Exact Bosonization of Nonrelativistic Fermions and Applications in String Theory

Avinash Dhar

Tata Institute of Fundamental Research, Mumbai

Indian Strings Meeting, Toshali Sands, Puri,

December 12, 2006

References

hep – th/0505084 A. D.

hep – th/0509164 A. D., G. Mandal and N.V. Suryanarayana

hep – th/0512312 A. D., G. Mandal and M. Smedback

hep – th/0603154 A. D., G. Mandal

Contents

- Introduction and Motivation
- Exact bosonization of nonrelativistic fermions in 1-dim.
- Applications to problems in string theory
- Summary

Introduction and Motivation

- Non-relativistic (NR) fermions in 1-d appear in many situations in string theory and quantum field theory.

Introduction and Motivation

- Non-relativistic (NR) fermions in 1-d appear in many situations in string theory and quantum field theory.
  - Non-critical string theory in 2-d $\rightarrow$ NR fermions in an inverted harmonic potential

Introduction and Motivation

- Non-relativistic (NR) fermions in 1-d appear in many situations in string theory and quantum field theory.
  - Non-critical string theory in 2-d $\rightarrow$ NR fermions in an inverted harmonic potential
  - Half-BPS sector of $\mathcal{N} = 4$ SYM theory in 4-d $\rightarrow$ NR fermions in a harmonic potential

Introduction and Motivation

- Non-relativistic (NR) fermions in 1-d appear in many situations in string theory and quantum field theory.
  - Non-critical string theory in 2-d $\rightarrow$ NR fermions in an inverted harmonic potential
  - Half-BPS sector of $\mathcal{N} = 4$ SYM theory in 4-d $\rightarrow$ NR fermions in a harmonic potential
  - YM theory on a cylinder $\rightarrow$ Free NR fermions on a circle

Introduction and Motivation

- Non-relativistic (NR) fermions in 1-d appear in many situations in string theory and quantum field theory.

  - Non-critical string theory in 2-d $\rightarrow$ NR fermions in an inverted harmonic potential
  - Half-BPS sector of $\mathcal{N} = 4$ SYM theory in 4-d $\rightarrow$ NR fermions in a harmonic potential
  - YM theory on a cylinder $\rightarrow$ Free NR fermions on a circle

  Its partition function counts certain $D0 - D2 - D4$ brane black holes

Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

$$S = \int dt \, \{\frac{1}{2}\dot{M}^2 - V(M)\}$$

Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

$$S = \int dt \, \{\frac{1}{2}\dot{M}^2 - V(M)\}$$

M is an $N \times N$ matrix

Introduction and Motivation

- A common feature of all these examples is that the fermionic system arises from an underlying matrix quantum mechanics problem

$$S = \int dt \, \{\frac{1}{2}\dot{M}^2 - V(M)\}$$

- In the $U(N)$ invariant sector, the matrix model is equivalent to a system of N NR fermions [a]
- Jevicki and Sakita [b] used this equivalence to develop a bosonization in the large-N limit - collective field theory

[a] Brezin, Itzykson, Parisi and Zuber, Comm. Math. Phys.59, 35, 1978

[b] Nucl.Phys.B165, 511, 1980

Introduction and Motivation

- Bosonization in terms of Wigner phase space density [a]

$$u(p,q,t) = \int dx\, e^{-ipx} \sum_{i=1}^{N} \psi_i^\dagger(q - x/2, t)\psi_i(q + x/2, t)$$

- $u(p,q,t)$ satisfies two constraints:
  - $\int \frac{dpdq}{2\pi} u(p,q,t) = N$
  - $u * u = u$

[a] Dhar, Mandal and Wadia, hep-th/9204028; 9207011; 9309028

Introduction and Motivation

- Bosonization in terms of Wigner phase space density [a]

$$u(p,q,t) = \int dx\, e^{-ipx} \sum_{i=1}^{N} \psi_i^\dagger(q - x/2, t)\psi_i(q + x/2, t)$$

- $u(p,q,t)$ satisfies two constraints:
  - $\int \frac{dpdq}{2\pi} u(p,q,t) = N$
  - $u * u = u$

Many more variables than are necessary

[a] Dhar, Mandal and Wadia, hep-th/9204028; 9207011; 9309028

Exact Bosonization

The Setup:

- each fermion can occupy a state in an infinite-dimensional Hilbert space $\mathcal{H}_f$
- there is a countable basis of $\mathcal{H}_f : \{|m\rangle, m = 0, 1, \cdots, \infty\}$
- creation and annihilation operators $\psi^\dagger_m$, ψ_m create and destroy particles in the state $|m\rangle, \quad \{\psi_m, \psi^\dagger_n\} = \delta_{mn}$
- total number of fermions is fixed:

