Quantum Games

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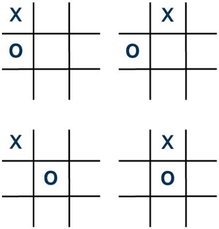
Introduction

What is a game?

One definition: A form of competitive sport or activity played according to rules.



TICK-TACK-TOE

CHESS

Von Neumann's definition

Von Neumann's idea when talking about games is only tangentially about sport.
Jacob Bronowski in *Ascent of Man* writes "To VonNeumann, games meant not really Chess which are amenable to a solution given a particular position. For him, games mimicked real life, wherein real life situations like bluffing, deception, etc, hold centre stage"

What is game theory really?

Game theory is a rigorous branch of Mathematical logic that underlies real conflicts among (not always rational) humans.

Why should we study game theory?

- Biology-evolutionary game theory: Survival of the fittest, Contribution of Axelrod "Evolution of Coperation" took game theory into biology.
- Quantum Physics (Quantum game theory, Quantum algorithms)
- Statistical Physics-Minority games: El Ferrol Bar problem.
- Social Sciences- Politics (Diplomacy, Election, etc.), Economics (Auctions, mergers & acquisitions, etc.)

Basic definitions

- Players: Game theory is about logical players interested only in winning.
- Actions : The set of all choices available to a player.
- Payoff : With each action we associate some value(a real number) such that higher values(i.e. payoff) are preferred.
- Optimal Strategy : Strategy that maximizes a player's expected payoff.

Types of games

Cooperative and non-cooperative

A game is cooperative if the players are able to form binding agreements i.e. the optimal strategy is to cooperate, players can coordinate their strategies and share the payoff.
Example of a cooperative game : Treasure Hunt- An expedition of n people have found a treasure in the mount; each pair of them can carry out one piece, but not more. How will they pair up?
Example of a non-cooperative game: Chess(Sports), Matching pennies, Penny flip

Types of games

Zero sum and Non-zero sum

If one player wins exactly the same amount the other player looses then the sum of their payoff's is zero. Since the payoff's are against each other these games are also known as non-cooperative games.
Example of a zero sum game : Matching pennies
Example of a non-zero sum game : Prisoner's dilemma

Types of games

Simultaneous and sequential

In simultaneous games players play simultaneously or say the players do not know of the other player's actions it makes the game effectively simultaneous.
Sequential games are where players play one after the another.
Example of a sequential game : Chess
Example of a simultaneous game : Matching pennies

Von-Neumann's Minimax theorem for zero sum games

Minimax via cake division

Cutter goes for nearly half the cake by electing to split the cake evenly. This amount, the maximum row minimum, is called "maximin". Cutter acts to maximize the minimum the chooser will leave him-"maximin".
Chooser looks for minimum column maximum-"minimax".

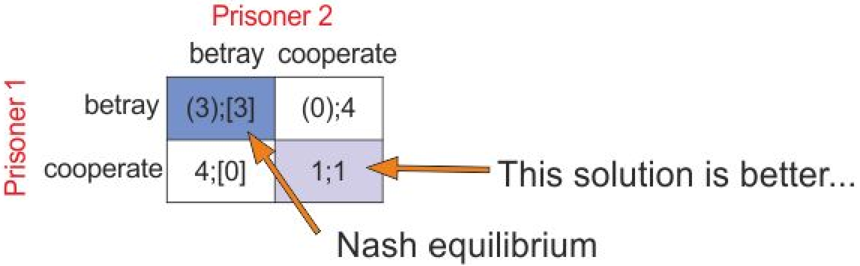
Chooser's strategies

		Choose bigger piece	Choose smaller piece
Cutter's strategies	Cut cake as evenly as possible	Half the cake minus a crumb	Half the cake plus a crumb
	Make one piece bigger than the other	Small piece	Big piece

Nash Equilibrium for zero and non-zero sum games

Nash Equilibrium via Prisoner's dilemma

Nash equilibrium: A set of strategies is a Nash equilibrium if no player can do better by unilaterally changing their strategy



Matching pennies

The game is played between two players, Players A and B.
Each player has a penny and must secretly turn the penny to heads or tails.
The players then reveal their choices simultaneously.
If the pennies match both heads or both tails then player A keeps both pennies,(so wins 1 from B i.e. +1 for A and -1 for B).
If they don't match player B keeps both the pennies.

B

A		Heads	Tails
A	Heads	1,-1	-1,1
A	Tails	-1,1	1,-1

A zero sum, non-cooperative and simultaneous game without a fixed Nash equilibrium.

Matching pennies

	Heads	Tails
Heads	1 cent *MINIMAX*	−1 cent *MAXIMIN*
Tails	−1 cent *MAXIMIN*	1 cent *MINIMAX*

	Heads	Tails	Random
Heads	1 cent	−1 cent	0
Tails	−1 cent	1 cent	0
Random	0	0	0 *MINIMAX & MAXIMIN*

Pure vs. Mixed strategies

Pure: Playing heads or tails with certainty.
Mixed: Playing heads or tails randomly (with 50% probability for each)

Nash equilibrium(NE) for Matching Pennies

Alice and Bob, have a penny that each secretly flips to heads H or tails T. No communication takes place between them and they disclose their choices simultaneously to a referee. If referee finds that pennies match (both heads or both tails), he takes 1$ from Bob and gives it to Alice (+1 for Alice, -1 for Bob). If the pennies do not match he does the opposite. As one players gain is exactly equal to the other players loss, the game is zero-sum and is represented with the payoff matrix:

$$\begin{array}{ccc} & & \text{Bob} \\ & & \begin{array}{cc} \mathcal{H} & \mathcal{T} \end{array} \\ \text{Alice} & \begin{array}{c} \mathcal{H} \\ \mathcal{T} \end{array} & \left(\begin{array}{cc} (a_1, b_1) & (a_2, b_2) \\ (a_3, b_3) & (a_4, b_4) \end{array} \right) \end{array}$$

$a_1 = +1$, $b_1 = -1$; $a_2 = -1$, $b_2 = +1$; $a_3 = -1$, $b_3 = +1$; and $a_4 = +1$, $b_4 = -1$.

