Monogamy of Quantum Correlations

R Prabhu

QIC Group
Harish-Chandra Research Institute
Allahabad, India

Coauthors: Arun K Pati, Aditi Sen (De), and Ujjwal Sen
Can quantum correlations be created in well controlled environments between distinct quantum systems?

1950’s inconceivable question

Early attempt to quantify the quantum correlations was through Bell’s inequalities

“As it is well known John Bell's questioning orthodox quantum mechanics was just his hobby, and it is this hobby John Bell is most famous for.”

Quantum [un]speakables: from Bell to quantum information
- R. A. Bertlmann & A. Zeilinger
Can quantum correlations be created in well controlled environments between distinct quantum systems?

1950’s inconceivable question

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Theory of Entanglement

- Characterization
- Quantification
- Manipulation
Let A and B be a pair of qubits and let the density matrix of the pair be $\rho_{AB}$.

**Concurrence:** $C_{AB} = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$

$\lambda$'s are the square root of the eigenvalues of $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y) \rho_{AB}^*(\sigma_y \otimes \sigma_y)$

- $C_{AB} = 0$ $\Rightarrow$ Unentangled
- $C_{AB} = 1$ $\Rightarrow$ Maximally entangled
In a multipartite setting, sharing entanglement between several parties is restricted by the monogamy of entanglement.
Monogamy of Entanglement

- If two qubits $A$ and $B$ have maximal quantum correlations, they cannot be correlated, at all with third qubit $C$
  

- Trade-off between the amount of entanglement between qubits $A$ and $B$ and the same qubit $A$ and qubit $C$
  

- In the classical world: If $A$ and $B$ bits are perfectly correlated, then there are no constraints on correlations between bits $A$ and $C$
Entangled

Monogamy satisfied

Polygamous
If two subsystems are highly entangled, then they cannot share a substantial amount of entanglement with other subsystems.
Monogamy for Pure States

\[ C_{AB}^2 + C_{AC}^2 \leq 4 \det \rho_A \]

Concurrence between qubit A and the pair BC: \( C_{A(BC)} \)

\[ C_{AB}^2 + C_{AC}^2 \leq C_{A(BC)}^2 \]

**NOTE:** Even though the state space of BC is four dimensional, only two of those dimensions are necessary to express the combined state ABC.

Monogamy for Mixed States

Let $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$, with set of all pure state decomposition $\{(\psi_i, p_i)\}$

$$C^2_{AB} + C^2_{AC} \leq (C^2)_{\min}^{A(BC)}$$

$$\min\langle C^2_{A(BC)} \rangle = \sum_i p_i C^2_{A(BC)}(\psi_i)$$


Recent!

For more on monogamy of entanglement:
Concept of monogamy to an information-theoretic quantum correlation measure
Several quantum phenomena have been discovered in which entanglement is absent.

- “quantum nonlocality without entanglement” — locally indistinguishable orthogonal product states
- Deterministic quantum computation with one quantum bit
Quantify quantum correlations independent of the entanglement-separability paradigm

- **Quantum discord**
  
  

- **Quantum deficit**
  
  
  
Quantum Discord

“Quantum discord is the difference between two classically equivalent expressions for the mutual information, when extended to the quantum regime.”

Mutual information

**Classical**

One form:
\[ H(X : Y) = H(X) + H(Y) - H(X, Y) \]

where \( X, Y \) are random variables

Shannon entropy

\[ H(\{p_i\}) = - \sum p_i \log_2 p_i \]

Second form:
\[ H(X : Y) = H(X) - H(X|Y) \]

where

\[ H(X|Y) = \sum_y p(y) H(X|Y = y) \]

conditional entropy

**Quantum**

One form:
\[ I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \]

Second form:
\[ S(\sigma) = - \text{tr} (\sigma \log_2 \sigma) \]

Von Neumann entropy

Physical quantity that can be negative for some quantum states!!!

What next then?
Fixing second form:

Classical correlation: the difference between the von Neumann entropies before and after a complete measurement on the subsystem B

$$J(\rho_{AB}) = S(\rho_A) - S(\rho_{A|B})$$

where quantum conditional entropy

$$S(\rho_{A|B}) = \min_{\{\Pi_i^B\}} \sum_i p_i S(\rho_{A|i})$$

minimization being over all projection-valued measurements performed on subsystem B

$$\rho_{A|i} = \frac{1}{p_i} \text{tr}_B(\Pi_A \otimes \Pi_i^B \rho_A \otimes \Pi_i^B)$$
Mutual information

One form:

$$H(X : Y) = H(X) + H(Y) - H(X, Y)$$

where $X$, $Y$ are random variables

$$H(\{p_i\}) = - \sum p_i \log_2 p_i$$

Shannon entropy

Second form:

