Nonclassical properties of single systems

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Overview

- Nonclassicality of monopartite systems in finite-dimensional operational theories treated as a composite of individual properties.
- Classical theory: One where these properties aggregate independently.
- Nonclassicality: One where is "tension" in aggregating properties.
- Two kinematic aspects of tension: *obstruction* and *frustration*.
- Obstruction: Intuitively "A AND B" replaced by "A OR B" $\Rightarrow$ uncertainty and non-simpliciality of the state space in the convex framework; Latter $\Rightarrow$ Measurement disturbance, non-comeasurability, non-commutativity, multiple pure-state decomposability, no-cloning, cryptographic security and intrinsic randomness.
- Lack of transitivity of non-obstruction can lead to logical conflict, which is resolved through frustration $\Rightarrow$ contextuality.

Introduction

- Nonclassical properties of QM: Uncertainty, no-cloning, complementarity, $\cdots$. In theoryspace, are these special to QM or is their absence special to classical mechanics (CM)?
- Generalized probability theory (GPT) or the convex framework are among recent approaches to try to answer such questions. In such approaches[6], typically a multipartite system is considered which satisfies nonlocality and no-signaling as axioms.
- In our approach: (I) Only single (monopartite) systems considered. (II) We are motivated to understand why QM is somehow "natural". What is the most elementary surprise about QM? Getting this right gives us "the right ignorance", which will motivate fruitful questions to understand physical reality efficiently!

[6] Masanes, Acin and Gisin 2006; Oppenheim and Wehner 2010; Banik, Gazi, Ghosh & Kar 2013

Operational framework: GCT

- Our approach is "operational" in that we deal with systems, states and measurement probabilities. To each system, we associate convex set Σ of all possible states.
- A state $\equiv$ list of co-measurable properties and their outcome probabilities. Each entry in a row is an *effect* and a row represents a (possibly joint) measurement.
- Here: Σ a polytope, given as convex hull of finite pure points. Thus we have a discrete theory. Transformations $\equiv$ affine maps $T : \Sigma \to \Sigma'$.

Monopartite systems

- Consider theory $\mathcal{T}$ featuring two properties A and B of each of dimension d. "Property space" $\Sigma_A = \mathrm{span}(a_j)$ and $\Sigma_B = \mathrm{span}(a_k)$.
- Classical $\mathcal{T}$: A and B aggregate independently. Any conjunction a_j AND b_k valid state in $\mathcal{T}$. Therefore $\dim(\Sigma_{AB})$ is $D_\otimes \equiv d^2 - 1$. Moreover, each state $a_j \otimes b_k$ is a pure state of the system, so that

$$\Sigma_{AB} \equiv \Sigma_A \otimes_{\min} \Sigma_B, \tag{1}$$

  i.e., the minimal tensor product (Barnum et al. 2007) of the individual property spaces Σ_A and Σ_B. Thus Σ_{AB} is the convex hull of these d^2 independent extreme points: hence a $(d^2 - 1)$-simplex.
- Nonclassiclity arises when properties A and B don't aggregate independently. This lack of independence is attributed to a "tension" that causes Σ_{AB} to depart from the form (1). This tension is kinematic, with dynamic implications.
- Two types of tension: obstruction and frustration.

Kinematic obstruction

- Given properties A and B, in the most dramatic version of tension, "a_j OR b_k" replaces "a_j AND b_k" for elements of Σ_{AB}. We replace tensor product (1) by:

$$\Sigma_{AB} \equiv \Sigma_A \oplus \Sigma_B. \tag{2}$$

  $\dim(\Sigma_{AB}) = D_\oplus \equiv 2 \times (d-1)$.
- Pure states are the d "eigenstates of A" (states with definite a_j and indefinite value of B), and similarly d eigenstates of B. Indefiniteness realizable by requiring eigenstates of A to be of form $|a'_j) \equiv \sum_k \alpha_{jk}|a_j \otimes b_k)$, and similarly for B. Thus the number of pure states is $P_\oplus \equiv 2d$.
- That $P_\oplus > D_\oplus + 1$ makes Σ_{AB} non-simplicial. Typically, the extra constraint satisfied by the pure states: $\sum_j |a'_j) = \sum_k |b'_k)$. Further, $\mathcal{T}_\oplus$ has uncertainty (A and B can't be together value-definite.)
- Aggregating n pairwise mutually obstructive properties: $D_\oplus = nd - n$ and $P_\oplus = nd$. More generally, pure states can be added to $\Sigma_\mathcal{T}$ keeping dimension at $D_\oplus$. Such a theory is denoted $\mathcal{T}^*_\oplus$.