$$\sum_n \psi^\dagger_n \psi_n = N$$

Exact Bosonization

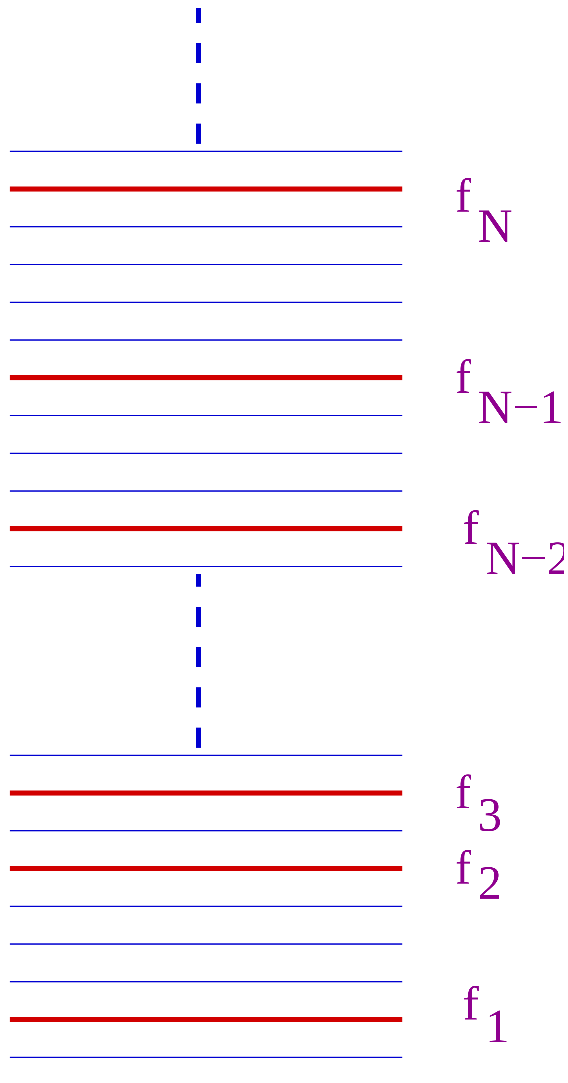
- The N-fermion states are given by (linear combinations of)

$$|f_1, \cdots, f_N\rangle = \psi^\dagger_{f_N} \cdots \psi^\dagger_{f_2} \psi^\dagger_{f_1} |0\rangle_F,$$

  - $|0\rangle_F$ is Fock vacuum
  - f_k are ordered $0 \leq f_1 < f_2 < \cdots < f_N$

- Repeated applications of the bilinear $\psi^\dagger_m \psi_n$ gives any desired state

Exact Bosonization



Exact Bosonization

Bosonization: [a]

