Quantum simulations using split-step quantum walks

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Aim of the talk

1. Discretization of quantum field theories in the era of QIT/QIP
2. Artificial synthesis of topological insulators

using discrete-time quantum walks

Outline

1. Discretization of quantum field theories : need and approaches
   - Quantum Cellular Automaton (QCA) and Dirac Cellular Automaton (DCA)
   - Discrete-time quantum walk and Dirac Hamiltonian (Dirac Equation)
   - Split-step quantum walk and DCA
     - Zitterbewegung oscillations
     - Entanglement spectrum

   arXiv:1509.08851 (with Arindam Mallick)

2. Artificial synthesis of topological insulators
   - Topological quantum walks and localized states
     - Two split-step
     - Four split-step
   - Entanglement spectrum of topological quantum walks and localized states

   arXiv:1502.00436 (with H. Obuse & T. Busch)

Discretization of space and time

- Early Proposal to simplify the computation of field theories
  - Divisibility of Space and Time., Yukawa, H. Atomistics and the Prog. Theor. Phys. Suppl. 37 and 38, 512 (1966)
  - Quantum field theory on discrete space-time, Yamamoto, H., Phys. Rev. D 30 1127 (1984)
- Discretization of Dirac equation describing the relativistic motion of a spin 1/2 particle (one prominent example)
  - Confinement of quarks, Wilson, K. G., Phys. Rev. D 10, 2445 (1974)
  - Weyl, Dirac, and Maxwell equations on a lattice as unitary cellular automata, Bialynicki-Birula, I., Phys. Rev. D 49, 6920 (1994)
- Lattice guage theories
  - An introduction to lattice gauge theory and spin systems, Kogut, J. B., Rev. Mod. Phys. 51, 659 (1979)
- Quantum cellular automaton and quantum lattice gas
  - From quantum cellular automata to quantum lattice gases, Meyer, D. A. J., Stat. Phys. 85, 551 (1996)
  - The Feynman path integral for the Dirac equation, Riazanov, G. V., Sov. Phys. JETP 6 1107-1113 (1958)

Quantum Cellular automaton and Dirac Cellular Automaton

- Lattice gauge theory

Evolution is described by the unitary operator which is an exponential of an Hamiltonian involving the whole system at a time

- Quantum Cellular Automaton

  - Evolution (update) rule of the system is described by a local unitary operators each involving few subsystems.
  - It can be regarded as a microscopic mechanism for an emergent quantum fields and as a framework to unify a hypothetical Planck scale with the usual Fermi scale of the high-energy physics
  - The QCA which is not derivable by quantizing classical theory can also be used as a framework for quantum theory of gravity

- Dirac Cellular Automaton

Free field QCA models emerging to Dirac Hamiltonian (DH) for spinor with non-zero mass and massless particles.

From discrete-time quantum walk to relativistic equations
:Klein-Gordon, Dirac

Discrete-time quantum walk in 1D

- Walk is defined on the Hilbert space $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$

  $\mathcal{H}_c$ (particle) is spanned by $|\uparrow\rangle$ and $|\downarrow\rangle$
  $\mathcal{H}_p$ (position) is spanned by $|x\rangle, x \in \mathbb{Z}$

- Initial state :$|\Psi_{in}\rangle = [\cos(\delta)|\uparrow\rangle + e^{i\eta}\sin(\delta)|\downarrow\rangle] \otimes |x=0\rangle$

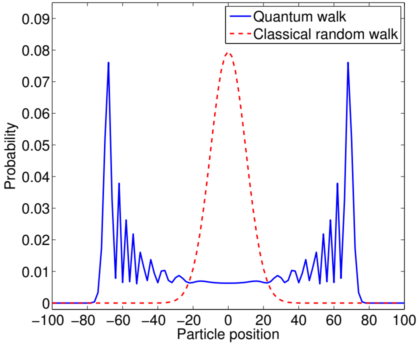
- Evolution :
  - Coin operation - Hadamard operation : $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
  - Conditional unitary shift operation S:

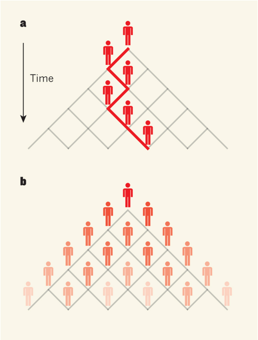
$$S = \sum_{x\in\mathbb{Z}} \left[|\uparrow\rangle\langle\uparrow| \otimes |x-1\rangle\langle x| + |\downarrow\rangle\langle\downarrow| \otimes |x+1\rangle\langle x| \right]$$

state $|\uparrow\rangle$ moves to the left and state $|\downarrow\rangle$ moves to the right