It is well known that MP has no pure strategy Nash equilibrium but instead has a unique mixed strategy NE.

Mixed strategy NE for Matching Pennies

Consider repeated play of the game in which x and y are the probabilities with which H is played by Alice and Bob, respectively. The pure strategy T is then played with probability (1-x) by Alice, and with probability (1-y) by Bob, and the players payoff relations read

$$\Pi_{A,B}(x,y) = \begin{pmatrix} x \\ 1-x \end{pmatrix}^T \begin{pmatrix} (a_1,b_1) & (a_2,b_2) \\ (a_3,b_3) & (a_4,b_4) \end{pmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix}.$$

A strategy pair $(x^\star, y^\star)$ is a NE when

$$\Pi_A(x^\star,y^\star) - \Pi_A(x,y^\star) \geq 0, \quad \Pi_B(x^\star,y^\star) - \Pi_B(x^\star,y) \geq 0.$$

For the payoff matrix these inequalities read:

$$2(x^\star - x)(2y^\star - 1) \geq 0 \text{ and } 2(y^\star - y)(-2x^\star + 1) \geq 0$$

generate the strategy pair $(x^\star, y^\star) = (1/2, 1/2)$ as the unique NE of the game.

At the NE, the player's payoff's work out as:

$$\Pi_A(1/2,1/2) = 0 = \Pi_B(1/2,1/2).$$

Meyer's Penny flip game

The PQ penny flip was designed by David Meyer, its a close cousin of the Matching pennies game and has the following rules:
Players P and Q each have access to a single penny.
Initial state of the penny is heads(say). Each player can choose to either flip or not flip the penny.
Players cannot see the current state of the penny.
Sequence of actions :$Q \rightarrow P \rightarrow Q$
If final state is heads, Q wins else P wins

Payoff's for Meyer's Penny flip game

The payoff matrix for the game is as follows with the first entry as the payoff of P and the second is the payoff of Q.

	FF	FN	NF	NN
F	+1,-1	-1,+1	-1,+1	+1,-1
N	-1,+1	+1,-1	+1,-1	-1,+1

Nash equilibrium for Meyer's penny flip

No pure strategy NE but a mixed strategy NE exists. The pair of mixed strategies with P flipping or not flipping with prob. 1/2 and Q playing each of the available four strategies with prob. 1/4 is the NE with payoff zero.

Quantum Games - An Introduction

Quantization Rules:

Superposed initial states
Quantum entanglement of initial states.
Superposition of strategies to be used on the initial states

Quantum Penny Flip game

Quantum Penny flip game Rules:

The penny of the game is represented as a qubit(two-level system), with Heads maps to $|0>$ and tails mapped to $|1>$.
Player P does classical moves i.e. Flip (X) or not flip (I).
Player Q does quantum moves i.e. any general unitary U (say Hadamard).

Quantum vs. Classical moves

$$|0\rangle \quad \overset{\text{Q does } H}{\longmapsto} \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\overset{\text{P does } X \text{ or } I}{\longmapsto} \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\overset{\text{Q does } H}{\longmapsto} \quad |0\rangle$$

Why does Q win?

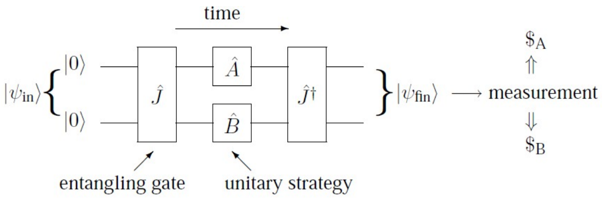
Q's quantum strategy puts the penny into the equal superposition of 'head' and 'tail'.
This state is invariant under X or I, Q always wins.

D. Mayer, Quantum Strategies, Phys. Rev. Lett. 82, 1052 (1999).

Quantum Prisoner's dilemma

Quantum Prisoner's dilemma: The steps

Step 1: Initial state $|\psi_{in}>=|00>$
Step 2: Generate entanglement via
$\hat{J}(\gamma)=\cos(\frac{\gamma}{2})\hat{I}\otimes\hat{I}+i\sin(\frac{\gamma}{2})\hat{X}\otimes\hat{X}$
Step 3: $|\psi_{fin}>=\hat{J}^{\dagger}(\gamma)(\hat{A}\otimes\hat{B})\hat{J}(\gamma)|00>$, $\hat{A}$ and $\hat{B}$ represent Alice and Bob's strategies.