- Introduce the bosonic operators

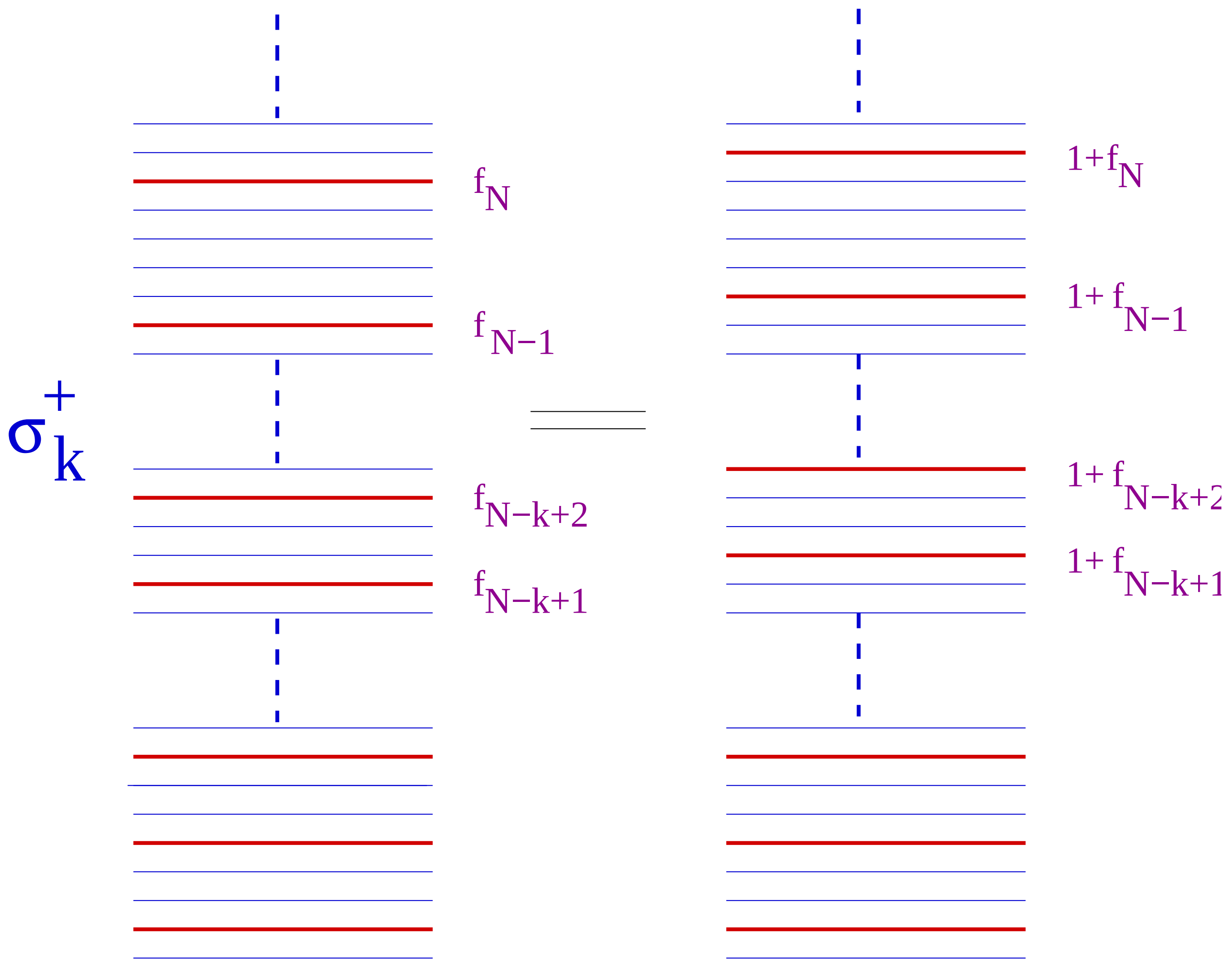
$$\sigma_k, \; k = 1, 2, \cdots, N$$

- and their conjugates

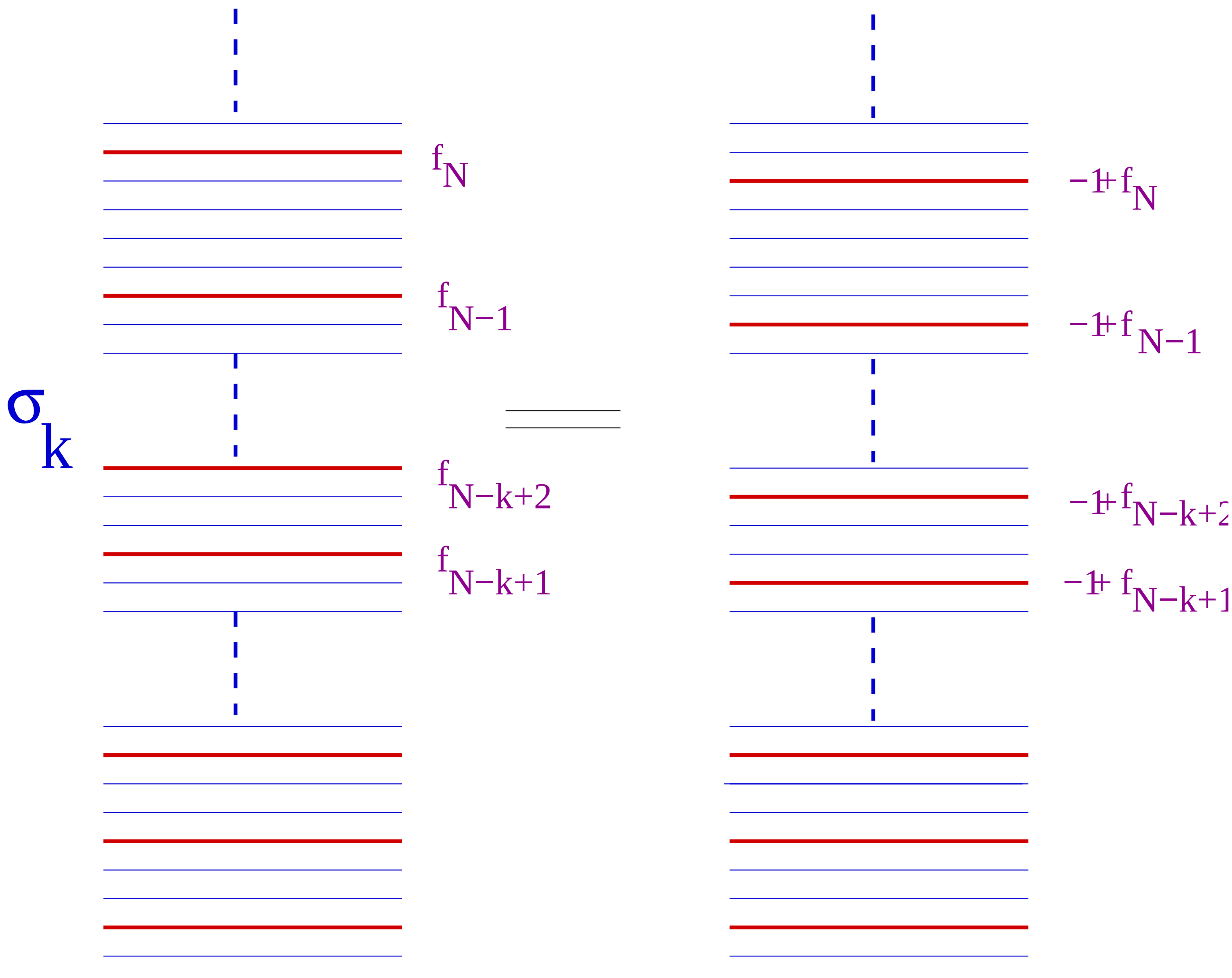
$$\sigma_k^\dagger, \; k = 1, 2, \cdots, N$$

[a] Dhar, Mandal and Suryanarayana, hep-th/0509164

Exact Bosonization



Exact Bosonization

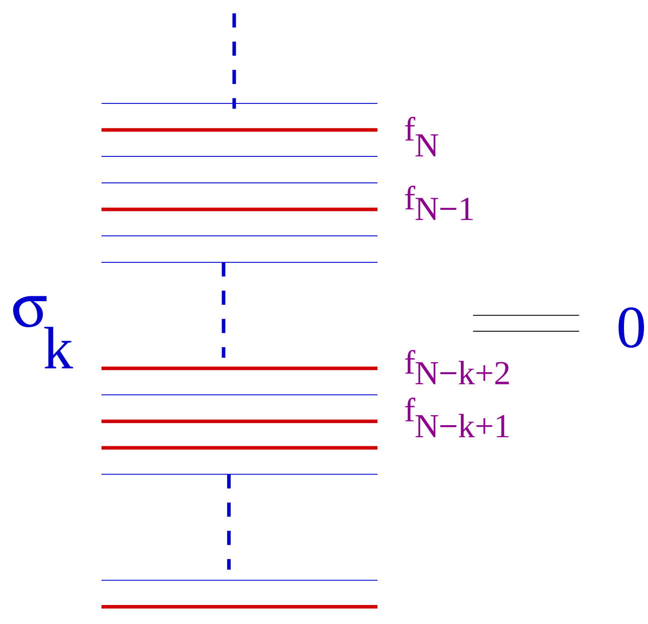


Exact Bosonization

- By definition:
  - $\sigma_k \sigma_k^\dagger = 1$
  - $\sigma_k^\dagger \sigma_k = 1$, if σ_k does not annihilate the state