Hadamard walk

- Each step of QW (Hadamard walk) : $W = S(H \otimes \mathbb{1})$





100 step of CRW and QW $[S(H \otimes \mathbb{1})]^{100}$on a particle with initial state $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$

- G. V. Riazanov (1958), R. Feynman (1986)
- K.R. Parthasarathy, Journal of applied probability 25, 151-166 (1988)
- Y. Aharonov, L. Davidovich and N. Zugury, Phys. Rev. A, 48, 1687 (1993)
- Use of word Quantum ~~random~~ walk
- Salvador E. Venegas-Andraca, Quantum Information Processing vol. 11(5), pp. 1015-1106 (2012)

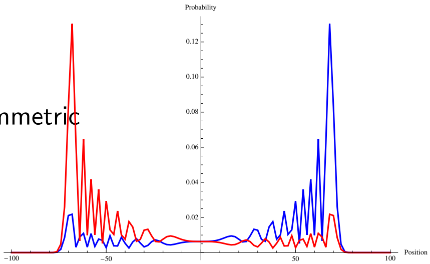
QW using generalized quantum coin operation

- Hadamard walk :

  $|\Psi_{in}\rangle = |\uparrow\rangle \otimes |x=0\rangle \rightarrow$ peak to left

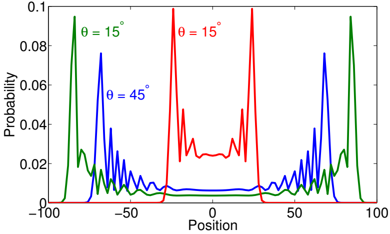
  $|\Psi_{in}\rangle = |\downarrow\rangle \otimes |x=0\rangle \rightarrow$ peak to right

  $|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\Big[|\uparrow\rangle \pm i|\downarrow\rangle\Big] \otimes |x=0\rangle \rightarrow$ symmetric



- SU(2) operation :

$$B_{\xi,\theta,\zeta} \equiv \begin{bmatrix} e^{i\xi}\cos(\theta) & e^{i\zeta}\sin(\theta) \\ -e^{-i\zeta}\sin(\theta) & e^{-i\xi}\cos(\theta) \end{bmatrix}$$



- Each step of generalized QW :

  $W_{\xi,\theta,\zeta} = S(B_{\xi,\theta,\zeta} \otimes \mathbb{1})$

  $\left(W_{\xi,\theta,\zeta}\right)^t |\Psi_{in}\rangle$ implements t steps of generalized DQW

Symmetric evolution of DQW and hyperbolic PDE

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\Big[|\uparrow\rangle \pm i|\downarrow\rangle\Big] \otimes |x=0\rangle \qquad B(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}\Big[|\uparrow\rangle \pm |\downarrow\rangle\Big] \otimes |x=0\rangle \qquad B(\theta) = \begin{bmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{bmatrix}$$

In the form of left moving and right moving component

$$\psi^0_{x,t+1} = \cos(\theta)\psi^0_{x+1,t} - i\sin(\theta)\psi^1_{x-1,t}$$
$$\psi^1_{x,t+1} = \cos(\theta)\psi^1_{x-1,t} - i\sin(\theta)\psi^0_{x+1,t}$$

Differential equation form in continuum limit :Klein-Gordon equation

$$\left[\frac{\partial^2}{\partial t^2} - \cos(\theta)\frac{\partial^2}{\partial x^2} + 2[1-\cos(\theta)]\right]\psi^{0(1)}_{x,t} = 0$$

CMC, SB and RS, PRA, **81** 062340 (2010)

Dirac equation from Discrete-time QW

Dirac equation

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{\mathbf{H}}_{\mathbf{D}}\right)\Psi = \left(i\hbar\frac{\partial}{\partial t} + i\hbar c\hat{\alpha}\cdot\frac{\partial}{\partial x} - \hat{\beta}mc^2\right)\Psi = 0$$

From DTQW when $\theta = 0$, the expression in continuum limit takes the form

$$\left[i\hbar\frac{\partial}{\partial t} - i\hbar\sigma_3\frac{\partial}{\partial x}\right]\Psi(x,t) = 0$$