Non-comeasurability and measurement disturbance

- States in theories $\mathcal{T}_\oplus$ or $\mathcal{T}^*_\oplus$, have geometric representation in $\mathcal{T}_\otimes$, which can serve an ontological model, implying an underlying joint distribution (JD) exists for obstructive pair A and B. They are non-simplexes within the minimal tensor product.
- However, they will lack co-measurability and possess measurement disturbance.
- While nonsimpliciality and uncertainty follow "naturally" from obstruction, uncertainty is not necessary for non-comeasurability and measurement disturbance.
- A simple such theory is "gdit theory", where $D_\mathcal{T} = D_\oplus$ while $P_\mathcal{T} = P_\otimes$. But this theory is, arguably, not "natural" in that its dimensionality reflects disjunction (A OR B), but its pure states reflect conjunction (A AND B).

Non-comeasurability from non-simpliciality

- (Theorem) Given d-valued properties A and B that are mutually obstructive in theory $\mathcal{T}_\oplus$ as defined above, characterized by a non-simplicial state space Σ_{AB}, the properties A and B will not be jointly measurable in $\mathcal{T}_\oplus$.
- Illustrative example: Two properties "Color" and "Size", taking values $\texttt{blue}$ / $\texttt{red}$ and $\texttt{big}$ / $\texttt{small}$.
- Classical theory $\mathcal{T}_\otimes$: the state space $\Sigma_{CS} = \Sigma_{\rm color} \otimes_{\rm min} \Sigma_{\rm size} =$ convex hull of $|\texttt{blue} * \texttt{big}) \equiv (1, 0, 0, 0)$, $|\texttt{blue} * \texttt{small}) \equiv (0, 1, 0, 0)$, $|\texttt{red} * \texttt{big}) \equiv (0, 0, 1, 0)$ and $|\texttt{red} * \texttt{small}) \equiv (0, 0, 0, 1)$.
  4 vertices of the 3-simplex, a pyramid.

- A nonclassical theory $\mathcal{T}_{\oplus}$, whose extreme points are

$$|\texttt{blue}') \equiv \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0.25 & 0.75 \\ \hline \end{array}; \quad |\texttt{red}') \equiv \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0.75 & 0.25 \\ \hline \end{array},$$
$$|\texttt{big}') \equiv \begin{array}{|c|c|}\hline 0.25 & 0.75 \\ \hline 1 & 0 \\ \hline \end{array}; \quad |\texttt{small}') \equiv \begin{array}{|c|c|}\hline 0.75 & 0.25 \\ \hline 0 & 1 \\ \hline \end{array}. \qquad (3)$$

  In representation of state as "list" of probabilitiy distributions (first row $\equiv$ blue/red, and second row $\equiv$ big/small):

- Σ_{AB} non-simpliciality seen by noting that the pure points are not linearly independent, in that

$$\frac{1}{2}(|\texttt{blue}') + |\texttt{red}')) = \frac{1}{2}(|\texttt{big}') + |\texttt{small}')). \qquad (4)$$

- The state space $\mathcal{T}_{\otimes}$ can be used to provide an ontological model, and hence JD, for properties A and B. We have:
  $|\texttt{blue}') \equiv (0.25, 0.75, 0, 0)$, $|\texttt{red}') \equiv (0, 0, 0.25, 0.75)$,
  $|\texttt{big}') \equiv (0.25, , 0, 0.75, 0)$, and $|\texttt{small}') \equiv (0, 0.75, 0.25)$.

- Suppose $\exists$ color-size joint observable M_{CS}. Marginalizations require:

$$\begin{aligned} M_{CS}(0,0|\texttt{blue}') + M_{CS}(0,1|\texttt{blue}') &= 1 \\ M_{CS}(1,0|\texttt{blue}') + M_{CS}(1,1|\texttt{blue}') &= 0 \\ M_{CS}(0,0|\texttt{blue}') + M_{CS}(1,0|\texttt{blue}') &= 0.25 \\ M_{CS}(0,1|\texttt{blue}') + M_{CS}(1,1|\texttt{blue}') &= 0.75. \end{aligned} \tag{5}$$

from which it follows that $M_{CS}(1,0|\texttt{blue}') = M_{CS}(1,1|\texttt{blue}') = 0$ whereas $M_{CS}(0,0|\texttt{blue}') = 0.25$ and $M_{CS}(0,1|\texttt{blue}') = 0.75$.

