Monogamy of Bell Inequality violations

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Motivation

• Bell inequality violation gives quantum advantage:
  • Device independent Cryptography,
  • Communication Complexity,
  • Randomness amplification, etc.

• Multiple inequalities - Quantum correlations are ‘monogamous’!

• Causes of monogamy: no-signaling, complementarity.

• Implications:
  • Security of key distribution,
  • Emergence of macroscopic local realism,
  • Properties of condensed matter.
Outline

- Qubit Bell inequalities - formalism

- Correlation complementarity.

- Derivation of Tsirelson bounds, Bell monogamies from
  - Correlation complementarity,
  - No-signaling.

- Bipartite and multipartite scenarios – graph formalism.

- Conclusions and Outlook
Qubit Bell inequalities

- Complete two-setting correlation inequalities for N qubits — Zukowski, Brukner (PRL (2002))
- Sufficient condition for existence of a local hidden variable model for N qubit correlations:

\[
\sum_{x_1,\ldots,x_N=1}^{2} T_{x_1\ldots x_N}^2 \leq 1,
\]

- Quantum value has upper bound: \( L^2 \leq \sum_{x_1,\ldots,x_N=1,2} T_{x_1\ldots x_N}^2 \), where \( T_{\mu_1\ldots\mu_N} = \text{Tr}[\rho(\sigma_{\mu_1} \otimes \cdots \otimes \sigma_{\mu_N})] \)
- \( x_i = 1,2 \): orthogonal local directions, sum and difference of vectors parametrising local settings.
- \( L \leq 1 \) : LHV
- \( L \leq 2^{(N-1)/2} \) : QM
- \( L \leq 2^{N/2} \) : NS
- Note:

\[
\rho = \frac{1}{2^N} \sum_{x_1,\ldots,x_N=0}^{3} T_{x_1\ldots x_N} \sigma_{x_1}^1 \otimes \cdots \otimes \sigma_{x_N}^N,
\]
Correlation Complementarity

- **Complementarity**: If expectation value of one measurement is $\pm 1$, then that for complementary measurement is zero.

- Operators corresponding to dichotomic complementary observables anti-commute.

- Proof: For two dichotomic complementary observables $A$ and $B$

\[
\langle A \rangle = 1 \Rightarrow A |a\rangle = |a\rangle \Rightarrow \langle a | B | a \rangle = 0 \Rightarrow B |a\rangle = |a_{\perp}\rangle \\
B^2 - 1 \Rightarrow B |a_{\perp}\rangle - |a\rangle \Rightarrow |b\rangle - \frac{1}{\sqrt{2}} (|a\rangle + |a_{\perp}\rangle) \\
\langle b | A | b \rangle = 0 \Rightarrow A |a_{\perp}\rangle = -1 |a_{\perp}\rangle
\]

- Argument applies to all $+1$ eigenstates, the two eigenspaces have equal dimension.

\[
A = \sum_{a} (|a\rangle \langle a| - |a_{\perp}\rangle \langle a_{\perp}|) \\
B = \sum_{a} (|a_{\perp}\rangle \langle a| + |a\rangle \langle a_{\perp}|) \\
\{A, B\} = 0
\]
Correlation Complementarity

- **Lemma:** Consider a set of traceless and trace orthogonal dichotomic Hermitian operators $A_k$ that obey $\{A_k, A_j\} = 2 \delta_{kj}$. Their expectation values for any state $\rho$ obey $\Sigma \langle A_k \rangle^2 \leq 1$.

- **Proof:**

  \[
  A = \sum_k \langle A_k \rangle A_k, \quad \langle A^2 \rangle - \langle A \rangle^2 \geq 0.
  \]

  \[
  \langle A^2 \rangle = \sum_k \langle A_k \rangle^2, \quad \langle A \rangle^2 = \langle A^2 \rangle.
  \]

  \[
  \sum_k \langle A_k \rangle^2 \leq 1.
  \]

- **Tight:** There exists a state having these nos. as expectation values for anti-commuting observables - Wehner, Winter (J. Math. Phys. (2008)).

- **Method:** Find quantum bounds for Bell violations using Correlation Complementarity. Identify sets of anti-commuting operators for Bell parameters, bound from Lemma.

- **Remember:** $L^2 \leq \sum_{x_1,\ldots,x_N=1,2} T_{x_1,\ldots,x_N}^2$
Tsirelson bounds

- Application: CHSH Tsirelson bound.

- For two qubits and two setting inequalities, single Bell parameter is upper bounded
  \[ L^2 \leq T_{xx}^2 + T_{xy}^2 + T_{yx}^2 + T_{yy}^2. \]

  \[
  \mathcal{L}^2 \equiv \frac{1}{4} S^2 \leq T_{xx}^2 + T_{xy}^2 + T_{yx}^2 + T_{yy}^2 \\
  \leq 1 + 1
  \]

- Identifying two sets of anti-commuting observables \((T_{xx}, T_{xy})\) and \((T_{yx}, T_{yy})\) \(L \leq \sqrt{2}\), Tsirelson bound.

- Get Tsirelson bounds of multi-setting inequalities: Laskowski et al. (PRL, (2004)) and many-qubit two-setting inequalities.

- Tsirelson bound born out of complementarity! 😊
Bipartite Bell monogamies

- Three qubits A, B, C. If AB violate two-qubit BI, then correlations AC admit LHV description.

