}$$
$$= -\frac{1}{16\pi}\frac{1}{(3!)^2}\lambda\Bigg[6\pi\left(\int d\theta\,d\phi\,\epsilon^{mn}\,K^{(6)}_{5mn}\right)$$
$$\epsilon^{\alpha\beta}\left[R_g\mathcal{F}^{(1)}_{\alpha\beta}+4\,g^{\alpha'\gamma'}\,g^{\delta'\delta}\,\mathcal{F}^{(1)}_{\alpha\alpha'}\mathcal{F}^{(1)}_{\gamma'\delta'}\,\mathcal{F}^{(1)}_{\delta\beta}\right]$$
$$-6\pi\left(\int d\theta\,d\phi\,\epsilon^{mn}\left[R_h\,\mathcal{F}^{(2)}_{mn}+4\,h^{m'p'}\,h^{q'q}\,\mathcal{F}^{(2)}_{mm'}\,\mathcal{F}^{(2)}_{p'q'}\,\mathcal{F}^{(2)}_{qn}\right]\right.$$
$$\left.\epsilon^{\alpha\beta}K^{(6)}_{4\alpha\beta}\Bigg]\,, \qquad (5$$

- where $R_h$ and $R_g$ denotes the scalar curvature associated with the metrics $h_{mn}$ and $g_{\alpha\beta}$ respectively.

- Since the lagrangian density now has manifest covariance, we can apply the entropy function formalism.
- For the six dimensional field configuration we have taken

$$h_{mn}dx^m dx^n = v_2\, u_2^{-2}\,(d\theta^2 + \sin^2\theta d\phi^2)\,,$$
$$g_{\alpha\beta}dx^\alpha dx^\beta = v_1\, u_1^{-2}\,(-r^2dt^2 + dr^2/r^2)\,, \qquad (55)$$

- we get

$$\sqrt{-\det G^{(2)}}\,\widetilde{\mathcal{L}}^{(2)\prime\prime} = \frac{2\lambda\pi}{3}\left[\frac{\widetilde{p}_4}{4\pi}\left(\frac{u_1^2}{v_1}\widetilde{e}_1 - 2\frac{u_1^4}{v_1^2}\,\widetilde{e}_1^3\right) + \widetilde{e}_3\left(\frac{u_2^2}{v_2}\,\frac{\widetilde{p}_2}{4\pi} - 2\frac{u_2^4}{v_2^2}\,\left(\frac{\widetilde{p}_2}{4\pi}\right)^3\right)\right] \qquad (56)$$

- Evaluating this for the leading order solution of the near horizon fields we get

$$\mathcal{E}_1'' = \frac{1}{6}\,\lambda\,\pi^2 \left( \frac{q_1 q_3}{\sqrt{|p_2 p_4 q_1 q_3|}} + \frac{\sqrt{|p_2 p_4 q_1 q_3|}}{p_2 p_4} \right) \tag{57}$$

- We shall now consider the range of values

$$p_2 > 0, \qquad p_4 > 0, \qquad q_3 > 0\,. \tag{58}$$

- In this case the full black hole entropy, given by the value of the entropy function at its extremum becomes

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1' + \mathcal{E}_1'' = \sqrt{|p_2 p_4 q_1 q_3|}\left[1 + \frac{\pi^2}{p_2 p_4}\left\{16 + \frac{1}{6}\,\lambda\,\left(1 + \frac{q_1}{|q_1|}\right) \tag{59}$$

- In order to determine the parameter $\lambda$. we define

$$a = 8\pi\, C^{(6)}_{45}\,, \tag{60}$$

- then after elimination of $H^{(6)}_{MNP}$ and dimensional reduction to four dimensions, the action contains the terms:

$$\frac{1}{32\pi}\int d^4x\,\Big[-\frac{1}{2}\sqrt{-\det G}\,e^{2\Phi}\,G^{\mu\nu}\partial_\mu a\partial_\nu a + \frac{\lambda}{48}\,a\,\epsilon^{\mu\nu\rho\sigma}\,R^a{}_{b\mu\nu}\,R^b \tag{61}$$

- $a$ plays the role of the axion field. Comparing this with the standard action for tree level heterotic string theory hep-th/0603149 compactified down to four dimensions, we get

$$\lambda = 48\,. \tag{62}$$

- Hence the entropy now becomes

$$
\begin{aligned}
\mathcal{E} &= \sqrt{|p_2 p_4 q_1 q_3|}\left[1 + 32\,\frac{\pi^2}{p_2 p_4}\right] \qquad \text{for } q_1 > 0 \\
&= \sqrt{|p_2 p_4 q_1 q_3|}\left[1 + 16\frac{\pi^2}{p_2 p_4}\right] \qquad \text{for } q_1 < 0\,.
\end{aligned}
\tag{63}
$$

# Conclusion

- The result for the entropy agrees with the result obtained by
  - including only the Gauss-Bonnet term in the four dimensional effective action hepth/9711053,0508042,
  - including a fully supersymmetrized version of the curvature squared correction in the four dimensional effective action in hepth/9812082,9904005,9906094,9910179,0007195, 0009234,0012232
  - the argument based on the existence of an $AdS_3$ component of the near horizon geometry and supersymmetry of the associated boundary theory hep-th/0506176,0508218.

- Since the last result makes use of supersymmetry to relate the gauge anomaly to the trace anomaly in the boundary theory, our result provides an indirect evidence that the bosonic effective action of heterotic strings we have used can be consistently supersymmetrized to this order in $\alpha'$.