David Mayer (1996) and Fredrick Strauch (2006)
For $\theta \neq 0$
Giuseppe Molfetta - Fabrice Debbasch (2013) and CMC (2013)

Dirac Cellular Automaton

DH from the QCA by constructing the evolution operator for a system which is (1) unitary, (2) invariant under space translation, (3) covariant under parity transformation, (4) covariant under time reversal and (5) has a minimum of two internal degrees of freedom (spinor). This QCA evolution which recovers DE is named as DCA and is in the form,

$$U_{DA} = \begin{pmatrix} \alpha T_- & -i\beta \\ -i\beta & \alpha T_+ \end{pmatrix} = \alpha\{T_- \otimes |\uparrow\rangle\langle\uparrow| + T_+ \otimes |\downarrow\rangle\langle\downarrow|\} - i\beta(I \otimes \sigma_x)$$

where α corresponds to the hopping strength, β corresponds to the mass term.Associated Hamiltonian in momentum basis, produces DH,

$$H(k) = \frac{a}{c\tau}\begin{pmatrix} -kc & mc^2 \\ mc^2 & kc \end{pmatrix}$$

with the identification $\beta = \frac{mac}{\hbar}$, k is a eigenvalue of momentum operator.

•Derivation of the Dirac equation from principles of information processing, D Ariano, G. M. and Perinotti, P. Phys. Rev. A 90, 062106 (2014)

•Quantum field as a quantum cellular automaton: The Dirac free evolution in one dimension, Bisio, A., DAriano,G. M., Tosini, A. Annals of Physics 354, 244264 (2015)

$DTQW$

The general form of C is,

$$C = C(\xi,\theta,\phi,\delta) = e^{i\xi}e^{-i\theta\sigma_x}e^{-i\phi\sigma_y}e^{-i\delta\sigma_z} = e^{i\xi}\times$$
$$\begin{pmatrix} e^{-i\delta}(\cos(\theta)\cos(\phi) - i\sin(\theta)\sin(\phi)) & -e^{i\delta}(\cos(\theta)\sin(\phi) + i\sin(\theta)\cos(\phi)) \\ e^{-i\delta}(\cos(\theta)\sin(\phi) - i\sin(\theta)\cos(\phi)) & e^{i\delta}(\cos(\theta)\cos(\phi) + i\sin(\theta)\sin(\phi)) \end{pmatrix}$$
$$= e^{i\xi}\begin{pmatrix} F_{\theta,\phi,\delta} & G_{\theta,\phi,\delta} \\ -G^*_{\theta,\phi,\delta} & F^*_{\theta,\phi,\delta} \end{pmatrix}$$

The general form of the evolution operator

$$U_{QW} = e^{i\xi}\begin{pmatrix} F_{\theta,\phi,\delta}\ T_- & G_{\theta,\phi,\delta}\ T_- \\ -G^*_{\theta,\phi,\delta}\ T_+ & F^*_{\theta,\phi,\delta}\ T_+ \end{pmatrix}$$

$$U_{QW} = F_\theta\{T_-\otimes|\uparrow\rangle\langle\uparrow| + T_+\otimes|\downarrow\rangle\langle\downarrow|\} + G_\theta\{T_-\otimes|\uparrow\rangle\langle\downarrow|) + T_+\otimes|\downarrow\rangle\langle\uparrow|\}$$

By taking the value of $\theta\to 0$ the off-diagonal terms can be ignored and a massless DH can be recovered.

Split-step QW

$$C(\theta_1,\phi_1,\delta_1) = \begin{pmatrix} F_{\theta_1,\phi_1,\delta_1} & G_{\theta_1,\phi_1,\delta_1} \\ -G^*_{\theta_1,\phi_1,\delta_1} & F^*_{\theta_1,\phi_1,\delta_1} \end{pmatrix},$$

$$C(\theta_2,\phi_2,\delta_2) = \begin{pmatrix} F_{\theta_2,\phi_2,\delta_2} & G_{\theta_2,\phi_2,\delta_2} \\ -G^*_{\theta_2,\phi_2,\delta_2} & F^*_{\theta_2,\phi_2,\delta_2} \end{pmatrix}$$