Exact Bosonization

- By definition:
  - $\sigma_k \sigma_k^\dagger = 1$
  - $\sigma_k^\dagger \sigma_k = 1$, if σ_k does not annihilate the state



Exact Bosonization

- By definition:
  - $\sigma_k \sigma_k^\dagger = 1$
  - $\sigma_k^\dagger \sigma_k = 1$, if σ_k does not annihilate the state
- For $k \neq l$, $[\sigma_k, \sigma_l^\dagger] = 0$

Exact Bosonization

- Introduce creation (annihilation) operators $a_k^\dagger$ (a_k) which satisfy the standard commutation relations

$$[a_k, a_l^\dagger] = \delta_{kl}, \quad k, l = 1, \cdots, N$$

- The states of the bosonic system are given by (a linear combination of)

$$|r_1, \cdots, r_N\rangle = \frac{(a_1^\dagger)^{r_1} \cdots (a_N^\dagger)^{r_N}}{\sqrt{r_1! \cdots r_N!}} |0\rangle$$

Exact Bosonization

- Now, make the following identifications

$$\sigma_k = \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k; \qquad \sigma_k^\dagger = a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}}$$

- together with the map [a]

$$r_N = f_1; \quad r_k = f_{N-k+1} - f_{N-k} - 1, \quad k = 1, 2, \cdots N-1$$

- For the Fermi vacuum, $f_{k+1} = f_k + 1$ and so $r_k = 0$ for all k => Fermi vacuum = Bose vacuum

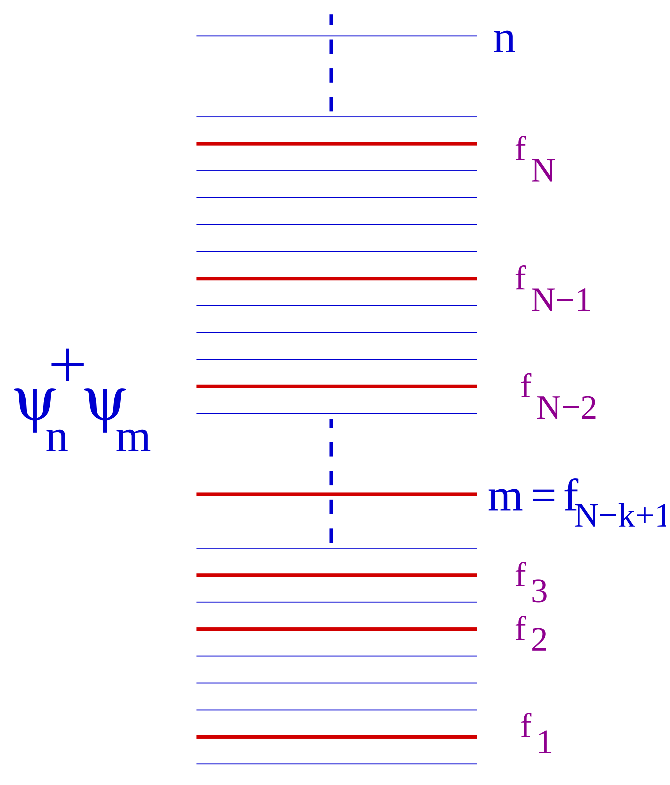
[a] Suryanarayana, hep-th/0411145

Exact Bosonization

- The σ_k, $k = 1, 2, \cdots, N$ are necessary and sufficient
  - Any bilinear $\psi_n^\dagger \psi_m$ can be built out of σ_k's