- Vertices represent observers violating Bell inequalities which are represented by edges.

- The upper bound reads $L^2_{AB} + L^2_{AC} \leq \sum_{k,l=x,y} T^2_{kl0} + \sum_{k,m=x,y} T^2_{k0m}$. Settings of A are same in both inequalities.

- Identifying two sets of mutually anti-commuting operators: \{XX0, XY0, Y0X, Y0Y\} and \{YX0, YY0, X0X, X0Y\}; $X = \sigma_x$ and $Y = \sigma_y$ gives $L^2_{AB} + L^2_{AC} \leq 2 \rightarrow B^2_{AB} + B^2_{AC} \leq 8!$
Bipartite B.I.’s – complete monogamies

- Consider N qubits trying to violate a set of bipartite B.I.s - Graph G (Black) with observers as vertices, inequalities as edges.

- **Method**: For arbitrary graph G, construct its line graph L(G) (Red) placing vertices of L on every edge of G & connecting vertices of L(G) whenever the corresponding edges of G share a vertex.

- Every edge of L(G) is an elementary monogamy: $L^2_{AB} + L^2_{AC} \leq 2$.

- $\sum_v d_v L^2_v \leq 2 \varepsilon$, where $d_v$ : degree of vertex v and $\varepsilon$: number of edges in L(G).

- Inequality is tight for arbitrary graph of bipartite inequalities! 😊
Multipartite EMRs

- Two-qubits: single EMR. Multiple qubits: $k-1$ EMR’s generate monogamies via the line graph.
- All EMRs are tight independent of the number of common observers: $L_1^2 + L_2^2 \leq 2^{k-1}$.

Monogamies for arbitrary graphs of $k$-qubit inequalities constructed via line graph.
- Condition for tightness: line graph of the multipartite graph must be bipartite!! 😊
Multipartite Polygamy – Complete hypergraph

• Consider parties A, B, C, D trying to violate a correlation Bell inequality in the graph shown.

\[ L_{ABC}^2 + L_{BCD}^2 + L_{CDA}^2 + L_{DAB}^2 \leq 4. \]

Mermin monogamy:

\[ M_{ABC}^2 + M_{BCD}^2 + M_{CDA}^2 + M_{DAB}^2 \leq 16. \]

Possibilities: two and three triples violate Mermin inequality non-maximally, for example:

• The state \( \frac{1}{2} (|0001\rangle + |0010\rangle + i \sqrt{2} |1111\rangle) \) allows ABC and ABD to obtain \( M = 2\sqrt{2} \).

• The state \( \frac{1}{\sqrt{6}} (|0001\rangle + |0010\rangle + |0100\rangle + i \sqrt{3} |1111\rangle) \) allows ABC, ABD and ACD to obtain \( M = 4/\sqrt{3} \).

• All four inequalities cannot be simultaneously violated.

• Tightness for three-qubit inequalities in complete graphs of arbitrary number of qubits!
Multipartite Polygamy – Tree hypergraph

- The $2^{k-1}$ inequalities obey $\sum L_i^2 \leq 2^{k-1}$ for arbitrary $k$.

- All patterns of violation except simultaneous violation of all.

- For any $m < 2^{k-1}$ of Bell inequalities, the state shows spherical tightness:

  $$ |\psi_n\rangle = \frac{1}{\sqrt{2^n}} |0\ldots\rangle + \frac{1}{\sqrt{2^m}} \sum_{j=1}^m |0\ldots01\ldots10\ldots0\rangle, $$

- $L_j^2 = 2^{k-1}/m$ for each Bell inequality $j = 1, \ldots, m$ - remaining Bell parameters vanish.

- Maximal violation in a branch of an “arbitrary graph” $\rightarrow$ no violation in any connected branch! 😊
Practical matters

- Construct the operator graph $H(G)$ from $G$: vertices correspond to the operators, edges connect anti-commuting vertices.
- **Clique partitioning**: partition the operator graph into sets of vertices that are fully connected.
- All spherically tight monogamy relations for given $k$ correspond to a single operator graph!
- The operator graph for the bipartite case $AB$ vs. $AC$:

  ![Operator Graph Diagram]

  - Compare with checking positivity of quantum states under different values of Bell parameters.
No-signaling monogamy

- **Method:** Decomposition of the graph $G$ (J B.I.’s of $N$ qudits each) into $J$ graphs each corresponding to a single B.I. – (Pawłowski, Brukner PRL (2009)).

- One measurement setting per qudit in each of the $J$ graphs $\rightarrow$ joint probability distribution.

- **Result:** For the graph with each qudit involved in as many B.I.’s as settings, violation of the inequality implies signaling.

\[
\sum_{p_1, p_2, \ldots, p_N} B(\bar{A}^{(1)}_{p_1} \bar{A}^{(2)}_{p_2} \ldots \bar{A}^{N}_{p_N}) \leq n^{N-1} R
\]

- Linear monogamies from no-signaling bound the spherical monogamies in quantum theory!
Conclusions & Outlook

- **Take-home**: Correlation complementarity implies tight bounds on violation of single and multiple Bell inequalities in quantum theory.

- Quadratic monogamies tighter than linear No-signaling monogamies.

- Possible applications: secure communication in tree networks, properties of condensed matter systems, etc.

- Possible extensions
  - Derivation of Entanglement monogamy.
  - Sub-determinants of density matrix to derive complementarities.
  - More measurement settings
  - Qudit inequalities