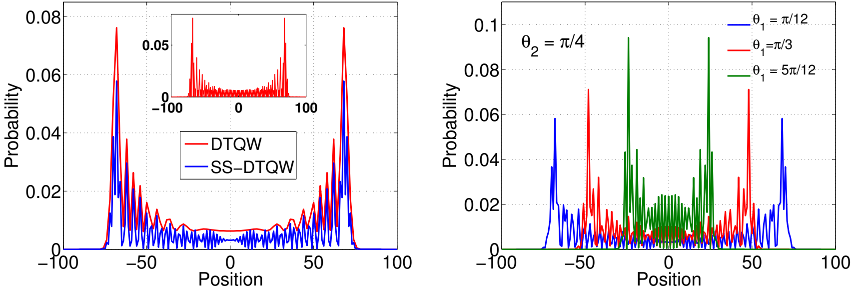
and a two half-shift operators,

$$S_- = \begin{pmatrix} T_- & 0 \\ 0 & I \end{pmatrix}, \qquad S_+ = \begin{pmatrix} I & 0 \\ 0 & T_+ \end{pmatrix}$$

$$U_{SQW} = S_+\Big(I \otimes C(\theta_2,\phi_2,\delta_2)\Big)S_-\Big(I \otimes C(\theta_1,\phi_1,\delta_1)\Big)$$

$$= \begin{pmatrix} F_{\theta_2,\phi_2,\delta_2}F_{\theta_1,\phi_1,\delta_1}\ T_- - G_{\theta_2,\phi_2,\delta_2}G^*_{\theta_1,\phi_1,\delta_1}\ I & F_{\theta_2,\phi_2,\delta_2}G_{\theta_1,\phi_1,\delta_1}\ T_- + G_{\theta_2,\phi_2,\delta_2}F^*_{\theta_1,\phi_1,\delta_1}\ I \\ -G^*_{\theta_2,\phi_2,\delta_2}F_{\theta_1,\phi_1,\delta_1}\ I - F^*_{\theta_2,\phi_2,\delta_2}G^*_{\theta_1,\phi_1,\delta_1}\ T_+ & -G^*_{\theta_2,\phi_2,\delta_2}G_{\theta_1,\phi_1,\delta_1}\ I + F^*_{\theta_2,\phi_2,\delta_2}F^*_{\theta_1,\phi_1,\delta_1}\ T_+ \end{pmatrix}$$

DCA and SS-QW



SSQW ($\theta_1 = 0, \theta_2 = \pi/4$) = DCA $\alpha = \beta = \frac{1}{\sqrt{2}}$ Substituting $\theta_1 = \phi_1 = \delta_1 = \delta_2 = 0$ we get,

$$U_{SQW} = \begin{pmatrix} \cos(\theta_2)T_- & -i\sin(\theta_2)I \\ -i\sin(\theta_2)I & \cos(\theta_2)T_+ \end{pmatrix}$$

which is in the same form as U_{DA} where $\beta = \sin(\theta_2) \equiv \frac{mca}{\hbar}$ and $\alpha = \cos(\theta_2)$.

DCA and SS-QW cont.

From the unitary operator we will recover the DH in the form,

$$H_{SQW} = -\frac{\hbar \cos^{-1}\left(\cos(\theta_2)\cos\left(\frac{ka}{\hbar}\right)\right)}{\tau\sqrt{1-\left(\cos(\theta_2)\cos\left(\frac{ka}{\hbar}\right)\right)^2}}\left[\cos(\theta_2)\sin\left(\frac{ka}{\hbar}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right.$$
$$\left. - \sin(\theta_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

For smaller mass, $\theta_2 \approx 0$ and for smaller momentum, $k \approx 0$, $\sin\theta_2 \approx \theta_2, \cos\theta_2 \approx 1, \sin\left(\frac{ka}{\hbar}\right) \approx \frac{ka}{\hbar}, \cos\left(\frac{ka}{\hbar}\right) \approx 1$.

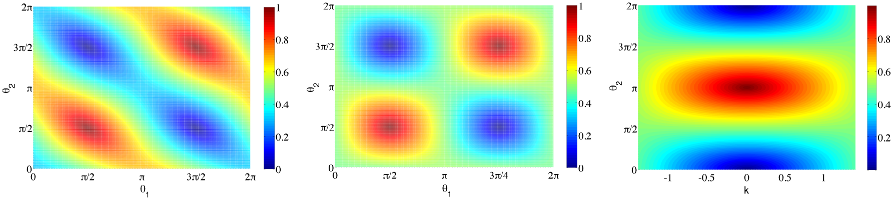
$$H_{SQW} \approx -\frac{a}{\tau}k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\hbar}{\tau}\theta_2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is in a form of one-dimensional Dirac equation for a $\frac{1}{2}$ spinor, with the identifications, $\frac{a}{\tau} = c$ and $\frac{\hbar\theta_2}{\tau} = mc^2$, so, $m = \frac{\hbar\theta_2\tau}{a^2}$.