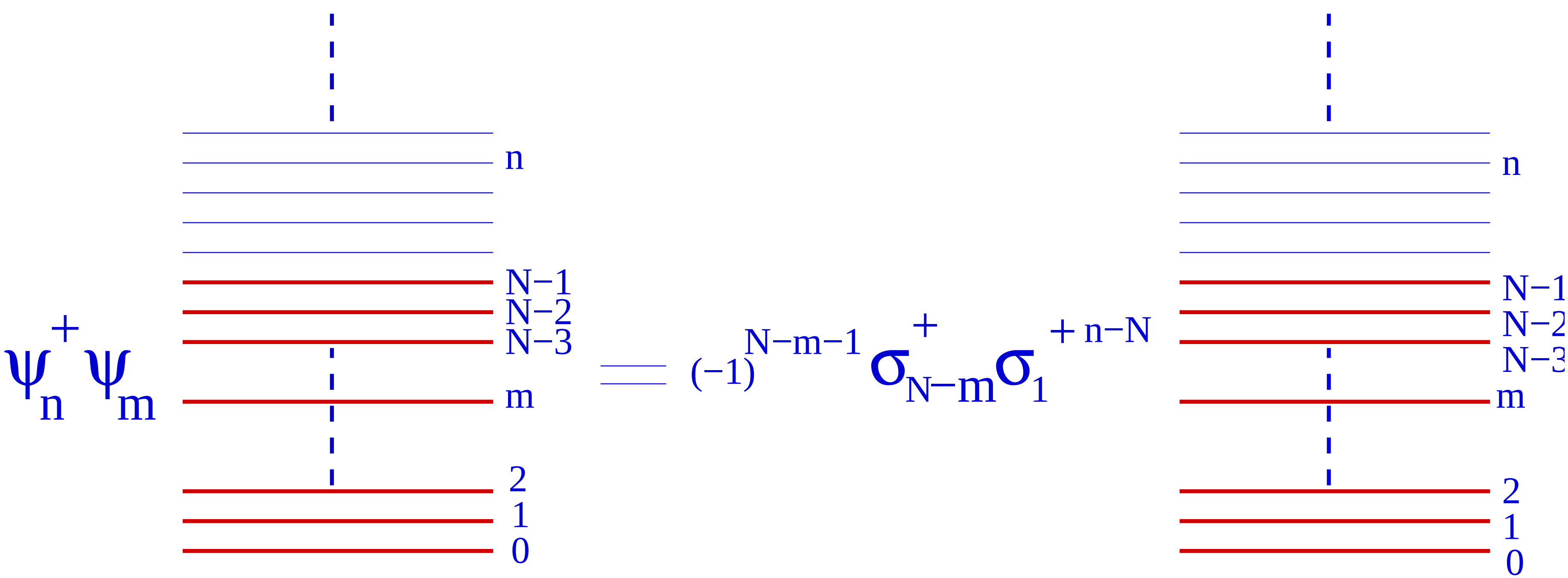
Exact Bosonization

- The σ_k, $k = 1, 2, \cdots, N$ are necessary and sufficient
  - Any bilinear $\psi_n^\dagger \psi_m$ can be built out of σ_k's



Exact Bosonization

- The σ_k, $k = 1, 2, \cdots, N$ are necessary and sufficient
  - Any bilinear $\psi_n^\dagger \psi_m$ can be built out of σ_k's



Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!

Exact Bosonization

Generic properties of the bosonized theory:

- Each boson can occupy only a finite number of different states, as a consequence of a finite number of fermions => a cut-off or graininess in the bosonized theory!
- There is no natural "space" in the bosonic theory - in the examples we will discuss, a spatial direction will emerge in the low-energy large-N limit.

Exact Bosonization

- The non-interacting fermionic Hamiltonian:

$$H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n$$

Exact Bosonization

- The non-interacting fermionic Hamiltonian:

$$H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n$$

- The bosonized Hamiltonian:

$$H = \sum_{k=1}^{N} \mathcal{E}(\hat{n}_k), \quad \hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$$

Exact Bosonization

- The non-interacting fermionic Hamiltonian:

$$H = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n$$

- The bosonized Hamiltonian:

$$H = \sum_{k=1}^{N} \mathcal{E}(\hat{n}_k), \quad \hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$$

- What about fermion interactions? These can also be included since the generic bilinear $\psi_n^\dagger \psi_m$ has a bosonized expression

Half-BPS states and LLM geometries

- SYM - half-BPS states are described by a holomorphic sector of quantum mechanics of an $N \times N$ complex matrix Z in a harmonic potential
- This system can be shown [a] to be equivalent to the quantum mechanics of an $N \times N$ hermitian matrix Z in a harmonic potential

[a] Takayama and Tsuchiya, hep-th/0507070

Half-BPS states and LLM geometries

- Gauge invariance => physical observables on boundary are $U(N)$-invariant traces:

$$\mathrm{tr} Z^k, \quad k = 1, 2, \cdots, N$$

- Physical states <=> operators

$$(\mathrm{tr} Z^{k_1})^{l_1} (\mathrm{tr} Z^{k_2})^{l_2} \cdots$$

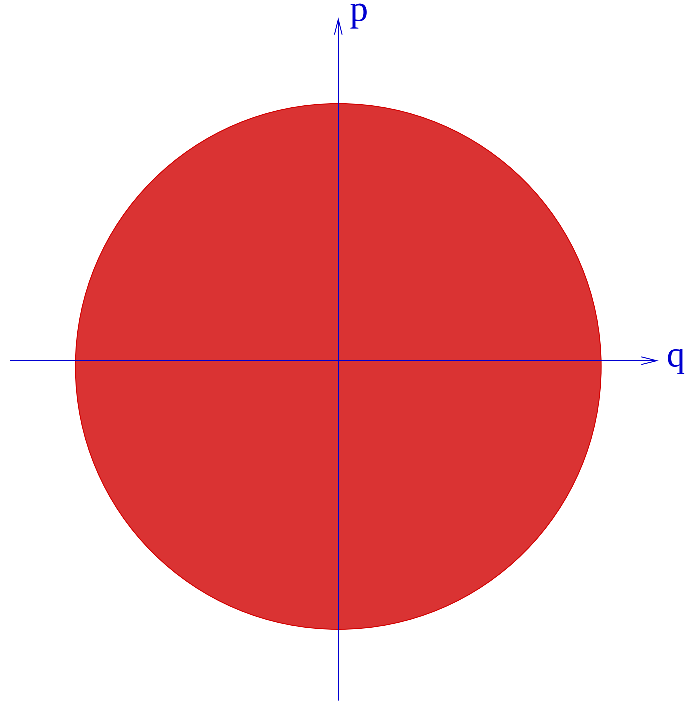
- Total number of Z's is a conserved RR charge $Q = \sum_i k_i l_i$. BPS condition => $E = Q$