Payoff's in Prisoners dilemma and Chicken

	Bob : C	Bob : D
Alice : C	$(3,3)$	$(0,5)$
Alice : D	$(5,0)$	$(1,1)$

Prisoner's Dilemma

	Bob : C	Bob : D
Alice : C	$(3,3)$	$(1,4)$
Alice : D	$(4,1)$	$(0,0)$

Game of chicken

Figure : Classical Payoff's

Quantum Payoff's for Alice or Bob

$$< \$ >= \sum_{i,j=1}^{2} \$_{ij} | < ij|\psi_{fin} > |^2$$

Alice's and Bob's strategies

Cooperate: |0>, Defect: |1>

General strategy:

$$\hat{M}(\theta, \alpha, \beta) = \begin{bmatrix} \cos(\theta/2)\, e^{i\alpha} & i \sin(\theta/2)\, e^{i\beta} \\ i \sin(\theta/2)\, e^{-i\beta} & \cos(\theta/2)\, e^{-i\alpha} \end{bmatrix}$$

Strategy- Always cooperate: $\hat{M}(0, 0, 0)$

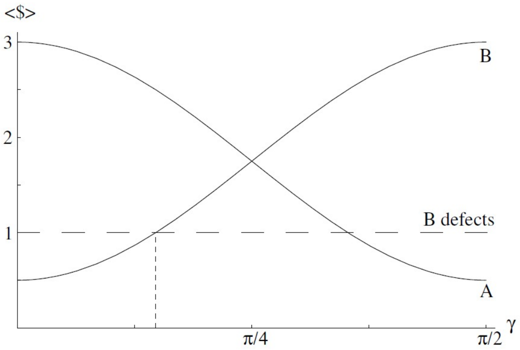
Strategy- Always defect: $\hat{M}(\pi, 0, 0)$

Alice- classical player: $\hat{M}(0, 0, 0)$ or $\hat{M}(\pi, 0, 0)$

Bob- quantum player: $\hat{M}(\theta, \alpha, \beta)$

Eisert's miracle move: $\hat{M}(\frac{\pi}{2}, \frac{\pi}{2}, 0) = \frac{i}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Payoff's for quantum prisoner's dilemma



classical Alice (A) playing "always defect"

Solid line: Bob plays Miracle move, Dashed line: Bob defects

Van Enk's criticism

PHYSICAL REVIEW A **66**, 024306 (2002)

Classical rules in quantum games

S. J. van Enk and R. Pike
Bell Labs, Lucent Technologies, 600-700 Mountain Avenue, Murray Hill, New Jersey 07974
(Received 11 April 2002; published 6 August 2002)

We consider two aspects of quantum game theory: the extent to which the quantum solution solves the original classical game, and to what extent the new solution can be obtained in a classical model.

Alice/Bob	C	D
C	$(3,3)$	$(0,5)$
D	$(5,0)$	$(1,1)$

(D,D) is the dominant equilibrium

Q is a superposition of C and D

Alice/Bob	C	D	Q
C	$(3,3)$	$(0,5)$	$(1,1)$
D	$(5,0)$	$(1,1)$	$(0,5)$
Q	$(1,1)$	$(5,0)$	$(3,3)$

Can classical strategies win against quantum strategies?

Motivation of our work

In the quantum penny flip game we see how the quantum player can outperform the classical player.
However, is the converse at all possible?

Quantum entangled penny flip game: Introduction

Introduction

A maximally entangled state of two qubits is the "penny" of the game.
It is shared by P and Q; each allowed to make moves on only the qubit in their possession.

Moves

Sequence of actions : $Q \rightarrow P \rightarrow Q$

Rules of winning

If the final state of the game is a maximally entangled state then Q wins, If it is a separable state then P wins.
If it is a non-maximally entangled state then its a draw.

Playing the Quantum entangled penny flip game

The classical pure strategy

In this case the classical player P is allowed only the pure strategy of either flipping or not flipping his qubits.
The initial state of the system:$|\psi>=\frac{1}{\sqrt{2}}(|10>-|01>)$

Moves

Sequence of actions : $Q \rightarrow P \rightarrow Q$

Q does a Hadamard

$H \otimes I|\psi>=\frac{1}{2}(|00>-|01>-|10>-|11>)$

The classical pure strategy

P's move: To flip or not to flip

$I \otimes X \frac{1}{2}(|00> -|01> -|10> -|11>) = \frac{1}{2}(|01> -|00> -|11> -|10>)$

OR

$I \otimes I \frac{1}{2}(|00> -|01> -|10> -|11>) = \frac{1}{2}(|00> -|01> -|10> -|11>)$

Q's move: H again

$H \otimes I \frac{1}{2}(|01> -|00> -|11> -|10>) = \frac{1}{\sqrt{2}}(|11> -|00>)$

OR

$H \otimes I \frac{1}{2}(|00> -|01> -|10> -|11>) = \frac{1}{\sqrt{2}}(|00> +|11>)$

In either case Q wins.

What did we learn?

Moral

In quantum entangled penny flip game, with one player having classical pure strategy, while the other player does quantum moves gives a definite win to quantum player.

Why is this important?

The game here is about whether player Q having all quantum strategies at his hand can keep the state maximally entangled, whereas P with classical moves can or cannot reduce/destroy the entanglement.

Algorithms

Strategy is similar to an algorithm: finite # of steps to solve a problem/win a game.

The classical mixed strategy

Defining mixed strategy

P can now flip or not flip with some probability "p".
A maximally entangled state of two qubits is the "penny" of the game.
It is shared by P and Q; each allowed to make moves on only the qubit in their possession.
Sequence of actions :$Q \rightarrow P \rightarrow Q$

Playing with classical mixed strategy

Initial state

The maximally entangled state "Penny":
$|\psi >= \frac{1}{\sqrt{2}}(|10 > -|01 >)$
In the form of density matrix:

$$\rho_0 = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Q's move

Q makes an unitary transformation on her part of the shared state.

$$U_{Q_1} = \begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix}.$$

The state after Q's move then is $\rho_1 = (U_{Q1} \otimes I)\rho_0(U_{Q1} \otimes I)^\dagger$.