- Proceeding thus, one finds

$$\begin{aligned} M_{CS}(1,0|\texttt{blue}') = 0; &\qquad M_{CS}(1,0|\texttt{small}') = 0 \\ M_{CS}(1,0|\texttt{red}') = 0.25; &\qquad M_{CS}(1,0|\texttt{big}') = 0.75. \end{aligned} \tag{6}$$

To satisfy the non-simpliciality condition, we must have $M_{CS}\left(j,k\,\middle|\,\frac{1}{2}(|\texttt{blue}') + |\texttt{red}'))\right) = M_{CS}\left(j,k\,\middle|\,\frac{1}{2}(|\texttt{big}') + |\texttt{small}'))\right)$,

- This is not satisfied by, for example, setting $(j,k) \equiv (1,0)$, since the lhs yields $0.25/2$, whereas the rhs yields $0.75/2$.
- Uncertainty not necessary. Gdit theory also exhibits non-comeasurability.

Measurement disturbance from non-simpliciality

- Theory $\mathcal{T}$ lacks no-cloning iff $\Sigma_{\mathcal{T}}$ forms a simplex with vertices given by one-shot distinguishable states (Barnum et al. 2007).
- That non-simpliciality entails measurement disturbance is similar, except we don't invoke a second system nor no-signaling.
- (Theorm) A theory $\mathcal{T}$ lacks measurement disturbance iff $\Sigma_{\mathcal{T}}$ space is a simplex with vertices given by states that are one-shot distinguishable.
- Intuitively, if $\Sigma_{\mathcal{T}}$ is such a simplex, then any unknown pure state can always be identified, and reconstructed even if there is some back-action. So, effectively no disturbance. Conversely, if there is no measurement disturbance, all properties can be measured repeatedly infinitely many times, and all pure states can be distinguished. Thus there are no indistinguishable mixtures of the type $\sum_{i=1}^{m} p_i\psi_i = \sum_{j=1}^{n} q_j\phi_j$, Hence all P pure states are linearly independent and $\Sigma_{\mathcal{T}}$ is a $(P-1)$-simplex.

An example

- Suppose we start with the equal-weighted Color mixture

$$M \equiv \frac{1}{2}(|\texttt{blue}') + |\texttt{red}')),$$

  which is identical to the equal-weighted Size mixture. Under measurement of Color, this returns blue/red with equal probability and leaves the actual state, and hence the mixture, undisturbed.
- If Size is measured, then per (3), both sizes are equiprobable, while the state after disturbance is:

$$\frac{1}{2}\left(\frac{1}{4}|\texttt{small}') + \frac{3}{4}|\texttt{big}')\right) + \frac{1}{2}\left(\frac{3}{4}|\texttt{small}') + \frac{1}{4}|\texttt{big}')\right) = M,$$

  meaning that the same mixture is returned.
- Any POVM is this theory (tossing a loaded coin and measuring Color or Size accoring to coin outcome) also does not help.

Kinematic frustration

- If pairwise unobstructiveness were transitive, then all properties $A, B, C, \cdots$ would form an equivalence class with any two properties in the same element of the class being unobstructive, and those in distinct elements being obstructive.
- Failure of transitivity of unobstructiveness leads to "frustration", which seems natural in a world where obstructiveness exists!! (In real life, if friendship were transitive, "saas-bahu" serials would not exist, and life would be boring !!!)
- E.g., (A, B) and (B, C) may be mutually unobstructive but (A, C) may not be. This sets up a potential logical contradition in value assignments. Nature responds by preventing A, B, C from being 3-way co-measurable.
- Thereby we only have states that fall outside $A \otimes_{\min} B \otimes_{\min} C$. By contrast, obstruction leads to non-simpliciality within $A \otimes_{\min} B \otimes_{\min} C$. This underscores the two fundamental ways in which nonclassicality enters the operational theory.

List of fiducial pairs

- Consider 5 binary properties A, B, C, D, E arranged in a cycle, with consecutive pairs co-measurable and observed to produce anticorrelated outcomes. Four deterministic assignments with 1-bit context dependence (the deterministic KCBS boxes), listing "fiducial pairs" and their value assignments:

Input	$\mathcal{Q}_1^A$	$\mathcal{Q}_2^A$	$\mathcal{Q}_1^B$	$\mathcal{Q}_2^B$
AB	01	10	01	10
BC	10	01	01	10
CD	01	10	10	01
DE	10	01	01	10
EA	01	10	10	01

(7)

Lack of JD follows from noting contradictory value assignments to A in $\mathcal{Q}_j^A$ and to B in $\mathcal{Q}_j^B$, and so on.