Half-BPS states and LLM geometries

- At large N there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space

Half-BPS states and LLM geometries

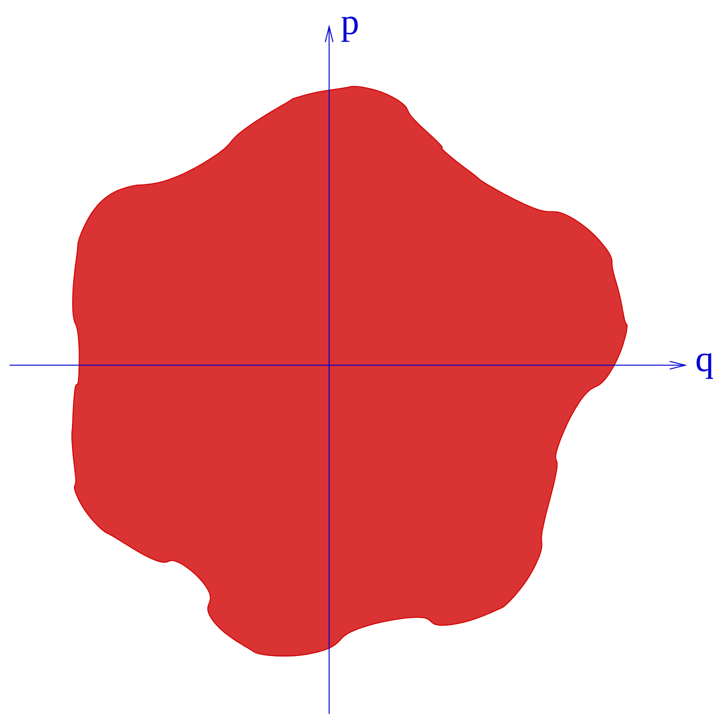
- At large N there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space



ground state distribution

Half-BPS states and LLM geometries

- At large N there is a semiclassical picture of the states of this system in terms of droplets of fermi fluid in phase space



small fluctuations around the ground state

Half-BPS states and LLM geometries

- By explicitly solving equations of type IIB gravity, LLM showed that there is a similar structure in the classical geometries in the half-BPS sector!
- LLM solutions - two of the space coordinates are identified with the phase space of a single fermion => noncommutativity in two space directions in the semicalssical description [a]
- Small fluctuations around AdS space, i.e low-energy graviton excitations [b] [c] $\equiv$ low-energy fluctuations of the fermi vacuum [d]

---

[a] Mandal, hep-th/0502104

[b] Grant, Maoz, Marsano, Papadodimas and Rychkov, hep-th/0505079

[c] Maoz and Rychkov, hep-th/0508059

[d] Dhar, hep-th/0505084

Half-BPS states and LLM geometries

- Motivation for our work [a] - on the CFT side the half-BPS system can be quantized exactly in terms of our bosons => window of opportunity to learn about aspects of quantum gravity.
- At finite N, only the low-energy excitations on the boundary can be identified with low-energy ($<< N$) gravitons in the bulk
- The single-particle graviton excitations are related to our bosons. On the boundary, these states are:

$$\beta_m^\dagger|0\rangle = \sum_{n=1}^{m}(-1)^{n-1}\sqrt{\frac{(N+m-n)!}{2^m(N-n)!}}\sigma_1^{\dagger\,m-n}\sigma_n^\dagger|0\rangle$$

[a] Dhar, Mandal and Smedback, hep-th/0512312

Half-BPS states and LLM geometries

- On the boundary, single-particle giant graviton states map to linear combinations of multi-graviton states [a]. Example:

$$|\text{giant graviton of energy } 2\rangle = ({\beta_1^\dagger}^2 - \beta_2^\dagger)|0\rangle$$

[a] Balasubramanian, Berkooz, Naqvi and Strassler, hep-th/0107119

Half-BPS states and LLM geometries

- On the boundary, single-particle giant graviton states map to linear combinations of multi-graviton states [a]. Example:

$$|\text{giant graviton of energy } 2\rangle = ({\beta_1^\dagger}^2 - \beta_2^\dagger)|0\rangle = a_2^\dagger|0\rangle$$

[a] Balasubramanian, Berkooz, Naqvi and Strassler, hep-th/0107119

Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a_k^\dagger |0\rangle$$

Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a_k^\dagger |0\rangle$$

- Hamiltonian: $H_F = \sum_n \mathcal{E}(n) \psi_n^\dagger \psi_n => H_B = \sum_{k=1}^N k a_k^\dagger a_k$

Half-BPS states and LLM geometries

- Boundary states corresponding to single-particle bulk giant states are our single-particle bosonic states:

$$|\text{giant graviton of energy } k\rangle = a_k^\dagger|0\rangle$$

- Hamiltonian: $H_F = \sum_n \mathcal{E}(n)\psi_n^\dagger\psi_n => H_B = \sum_{k=1}^N k a_k^\dagger a_k$

- Discrete space?

$$\phi(\theta_n) = \sum_{k=1}^{N}(e^{ik\theta_n}a_k + e^{-ik\theta_n}a_k^\dagger), \quad \theta_n = \frac{2\pi n}{N}$$

Black holes and 2-d YM on a circle

- Vafa [a] has argued that the partition function of $U(N)$ 2-d YM on a circle counts certain BPS D-brane black hole configurations in a CY compactification of type IIA string theory.

- The rank of the gauge group, N, maps to the number of $D4$-branes and combinations of the YM coupling and the theta-angle map to chemical potentials for $D2$ and $D0$-branes.

[a] Vafa, hep-th/0406058

Black holes and 2-d YM on a circle

- In the leading large N limit, the partition function satisfies the OSV [a] relation, $Z_{\rm bh} = |\psi|^2$. But at finite N there are nonperturbative O(e^{-N}) corrections to this relation [b] which can be attributed to multi-center black holes.