Playing with classical mixed strategy

P's move

P now plays a mixed strategy, which entails flipping the state of his qubit with probability "p" or not flipping. The state after P's move then is: $\rho_2 = p(I \otimes X)\rho_1(I \otimes X)^\dagger + (1-p)(I \otimes I)\rho_1(I \otimes I)^\dagger$.

Q's final move

At the end Q makes her final move, which as before has to be an unitary transformation, it further could be same as her first move or different. Thus $U_{Q2} = \begin{bmatrix} \alpha & \beta^* \\ \beta & -\alpha^* \end{bmatrix}$. The state after this final move then is $\rho_3 = (U_{Q2} \otimes I)\rho_2(U_{Q2} \otimes I)^\dagger$.

Analysing the game

When Q's moves are Hadamard

To understand this case of P using mixed, lets analyse this case for Q using the familiar Hadamard transform in both steps 2 and 4. In this special case,

$$\rho_3 = \frac{1}{2}\begin{bmatrix} p & 0 & 0 & -p \\ 0 & 1-p & -1+p & 0 \\ 0 & -1+p & 1-p & 0 \\ -p & 0 & 0 & p \end{bmatrix}.$$

Is the final state entangled or separable?

To check the entanglement content of this final state we take recourse to an entanglement measure- Concurrence.
Concurrence for a two qubit density matrix ρ_3 is defined as follows- we first define a "spin-flipped" density matrix, γ as $(\sigma_y \otimes \sigma_y)\rho_3^*(\sigma_y \otimes \sigma_y)$. Then we calculate the square root of the eignevalues of the matrix $\rho_3\gamma$ (say λ_1,λ_2,λ_3,λ_4) in decreasing order. Then, Concurrence is :

$$\max\,(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$$

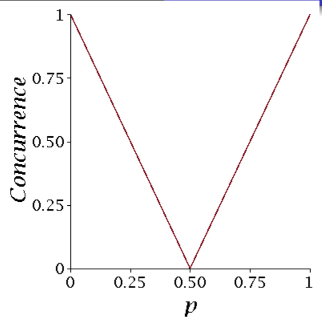


Figure : Concurrence vs p showing that entanglement vanishes at $p = 1/2$, so by P's classical moves entanglement is completely destroyed enabling him to win.

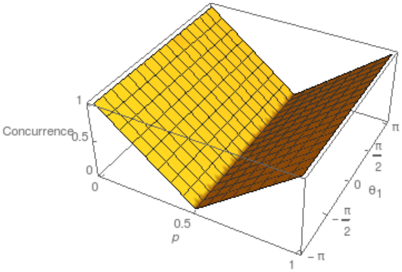
Classical random strategy wins against quantum strategy

Although the individual moves had no effect, a probabilistic move has enabled the classical player to win!!!.

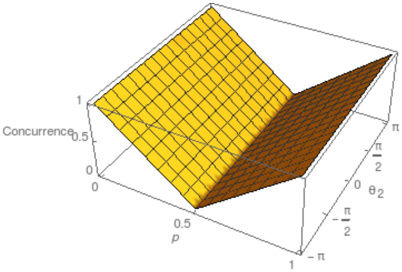
Quantum entangled penny flip game: General quantum strategy versus mixed classical strategy

What if the quantum player uses a general unitary and not just a Hadamard?
Further in successive turns he does not implement the same unitary, i.e., $U_{Q_1} \neq U_{Q_2}$

$$U_{Q_i} = \begin{bmatrix} \cos(\theta_i)e^{i\phi_i} & \sin(\theta_i)e^{i\phi_i'} \\ \sin(\theta_i)e^{-i\phi_i'} & -\cos(\theta_i)e^{-i\phi_i} \end{bmatrix}, i = 1, 2.$$



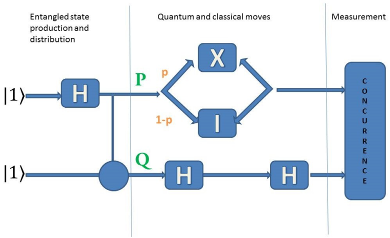
(a) Concurrence vs. θ_1, $\theta_2 = 0, \phi_1 = \pi/2, \phi_1' = 0, \phi_2 = \pi/2, \phi_2' = 0$



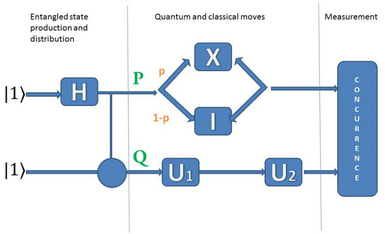
(b) Concurrence vs. θ_2, $\theta_1 = 0, \phi_1 = \pi/2, \phi_1' = 0, \phi_2 = \pi/2, \phi_2' = 0$

Figure : The Concurrence when quantum player plays a general unitary vs. classical players mixed strategy. The classical player always wins when $p = 1/2$, confirming that regardless of whether quantum player uses a Hadamard or any other unitary he always loses when classical player plays a mixed strategy of either flipping or not flipping with probability 50%.

Quantum circuit implementation



(a) Quantum player uses Hadamard.



(b) Quantum player uses a general unitary

Figure : The quantum circuit for the entangled penny flip game. M denotes measurement of entanglement content via concurrence.

Conclusion

Quantum entangled penny flip game

In a particular case where classical player uses a mixed strategy with p = "0.5", the quantum player indeed loses as opposed to the expected win for all possible unitaries!

Meyer's penny flip

Meyer showed that in the PQ penny flip if both players use Quantum strategies then there is no advantage. However, a player using a quantum strategy will win 100% of the time against a player using a classical strategy.