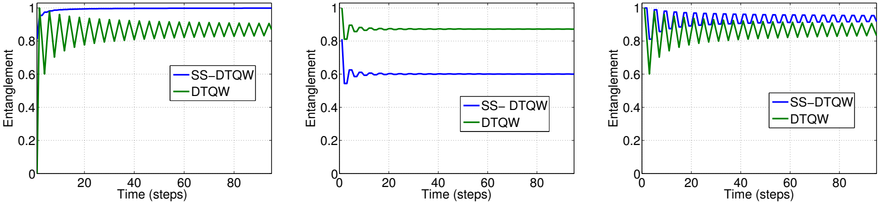
Zitterbewegung Oscillation

Any quantum mechanical observable $\hat{A}$ which doesn't commute with the Hamiltonian operator, that is, $[\hat{A}, H] \neq 0$, results in mixing of positive and negative energy eigenvalue solutions during the evolution. This mixing is responsible for oscillation of the expectation value of the observable and is known as Zitterbewegung oscillation.

$$Z_{SQW} = \frac{1}{\tau\pi}\cos^{-1}\left[\cos(\theta_1)\cos(\theta_2)\cos\left(\frac{ka}{\hbar}\right) - \sin(\theta_1)\sin(\theta_2)\right]$$



Entanglement between position space and internal degree



Standard QW the mean value of entanglement does not change with change in initial state but for SS-QW we see a noticeable change.

Recap and Summary

- QA $=$ a single quantum system, driven by some input, e.g Ambainis 98 "1-way quantum finite automata"... State space : H_d
- QCA $=$ a grid of interacting quantum systems,e.g. Watrous, Werner Schumacher, Arrighi-Nesme-Werner, Arrighi-Grattage, Meyer-Love-Shakeel. State space : $\bigotimes_Z H_d$.
- QW $=$ the single particle sector of QCA,e.g. Birula-Bialinicki, Meyer, Gross, Zeilinger, Aarhanov, Kempe, D'Ariano, CMC...State space: $\bigoplus H_d = H_Z \otimes H_d$.
- The DCA $=$ a multi-particle non interacting quantum system and in the continuum limit leads us to the (free) Dirac field equations, e.g. by Bisio, D'Ariano, Tosini, CMC.

Starting from single particle SS-QW we recover DCA for set of walk evolution parameters without loosing any intriguing features in the dynamics.

Simulation of Topological Insulator

Basic formalism

Basis states $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

Initial state $|\Psi_{in}\rangle = \frac{1}{\sqrt{2}}[|0\rangle + |1\rangle] \otimes |x = 0\rangle$

Coin operation $R_\theta \equiv \begin{bmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{bmatrix}$

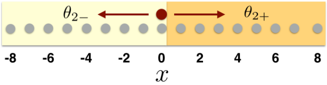
Shift operation

$$S_- = |0\rangle\langle 0| \otimes |x-1\rangle\langle x| + |1\rangle\langle 1| \otimes |x\rangle\langle x|$$

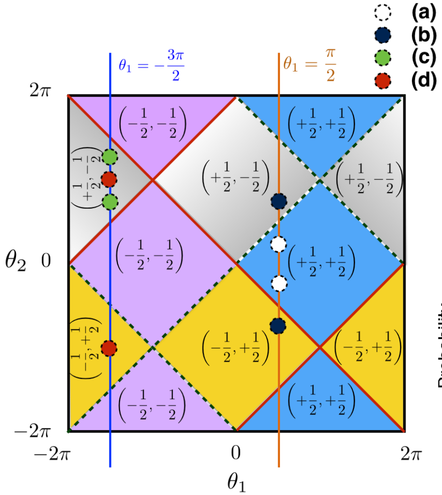
$$S_+ = |0\rangle\langle 0| \otimes |x\rangle\langle x| + |1\rangle\langle 1| \otimes |x+1\rangle\langle x|$$