- 2-d YM on a circle can be mapped to free NR fermions on a circle [c]. This relation was exploited by DGOV to obtain the nonperturbative corrections.

---

[a] Ooguri, Strominger and Vafa, hep-th/0405146

[b] Dijkgraaf, Gopakumar, Ooguri and Vafa, hep-th/0504221

[c] Minahan and Polychronakos, hep-th/9303153

Free fermions and 2-d YM on a circle

- In the gauge the gauge $A_0 = 0$, one solves the Gauss law constraint in terms of the Wilson line

$$W_{ab} = P\left(\exp[ig\int_a^b dx\, A_1]\right)$$

- One gets

$$E(x) \equiv \dot{A}_1(x) = W_{x0} V W_{Lx}.$$

- This expression for $E(x)$ and its periodicity on the circle lead to the constraint

$$[W, \dot{W}] = 0, \quad W \equiv W_{0L}.$$

Free fermions and 2-d YM on a circle

- The 2-d YM hamiltonian becomes

$$H = \frac{1}{2} \int_0^L dx \, \mathrm{Tr} E^2 = -\frac{1}{2g^2 L} \mathrm{Tr}(W^{-1}\dot{W})^2.$$

- The constraint and the canonical structure implied by the original Poisson bracket of the potential A_1 with its conjugate field E give the standard matrix model structure for the dynamics. Thus, this hamiltonian desribes the singlet sector of a unitary matrix quantum mechanics.

- Quantum mechanics of a unitary matrix is well-known to map to fermions on a circle.

Free fermions on a circle

- We will only discuss [a] the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.
- The free hamiltonian:

$$H = -\frac{\hbar^2}{2m}\int_0^L dx\ \chi^\dagger(x)\partial_x^2\chi(x)$$

[a] Dhar and Mandal, hep-th/0603154

Free fermions on a circle

- We will only discuss [a] the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

- Hamiltonian in terms of fourier modes:

$$H = \omega\hbar \sum_{n=-\infty}^{\infty} n^2 \chi_n^\dagger \chi_n, \qquad \omega \equiv \frac{2\pi^2\hbar}{mL^2}$$

[a] Dhar and Mandal, hep-th/0603154

Free fermions on a circle

- We will only discuss [a] the free fermion problem. Interactions can be taken into account once the free part has been dealt with properly.

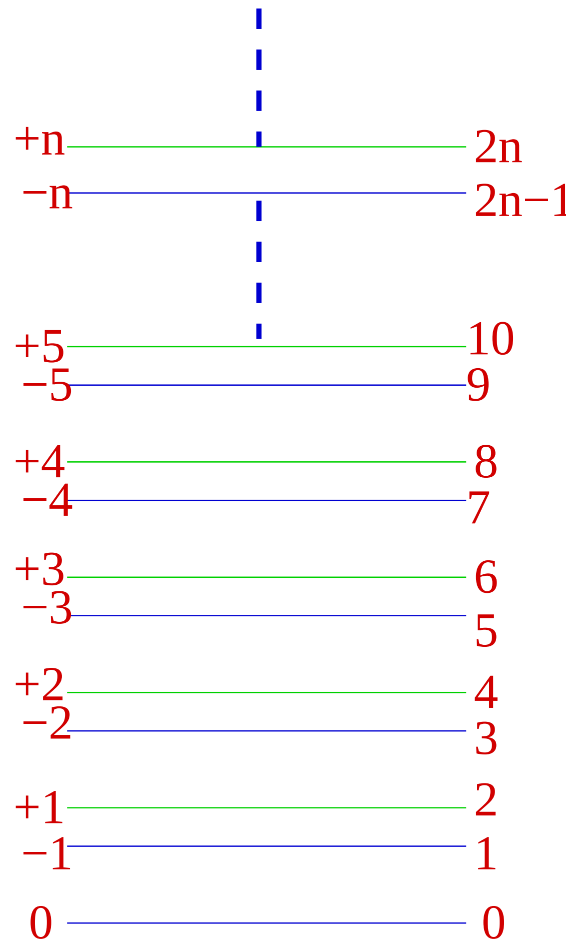
- Hamiltonian in terms of fourier modes:

$$H = \omega\hbar \sum_{n=-\infty}^{\infty} n^2 \chi_n^\dagger \chi_n, \qquad \omega \equiv \frac{2\pi^2\hbar}{mL^2}$$

- To apply our bosonization rules, need to introduce an ordering in the spectrum. For example, replace $n^2 \rightarrow (n+\epsilon)^2$

[a] Dhar and Mandal, hep-th/0603154

Free fermions on a circle



Free fermions on a circle

- Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$

Free fermions on a circle

- Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$
- Fermionic hamiltonian:

$$H = \omega\hbar \sum_{n=1}^{\infty} \left(\frac{n + e(n)}{2} \right)^2 \psi_n^\dagger \psi_n$$

Free fermions on a circle

- Effectively, we have set $\chi_{+n} = \psi_{2n}$ and $\chi_{-n} = \psi_{2n-1}$
- Fermionic hamiltonian:

$$H = \omega\hbar \sum_{n=1}^{\infty} \left(\frac{n + e(n)}{2} \right)^2 \psi_n^\dagger \psi_n$$

- Bosonized hamiltonian:

$$H = \omega\hbar \sum_{k=1}^{N} \left(\frac{\hat{n}_k + e(\hat{n}_k)}{2} \right)^2$$

where $\hat{n}_k = \sum_{i=k}^{N} a_i^\dagger a_i + N - k$

Free fermions on a circle

- Large-N low energy limit: $H = H_F + H_0 + H_1$

Free fermions on a circle

- Large-N low energy limit: $H = H_F + H_0 + H_1$

$$H_0 = \frac{\hbar\omega N}{2}\left(\sum_{k=1}^{N} k\ a_k^\dagger a_k + \hat{\nu}\right)$$