Perspective on quantum algorithms

Quantum algorithms have been shown to be more efficient than classical algorithms, for example Shor's algorithm. We in this work put forth a counter example which demonstrates that a particular classical algorithm can outwit the previously unbeatable quantum algorithm in the entangled quantum penny flip problem. On top of that the mixed strategy works against any possible unitary as we show by simulation on a strategy space for all possible parameters.

What we show

Quantum strategies are not(always) better than classical strategies.

N. Anand and Colin Benjamin, Quantum Information Processing, 14 (11), 4027-4038 (2015).

XOR games

- The simplest type of two-player games.
- Alice and Bob are two players playing a game with Charlie a referee.
- Charlie prepares $x, y \in \{0, 1\}$- x to Alice and y to Bob.
- Alice is to produce $a \in \{0, 1\}$, and Bob is to produce $b \in \{0, 1\}$.
- Alice and Bob are not permitted to communicate.
- Alice and Bob win the game if

$$a \oplus b = x \Lambda y \tag{1}$$

- Where $'\oplus'$ denotes the sum modulo 2 (the XOR gate) and Λ denotes the product (the AND gate).

XOR games

- Can Alice and Bob find a strategy that enables them to win the game every time, no matter how Charlie chooses the input bits?
- Let a_0, a_1 denote the value of Alice's output if her input is $x = 0, 1$ and let b_0, b_1 denote Bob's output if his input is $y = 0, 1$.
- For Alice and Bob to win for all possible inputs, their output bits must satisfy

$$a_0 \oplus b_0 = 0, \tag{2}$$
$$a_0 \oplus b_1 = 0, \tag{3}$$
$$a_1 \oplus b_0 = 0, \tag{4}$$
$$a_1 \oplus b_1 = 1. \tag{5}$$

XOR games: Shared randomness

Suppose that Charlie generates the input bits at random. Then there is a very simple strategy that enables Alice and Bob to win the game three times our of four: they always choose the output $a = b = 0$ so that they lose only if the input is $x = y = 1$. The CHSH inequality can be regarded as the statement that, if Alice and Bob share no entanglement, then there is no better strategy. We define random variables taking values $+1, -1$ as-

$$a = (-1)^{a_0}, a' = (-1)^{a_1}, \tag{6}$$

$$b = (-1)^{b_0}, b' = (-1)^{b_1}, \tag{7}$$

Then the CHSH inequality says that for any joint probability distribution governing $a, a_0, b, b_0 \in \{+1, -1\}$, the expectation values satisfy

$$\langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle \leq 2 \tag{8}$$

Shared randomness

If we denote by $p_{xy}$the probability that equation 2, 3, 4, 5 is satisfied when the input bits are (x, y), then

$$\langle ab \rangle = 2p_{00} - 1, \tag{9}$$
$$\langle ab' \rangle = 2p_{01} - 1, \tag{10}$$
$$\langle a'b \rangle = 2p_{10} - 1, \tag{11}$$
$$\langle a'b' \rangle = 1 - 2p_{11}; \tag{12}$$

for example $\langle ab \rangle = p_{00} - (1 - p_{00}) = 2p_{00} - 1$, because the value of ab is $+1$ when Alice and Bob win and -1 when they lose.

Shared randomness

The CHSH inequality equation 8 becomes

$$2(p_{00} + p_{01} + p_{10} + p11) - 4 \leq 2, \tag{13}$$

or

$$\langle p \rangle = \frac{1}{4}(p_{00} + p_{01} + p_{10} + p11) \leq \frac{3}{4} \tag{14}$$

where $\langle p \rangle$ denotes the probability of winning averaged over a uniform ensemble for the input bits. Thus, if the input bits are random, Alice and Bob cannot attain a probability of winning higher than $3/4$.

Shared Quantum entanglement

If Alice and Bob share quantum entanglement, they can devise a better strategy. Based on the value of her input bit, Alice decides to measure one of two Hermitian observables with eigenvalues $+1, -1$: a if x $= 0$ and a' if x$=$1. Similarly Bob measures b if y$=$0 and b' if y$=$1. Then the quantum mechanical expectation values of these observables satisfy

$$\langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle \leq 2\sqrt{2} \tag{15}$$

Shared Quantum entanglement

The probability that Alice and Bob win the game is constrained by

$$2(p_{00} + p_{01} + p_{10} + p11) - 4 \leq 2\sqrt{2}, \tag{16}$$

or

$$\langle p \rangle = \frac{1}{4}(p_{00} + p_{01} + p_{10} + p11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}} = 0.853 \tag{17}$$

Thus we have found that Alice and Bob can play the game more successfully with quantum entanglement than without it. At least for this purpose, shared quantum entanglement is a more powerful resource than shared classical randomness.

XOR: CHSH games

- Binary games are games in which Alice and Bob's answer are bits: $A = B = \{0, 1\}$.
- XOR games are binary games that are further restricted in that the winning condition depends only on $a \oplus b$ and not a and b independently.
- The CHSH games are example of XOR games.