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conditional entropy

One form:

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

Q. conditional entropy

Second form:

$$S(\sigma) = - \text{tr} (\sigma \log_2 \sigma)$$

Von Neumann entropy

$$J(\rho_{AB}) = S(\rho_A) - S(\rho_{A|B})$$

$$S(\rho_{A|B}) = \min_{\{\Pi_i^B\}} \sum_i p_i S(\rho_{A|i})$$
Quantum Discord

“Quantum discord is the difference between two classically equivalent expressions for the mutual information, when extended to the quantum regime.”

\[ D(\rho_{AB}) = I(\rho_{AB}) - J(\rho_{AB}) \]

Total correlations  Cl. correlations
Does quantum discord satisfy monogamy relation?

Does the sharing of quantum discord follow the same broad guidelines that are followed by entanglement?
For tripartite state $\rho_{ABC}$

$$D(\rho_{AB}) + D(\rho_{AC}) \leq D(\rho_{A:BC})$$

Monogamy relation

- satisfied: Monogamous
- violated: Polygamous
\[ D(\rho_{AB}) + D(\rho_{AC}) \leq D(\rho_{A:BC}) \]

OR

\[ \delta_M = D(\rho_{AB}) + D(\rho_{AC}) - D(\rho_{A:BC}) \]

- \( \delta_M \leq 0 \) \( \rightarrow \) Q. correlations \( \rightarrow \) Monogamous
- \( \delta_M > 0 \) \( \rightarrow \) Q. correlations \( \rightarrow \) Polygamous
The difference between the information shared by the subsystem AB when C is present, and when C is not present (traced out)

For tripartite state $\rho_{ABC}$

$$I(\rho_{ABC}) = I(\rho_{A:B|C}) - I(\rho_{AB})$$

$$S(\rho_{A|C}) - S(\rho_{A|BC})$$

(Conditional mutual information)
The difference between the information shared by the subsystem AB when C is present, and when C is not present (traced out),
Theorem 1

Tripartite states follow monogamy
iff
Interrogated $\Pi$ ≤ unmeasured $\Pi$

Theorem 2

Unmeasured $\Pi < 0$
Interrogated $\Pi > 0$

Polygamous

Unmeasured $\mathcal{U} < 0$
Interrogated $I > 0$
Tripartite states
Sir

GHZ or W state?

Quantum States Shop

Discord monogamy can be a test?
Generalized GHZ states

\[ |\psi_{GHZ}\rangle_{ABC} = \cos \Phi |000\rangle + \sin \Phi |111\rangle \]

\[ \delta_M < 0 \]  

Monogamous
Generalized W states

\[ |\psi_W\rangle_{ABC} = b|011\rangle + c|101\rangle + d|110\rangle \]

Here

\[ b = \sin \theta \cos \phi \]
\[ c = \sin \theta \sin \phi \]
\[ d = \cos \theta \]

Polygamous

\[ \delta_M > 0 \]
\[ |\psi_G\rangle = \cos \frac{\theta}{2} |000\rangle + |\psi_1\rangle|\psi_2\rangle|\psi_3\rangle \]

where \[ |\psi_i\rangle = \alpha_i |0\rangle + \beta_i |1\rangle \]

\[ i = 1, 2, 3 \]

Numerical results confirm that these states always violate monogamy.
Mixed state: Generalized GHZ states

\[ \rho_{GHZ} = (1 - p)\mathbb{I}/8 + p|\psi_{GHZ}\rangle\langle\psi_{GHZ}| \]

\[ \delta_M \leq 0 \]

Monogamous
Mixed state: Generalized W states

\[ \rho_W = \frac{(1 - p) \mathbb{I}}{8} + p |\psi_W \rangle \langle \psi_W| \]
Does quantum discord satisfy monogamy relation?

Does the sharing of quantum discord follow the same broad guidelines that are followed by entanglement?

No !!
## Results for discord monogamy test

<table>
<thead>
<tr>
<th>Tripartite states</th>
<th>Discord Monogamy</th>
<th>Monogamy test result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen GHZ</td>
<td>$&lt; 0$</td>
<td>Satisfy</td>
</tr>
<tr>
<td>Gen W</td>
<td>$&gt; 0$</td>
<td>Violate</td>
</tr>
</tbody>
</table>
| GHZ class         | 75%$>0$
0<25%         | Satisfy
Violate         |
| W class           | $> 0$             | Violate              |
When a state is subjected to test under monogamy of discord

Satisfying monogamy means the state belongs to **GHZ class**

Violating monogamy means the state belongs to **GHZ class** or **W class**

THANK YOU for your attention