Free fermions on a circle

- Large-N low energy limit: $H = H_F + H_0 + H_1$

$$H_0 = \frac{\hbar \omega N}{2} \left(\sum_{k=1}^{N} k \; a_k^\dagger a_k + \hat{\nu} \right)$$

  - $\hat{\nu} = N_- - N_{-F} = \sum_{k=1}^{N} (e(\hat{n}_k) - e(N-k))$ is the number of excess fermions in negative momentum states over and above the number in fermi vacuum

Free fermions on a circle

- Large-N low energy limit: $H = H_F + H_0 + H_1$

$$H_0 = \frac{\hbar\omega N}{2}\left(\sum_{k=1}^{N} k\; a_k^{\dagger} a_k + \hat{\nu}\right)$$

  - $\hat{\nu} = N_- - N_{-F} = \sum_{k=1}^{N}(e(\hat{n}_k) - e(N-k))$ is the number of excess fermions in negative momentum states over and above the number in fermi vacuum
  - H_1 is order one on excited states whose energy is low compared to N

Free fermions on a circle

- The operator $\hat{\nu}$ commutes with both H_0 and H_1 separately. States can therefore be labeled by the quantum number ν, the eigenvalue of this operator.

- These states can be explicitly constructed:

$$|\nu\rangle = \begin{cases} \sigma^\dagger_{2\nu-1}\sigma^\dagger_{2\nu-2}\cdots\sigma^\dagger_1|0\rangle, & \nu > 0 \\ \\ \sigma^\dagger_{2|\nu|}\sigma^\dagger_{2|\nu|-1}\cdots\sigma^\dagger_1|0\rangle, & \nu < 0. \end{cases}$$

- These states satisfy

$$\hat{\nu}|\nu\rangle = \nu|\nu\rangle, \quad H_0|\nu\rangle = \hbar\omega N\nu^2|\nu\rangle, \quad H_1|\nu\rangle = 0$$

Free fermions on a circle

- Generalized partition function (H_0 part only), which keeps track of ν as well, is

$$Z_N(q,y) = \sum_{r_1, r_2, \cdots, r_N = 0}^{\infty} q^{\frac{1}{2}\sum_{k=1}^{N} k r_k} \, y^{\sum_{k=1}^{N} (-1)^{N-k} e(\sum_{i=k}^{N} r_i)},$$

where $q = e^{-\hbar \omega N \beta}$, $y = e^{-\mu}$.

Free fermions on a circle

- Generalized partition function (H_0 part only), which keeps track of ν as well, is

$$Z_N(q,y) = \sum_{r_1,r_2,\cdots,r_N=0}^{\infty} q^{\frac{1}{2}\sum_{k=1}^{N} k r_k}\; y^{\sum_{k=1}^{N}(-1)^{N-k} e(\sum_{i=k}^{N} r_i)},$$

where $q = e^{-\hbar\omega N\beta}$, $y = e^{-\mu}$.

- The following recursion relation can be easily derived:

$$Z_N(q,y) = (1-q^N)^{-1}[Z_{N-1}(q,y^{-1}) + q^{N/2} Z_{N-1}(q,y)].$$

Free fermions on a circle

- Exact partition function for finite N:

$$Z_N(q,y) = \sum_{\nu=-\frac{N-1}{2}}^{\frac{N+1}{2}} y^{\nu} q^{\nu(\nu-\frac{1}{2})} \prod_{n=1}^{\frac{N+1}{2}-\nu} (1-q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1-q^n)^{-1}$$

Free fermions on a circle

- Exact partition function for finite N:

$$Z_N(q,y) = \sum_{\nu=-\frac{N-1}{2}}^{\frac{N+1}{2}} y^\nu q^{\nu(\nu-\frac{1}{2})} \prod_{n=1}^{\frac{N+1}{2}-\nu} (1-q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1-q^n)^{-1}$$

- Setting $y = q^{1/2}$, we get

$$Z_N(q) = \sum_{\nu=-\frac{N-1}{2}}^{\frac{N+1}{2}} q^{\nu^2} \prod_{n=1}^{\frac{N+1}{2}-\nu} (1-q^n)^{-1} \prod_{n=1}^{\frac{N-1}{2}+\nu} (1-q^n)^{-1}.$$

Free fermions on a circle

- Nonperturbative corrections for large but finite N:

$$Z_N(q) = \sum_{\nu=-\frac{N-1}{2}}^{\frac{N+1}{2}} \qquad q^{\nu^2}$$

$$\times \left[\prod_{n=1}^{\infty} (1 - q^{\frac{N+1}{2} - \nu + n}) \prod_{n=1}^{\infty} (1 - q^{\frac{N-1}{2} + \nu + n}) \right]$$

$$\times \left[\prod_{n=1}^{\infty} (1 - q^n)^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{-1} \right]$$

SUMMARY

- We have developed a simple and exact bosonization of a finite number of NR fermions; we discussed here applications to problems in string theory, but our techniques are applicable in other areas of physics as well, e.g. to problems in condensed matter physics.
- Our bosonization trades finiteness of the number of fermions for finite dimensionality of the single-particle boson Hilbert space
- The bosonized theory is inherently grainy; in the specific applications we discussed, a local space-time field theory emerges only in the large-N and low-energy limit
- Bosonization of finite number of fermions in higher dimensions?