XOR: CHSH games

- For shared randomness the probability of winning is $\leq \frac{3}{4} = 0.75$.
- For shared quantum entanglement the winning probability is $\leq \cos^2 \frac{\pi}{8} = 0.853$
- Without shared entanglement the maximum probability of winning is $= 75\%$
- With shared entanglement the maximum probability for winning is $= 85\%$

CHSH game: classical version

- Defining one of the optimal strategy i.e. they both produce 0.
- Three out of four winning conditions are satisfied.
- Overall probability to win the game $= \frac{3}{4}$

CHSH game: quantum version

- Let Alice and Bob share an entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{18}$$

- A_a^s and B_b^t be the set of operators for Alice and Bob respectively and are defined as follow:

$$\begin{aligned} A_0^a &= |\phi_a(0)\rangle\langle\phi_a(0)|, \\ A_1^a &= |\phi_a(\pi/4)\rangle\langle\phi_a(\pi/4)|, \\ B_0^b &= |\phi_b(\pi/8)\rangle\langle\phi_b(/pi/8)|, \\ B_1^b &= |\phi_b(-\pi/8)\rangle\langle\phi_b(-/pi/8)|. \end{aligned} \tag{19}$$

CHSH game: quantum version

- such that:

$$|\phi_0(\theta)\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle,$$
$$|\phi_1(\theta)\rangle = -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle. \tag{20}$$

- Now the probability that on question s, t Alice answer a and Bob answers b is given by :

$$P(a, b|s, t) = \langle\Psi|A_a^s \otimes B_b^t|\Psi\rangle \tag{21}$$

CHSH game: quantum version

$$
\begin{aligned}
A_0^0 = |\phi_0(0)\rangle\langle\phi_0(0)| &= (\cos(0)|0\rangle + \sin(0)|0\rangle)(\cos(0)\langle 0| + \sin(0)\langle 0|) \\
\Rightarrow A_0^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
A_1^0 = |\phi_1(0)\rangle\langle\phi_1(0)| &= (-\sin(0)|0\rangle + \cos(0)|0\rangle)(-\sin(0)\langle 0| + \cos(0)\langle 0|) \\
\Rightarrow A_0^1 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
A_0^1 = |\phi_0(\pi/4)\rangle\langle\phi_0(\pi/4)| &= (\cos(\pi/4)|0\rangle + \sin(\pi/4)|1\rangle)(\cos(\pi/4)\langle 0| + \sin(\pi/4)\langle 1|) \\
\Rightarrow A_1^0 &= \begin{pmatrix} \cos^2(\pi/4) & \sin(\pi/4)\cos(\pi/4) \\ \sin(\pi/4)\cos(\pi/4) & \sin^2(\pi/4) \end{pmatrix} \\
A_1^1 = |\phi_1(\pi/4)\rangle\langle\phi_1(\pi/4)| &= (-sin(\pi/4)|0\rangle + cos(\pi/4)|1\rangle)(-sin(\pi/4)\langle 0| + cos(\pi/4)\langle 1|) \\
\Rightarrow A_1^1 &= \begin{pmatrix} \sin^2(\pi/4) & -\sin(\pi/4)\cos(\pi/4) \\ -\sin(\pi/4)\cos(\pi/4) & \cos^2(\pi/4) \end{pmatrix} \\
B_0^0 = |\phi_0(\pi/8)\rangle\langle\phi_0(\pi/8)| &= (\cos(\pi/8)|0\rangle + \sin(/pi/8)|0\rangle)(cos(/pi/8)\langle 0| + \sin(/pi/8)\langle 0|) \\
\Rightarrow A_0^0 &= \begin{pmatrix} \cos^2(\pi/4) & \sin(\pi/4)\cos(\pi/4) \\ \sin(\pi/4)\cos(\pi/4) & \sin^2(\pi/4) \end{pmatrix} \\
B_1^0 = |\phi_1(/pi/8)\rangle\langle\phi_1(/pi/8)| &= (-\sin(\pi/8)|0\rangle + \cos(\pi/8)|0\rangle)(-\sin(/pi/8)\langle 0| + \cos(\pi/8)\langle 0|) \\
\Rightarrow A_0^1 &= \begin{pmatrix} \sin^2(\pi/4) & \sin(\pi/4)\cos(\pi/4) \\ \sin(\pi/4)\cos(\pi/4) & \cos^2(\pi/4) \end{pmatrix} \\
B_0^1 = |\phi_0(-\pi/8)\rangle\langle\phi_0(-\pi/8)| &= (\cos(-\pi/8)|0\rangle + \sin(-\pi/8)|1\rangle)(\cos(-\pi/8)\langle 0| + \sin(-\pi/8)\langle 1|) \\
\Rightarrow B_1^0 &= \begin{pmatrix} \cos^2(\pi/4) & -\sin(\pi/4)\cos(\pi/4) \\ -\sin(\pi/4)\cos(\pi/4) & \sin^2(\pi/4) \end{pmatrix} \\
B_1^1 = |\phi_1(-\pi/8)\rangle\langle\phi_1(-\pi/8)| &= (-\sin(-\pi/8)|0\rangle + \cos(-\pi/8)|1\rangle)(-\sin(-\pi/8)\langle 0| + \cos(-\pi/8)\langle 1|) \\
\Rightarrow B_1^1 &= \begin{pmatrix} \sin^2(\pi/4) & -\sin(\pi/4)\cos(\pi/4) \\ -\sin(\pi/4)\cos(\pi/4) & \cos^2(\pi/4) \end{pmatrix}
\end{aligned}
$$

CHSH game: quantum version

- Given our particular choice of Ψ, we have $\langle\Psi|A_a^s \otimes B_b^t|\Psi\rangle = \frac{1}{2}Tr(A^T)B$ for arbitrary matrices A and B.
- on doing calculations it is easy to show that the probability of wining is given as $\cos^2\frac{\pi}{8} = 0.853$

"Bell nonlocality", N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).