- Another way of expressing this: JD would imply a probability distribution over 2^5 strings $ABCDE$. But no string can exhibit the required pairwise anti-correlation, which is equivalent to violating the KCBS inequality:

$$\langle AB\rangle + \langle BC\rangle + \langle CD\rangle + \langle DE\rangle + \langle EA\rangle \geq -3. \tag{8}$$

- Lack of JD also means that such highly anticorrelated states lie outside the minimal tensor product $\bigotimes_{\min\ j} \Sigma_j$ ($j \in \{A, B, C, D, E\}$).
- The maximal tensor product $\Sigma_A \otimes_{\max} \Sigma_B$ can be considered as the set of non-signaling states that assigns a valid probability distribution to product properties. Now $\mathcal{Q}_j^A$ has a $A/BCDE$ signaling.
- This "Gleason signaling" can be cancelled by mixing with its anti-box: $\frac{1}{2}(\mathcal{Q}_1^A + \mathcal{Q}_2^A)$ is non-signaling. Thus it lies outside the minimal tensor product but within the maximal one. In this sense, contextual states are "states with entangled properties".
- The presence of obstructive pairs such as $(A, C), (B, D)$ etc. makes it non-simplicial and introduces randomness. Otherwise we would simply have (a mixture of) classical signaling correlations.

Contextuality/indeterminism complementarity

- "Gleason signal" in operational theory:

$$S_G \equiv \max_j \left| P(\xi_j = 0|\xi_j\xi_{j-1}) - P(\xi_j = 0|\xi_j\xi_{j+1}) \right|, \tag{9}$$

  Amount of context dependence.

- Intrinsic randomness:

$$I \equiv \min_j \left(P(0|\xi_j), P(1|\xi_j) \right). \tag{10}$$

  One can show that the mixture of 10 $\mathcal{Q}_j^K$ satisfies the complementarity:

$$S_G + 2I \geq 1. \tag{11}$$

  Contextuality equivalent of complementarity in the spatial scenario (Kar et al. 2011; Hall 2010; Aravinda & RS 2013)

Non-occurence of quantum "overprotective seer" correlations

- In Specker's "overprotective seer" (OS) correlations (Liang, Spekkens and Wiseman 2011), A, B and C are three two-valued properties, such that any two can be measured, and the outcomes will be anticorrelated with equal probability.
- No JD over properties ABC exists. If it did, it must be probability distribution of the 8 three-bit string ABC $\{0, 1\}^3$. Anticorrelation on the first two bits implies ABC is of pattern 01? or 10?. Anticorrelation on BC restricts this to the two possibilities 010 and 101, but in this case AC will not be anticorrelated.
- Any correlation explainable via JD has to satify the inequality

$$\langle AB\rangle + \langle BC\rangle + \langle CA\rangle \geq -1, \tag{12}$$

since, as noted above, at most only two pairs can be anticorrelated in a noncontextual way.

- The following deterministic 1-bit contextual "OS boxes" violate (12), by reaching algebraic maximum of -3:

Input	$\mathcal{Q}_1^A$	$\mathcal{Q}_2^A$	$\mathcal{Q}_1^B$	$\mathcal{Q}_2^B$	$\mathcal{Q}_1^C$	$\mathcal{Q}_2^C$
AB	01	10	01	10	01	10
BC	10	01	01	10	10	01
CA	01	10	10	10	10	01

. (13)

- The first pair of OS box and antibox are context dependence at A, the second pair at B and the third pair at C.
- In an operational theory that is OS contextual and Gleason noncontextual (i.e., probabilities are context-independent, $S_G = 0$), pure states that violate (12) must contain only equally weighted contributions of $\mathcal{Q}_0^K$ and $\mathcal{Q}_1^K$ in Eq. (13).

- Ineq. (12) is not violated in QM, where, if three projective measurements are pairwise compatible, then all three are jointly measurable.
- A noisy version of Ineq. (12), which is the Liang-Spekkens-Wiseman (LWS) inequality, can be violated by a suitable choice of POVM's (Kunjwal & Ghosh 2014).
- Question remains: what is the foundational (as against algebraic) reason that OS correlations (12) cannot be violated in QM?
- Our approach suggests an answer: Frustration is ultimately due to– and thus in our approach, implies– obstruction. In OS scenario, all pairs are co-measurable and hence non-obstructive. Thus, the physical basis for frustration does not arise.
- Thus we require the 5-property KCBS scenario as the simplest instance for demonstrating contextuality of this type.

Thank you!