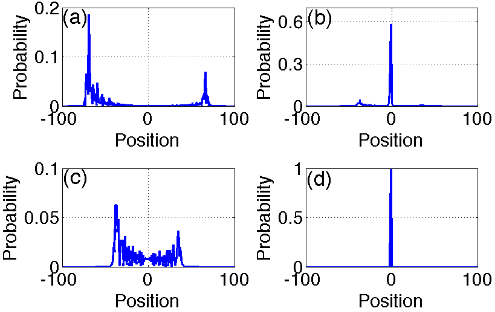
→ Two split-step evolution $W(\theta_1, \theta_2) = S_+ R_{\theta_2} S_- R_{\theta_1}$

→ Four split-step evolution $W(\theta_1, \theta_2, \theta_3, \theta_4) = S_+ R_{\theta_4} S_+ R_{\theta_3} S_- R_{\theta_2} S_- R_{\theta_1}$



Two split-step quantum walk and localized state



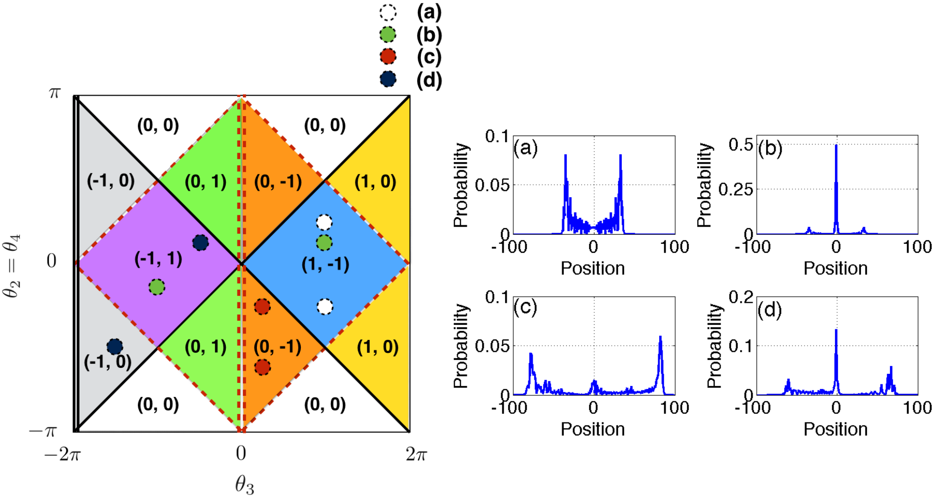


$W(\theta_1,\theta_2) = S_+ R_{\theta_2} S_- R_{\theta_1}$

Three variable parameter : θ_1, θ_{2-} and θ_{2+}

Phys. Rev. A **82**, 033429 (2010)

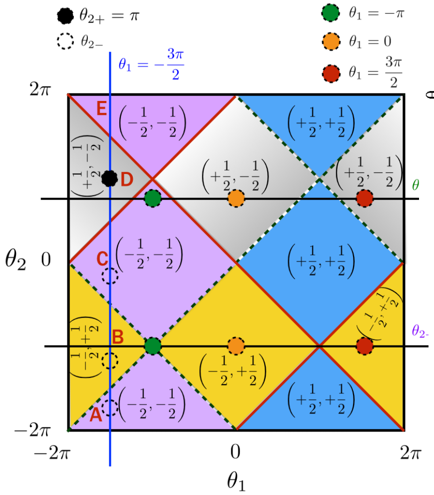
Four split-step quantum walk and localized state

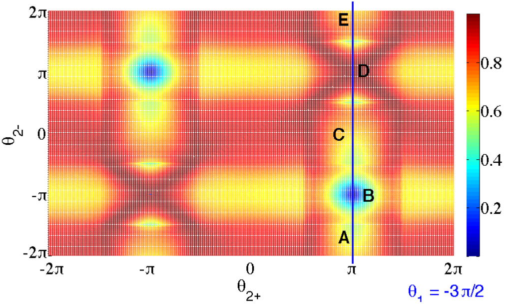


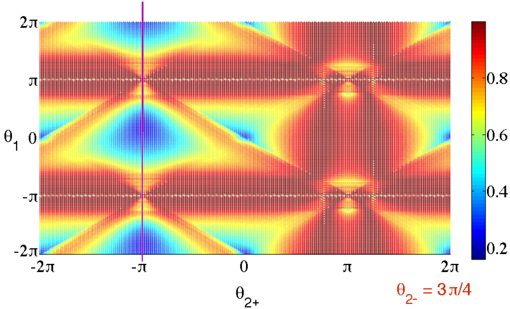
$W(\theta_1, \theta_2, \theta_3, \theta_4) = S_+ R_{\theta_4} S_+ R_{\theta_3} S_- R_{\theta_2} S_- R_{\theta_1}$ Phys. Rev. A **88**, 121406(R) (2013)