Parrondo's Paradox

Two games in which the probability of losing is greater than probability of winning, if played in a particular sequence makes the probability of winning greater than losing.
Example:
Game 1: If on a particular the amount of money that A has is even, then he gains Rs. 3 and if its odd then he loses Rs.5
Game 2: Every turn A would lose Rs.1

Losing or winning?

Game 1: He would surely lose. Suppose he has Rs. 200, then after 1 st turn he would have Rs. 203, then the turn after he would have Rs. 198. So finally he would lose all his money.
Game 2: Surely he would lose.

Parrondo's paradox

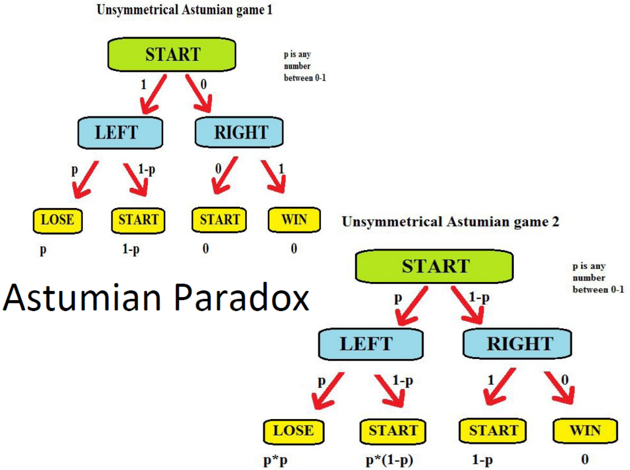
Suppose A plays the games in a sequence like

Game1 → Game 2 → Game 1 → ...

Suppose he starts with Rs. 200. Then after 1st turn he would have Rs.203, then after next turn he would have Rs. 202, then the turn after he would have Rs. 205 ...

So finally he would never lose.

Astumian's paradox



The probability to lose in each game is more than win. But suppose the game is played by tossing a faircoin at each turn such that if its head then game 1 is played and if its tails then game 2 is played.

Fair coin is tossed to decide the game

The possibilities	Probability to lose	Probability to win
Head → Head	p/4	0
Head → Tail	p/4	0
Tail → Head	p*p/4	(1-p)/4
Tail → Tail	p*p/4	0
Total	p(1+p)/2	(1-p)/4

Simple games to illustrate Parrondos paradox, H. Martin, H. Christian von Baeyer, Am. J. Phys. 72, 710 (2004).

History dependent Parrondo's game

To introduce the quantum version of Parrondo's game we have to first understand the history dependent classical Parrondo's game. The construction of the game is the following-

Game A:It involves tossing a weighted coin 1 with probability $p_w = 0.5 - \epsilon, 0 < \epsilon \ll 1$ for winning and $p_l = 1 - p_w$ for losing.

Game B:In this game 3 coins are used. One of them is tossed based on the outcome of the previous game.

The choice of the coin to be tossed at nth game.

Game_{n-2}	Game_{n-1}	Coin chosen
Loss	Loss	2
Loss	Win	3
Win	Loss	3
Win	Win	4

Coin 2: $p_w = .9 - \varepsilon$, Coin 3: $p_w = .25 - \varepsilon$, Coin 4: $p_w = .7 - \varepsilon$.

Evidently, coin 3 is tossed more often than the other coins, and hence B is a losing game.

In the Parrondos games both A and B are losing games for small positive values of ϵ. However, simulation of the games have predicted that switching between the losing games, e.g., playing two times A, two times B, two times A, and so on results in winning, i.e., a player can play the two losing games A and B in such an order to realize a winning expectation.

Quantum Parrondo's game

The coin tossing game can be quantized by an SU(2) operation on a qubit. A physical system may be a collection of polarized photons with $|0>$ and $|1>$ representing horizontal and vertical polarizations respectively.
An arbitrary SU(2) operation on a qubit is expressed as:

$$\begin{aligned} \widehat{A}(\theta,\gamma,\delta) &= \widehat{P}(\gamma)\widehat{R}(\theta)\widehat{P}(\delta) \\ &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\gamma+\delta)/2}\cos\theta & -\mathrm{e}^{-\mathrm{i}(\gamma-\delta)/2}\sin\theta \\ \mathrm{e}^{\mathrm{i}(\gamma-\delta)/2}\cos\theta & \mathrm{e}^{\mathrm{i}(\gamma+\delta)/2}\cos\theta \end{pmatrix} \end{aligned}$$

where $\theta \in [-\pi, \pi]$ and $\gamma, \delta \in [0, 2\pi]$

This is the quantum analogue of the game A− a single toss of a biased coin.

Quantum Parrondo's Games, A. P. Flitney, J. Ng, D. Abbott, Physica A, 314 (2002) 35-42.

Game B consists of four SU(2) operations, each of the form of $\widehat{A}$:

$$\widehat{B} = \begin{pmatrix} A(\phi_1,\alpha_1,\beta_1) & 0 & 0 & 0 \\ 0 & A(\phi_2,\alpha_2,\beta_2) & 0 & 0 \\ 0 & 0 & A(\phi_3,\alpha_3,\beta_3) & 0 \\ 0 & 0 & 0 & A(\phi_4,\alpha_4,\beta_4) \end{pmatrix}$$

This acts on the state $|\psi(t-2)\rangle \otimes |\psi(t-1)\rangle \otimes |i\rangle$,
where $|\psi(t-1)\rangle$ and $|\psi(t-2)\rangle$ represent the results of the two previous games. $|i\rangle$ is the qubit's initial state. We write $\widehat{B}|q_1q_2q_3\rangle = |q_1q_2b\rangle$,

where $q_i \in \{0,1\}$ and b is the output of the game B.

The result of n successive games of B is found by:

$$|\psi_f\rangle = \left(\widehat{I}^{n-1} \otimes \widehat{B}\right)\left(\widehat{I}^{n-2} \otimes \widehat{B} \otimes \widehat{I}\right)\cdots\left(\widehat{B} \otimes \widehat{I}^{n-1}\right)|\psi_i\rangle ,$$

where $|\psi_i\rangle$ is the initial state of $n+2$ qubits.

Suppose a player plays AAB n times. Then

$$
\begin{aligned}
|\psi_f\rangle &= \left\{\widehat{I}^{3n-3} \otimes \left[\widehat{B}\left(\widehat{A} \otimes \widehat{A} \otimes \widehat{I}\right)\right]\right\} \\
&\quad \times \left\{\widehat{I}^{3n-6} \otimes \left[\widehat{B}\left(\widehat{A} \otimes \widehat{A} \otimes \widehat{I}\right)\right] \otimes \widehat{I}^{3}\right\} \\
&\quad \cdots \left\{\left[\widehat{B}\left(\widehat{A} \otimes \widehat{A} \otimes \widehat{I}\right)\right] \widehat{I}^{3n-3}\right\} |\psi_i\rangle \\
&= \widehat{G}^n |\psi_i\rangle \ ,
\end{aligned}
$$

where $\widehat{G}^n = \widehat{B}\left(\widehat{A} \otimes \widehat{A} \otimes \widehat{I}\right)$ and $|\psi_i\rangle$ is an initial state of $3n$ qubits.

The classical game can be reproduced by $|\psi_i\rangle = |00\cdots 0\rangle$. Suppose $|\psi_i\rangle$ is the entangled state

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}\left(|00\cdots 0\rangle + |11\cdots 1\rangle\right) \ .$$

Let the payoff for a $|1\rangle$ state be 1 and for a $|0\rangle$ state be -1. Since quantum mechanics is a probabilistic theory $\langle$ payoff $\rangle$ is important and is given by

$$\langle \text{ payoff } \rangle = |\$\rangle = \sum_{j=0}^{n} \left[(2j-n) \sum_{j'} |\langle \psi_j^{j'} | \psi_f \rangle|^2 \right]$$

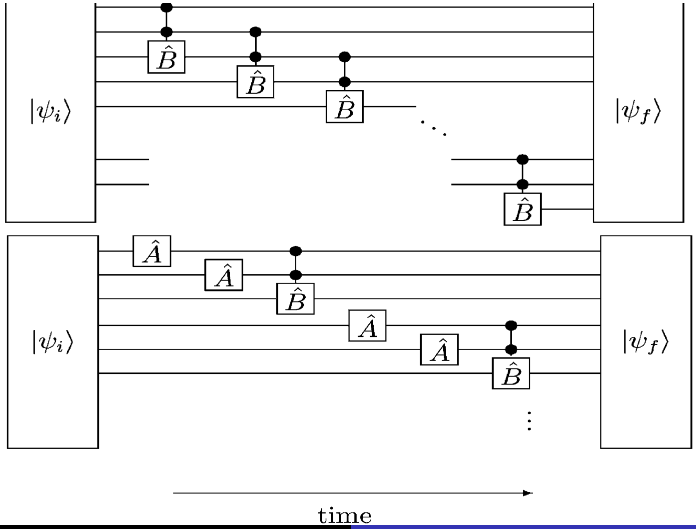
the second summation is over all basis states $\langle \psi_j^{j'}|$ with $n-j$ zero's and j ones.

For the sequence AAB with $|\psi_i\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ we have

$$\begin{aligned} \langle \$_{\mathrm{AAB}} \rangle &= \frac{1}{2} \cos 2\theta \left(\cos 2\phi_4 - \cos 2\phi_1 \right) \\ &+ \frac{1}{4} \sin^2 2\theta \left[\cos(2\delta + \beta_1) \sin 2\phi_1 - \cos(2\delta + \beta_2) \sin 2\phi_2 \right. \\ &\left. - \cos(2\delta + \beta_3) \sin 2\phi_3 + \cos(2\delta + \beta_4) \sin 2\phi_4 \right] . \end{aligned}$$

The maximum payoff is for $\beta_1 = \beta_4 = -2\delta$ and $\beta_2 = \beta_3 = \pi - 2\delta$. The result is minimum for $\beta_1 = \beta_4 = \pi - 2\delta$ and $\beta_2 = \beta_3 = -2\delta$. Observe that the values of ϕ_i's are irrelevant. $\langle \$_{\mathrm{AAB}} \rangle$ varies between $-0.812 + 0.03\epsilon$ and $0.812 + 0.24\epsilon$.

The classical payoff is $1/60 - 28\epsilon/15$.


$|\psi_i\rangle$
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Game theory in Greek crisis

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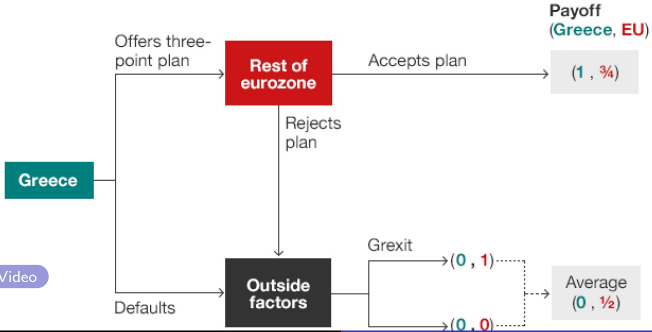
Can game theory explain the Greek debt crisis?

Greek crisis decision tree



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