Four variable parameter : $\theta_1 = 0$, $\theta_{2\pm} = \theta_{4\pm}$ and θ_3

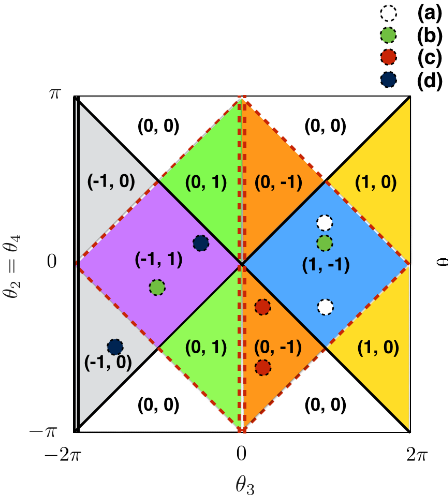
Entanglement spectrum of two split-step walk

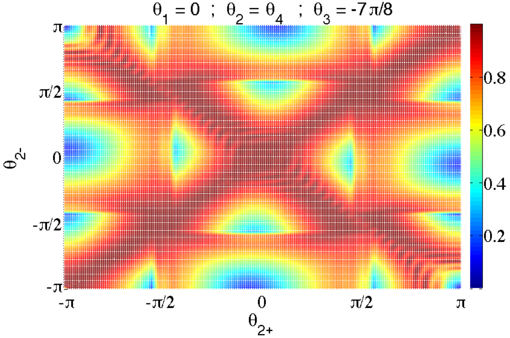






Entanglement spectrum of four split-step walk

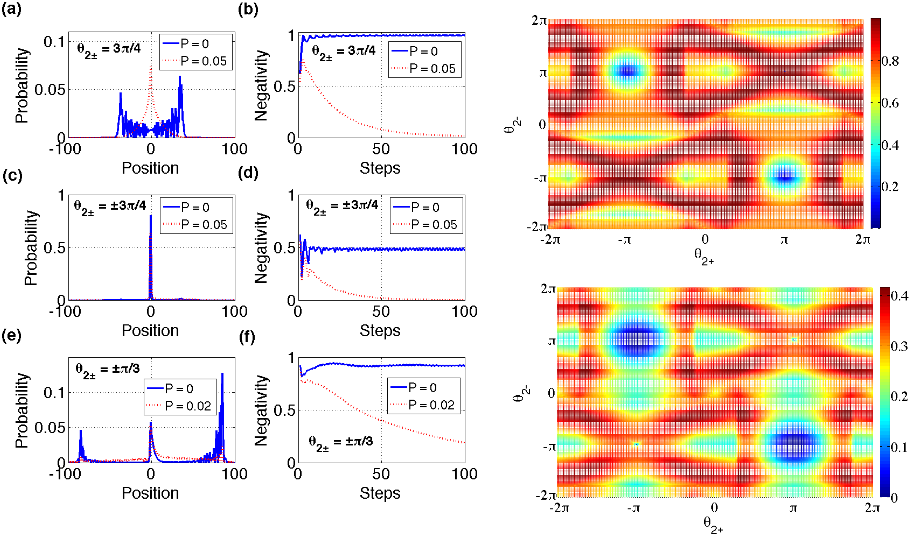




Valley in entanglement spectrum indicate the existence of localized state

Effect of noise on topologically localized state

$$\rho(t) = P\left[f_1 W_{\theta_1,\theta_2}\rho(t-1)W^{\dagger}_{\theta_1,\theta_2}f_1^{\dagger}\right]+(1-P)\rho(t-1)\ ;\ f_1 \equiv \sigma_x\otimes\mathbb{I}\ ;\ \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



Summary

- Disordered DTQW and topological QWs results in localized states.
- Entanglement is robust against localization due to disorder but results in a valley in entanglement profile localization due to topological effect.
- Localized states from topological effect is robust against noise.

Looks promising for artificial synthesis of topological insulators.

With a choice of evolution parameters we can use SS-QW to simulate both, free quantum field theory equation and topological insulators.

THANK YOU