On the spectra of the partial transpose

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Collaborators:
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In related previous work: J. Bandyopadhyay (NUS), S. Majumdar (Orsay), O. Bohigas (Orsay)

Motivating question and some answers

What is the entanglement within subsystems of a pure state?

- Bipartite states well understood; Tripartite states $(N_1 \otimes N_2) \otimes N_3$ less
- Tripartite states $(N_1 \otimes N_2) \otimes N_3$ such that if $4N_1N_2 < N_3$ dominantly PPT, $4N_1N_2 > N3$ dominantly NPT.
- Possible to calculate the average third moment after PT exactly.
- Quantify with average negativity/log-negativity.
- Classic spectra of random matrix theory (Wigner semicircle) arises prominently in the spectrum after PT.
- Applications of Extreme-Value statitstics at critical dimensions $4N_1N_2 = N_3$.

Purity and Entropy of bipartite pure states

$$\mathcal{H} = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}, \quad N_2 \geq N_1. \ |\psi\rangle = \sum_i \sum_\alpha a_{i\alpha} |i\alpha\rangle$$

Random states: choose uniformly from $2N_1N_2 - 1$ dimensional unit sphere.

$$P(\{a_{i\alpha}\}) = C\,\delta\left(\sum_{i\alpha} |a_{i\alpha}|^2 - 1\right)$$

Measure: Unitarily invariant Haar measure: Usual geometric hypersurface volume on the unit sphere $S^{2N_1N_2-1}$.

$$\langle \mathrm{Tr}(\rho_A^2)\rangle = \frac{N_1 + N_2}{N_1N_2 + 1} \approx \frac{1}{N_1} + \frac{1}{N_2}$$

$$\langle E \rangle \approx \log(N_1) - \frac{N_1^2 - 1}{2N_1N_2 + 2}, \quad N_1 \ll N_2 \ \text{(Lubkin 1978)}$$

The spectrum of the density matrix

j.p.d.f. ($\beta = 1, 2$ for real, complex states)

$$P_\beta(\lambda_1, \cdots, \lambda_{N_1}) = B\delta\left(\sum_{i=1}^{N_1} \lambda_i - 1\right) \prod_{i=1}^{N_1} \lambda_i^{\frac{\beta}{2}(N_2 - N_1 + 1) - 1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta.$$

S. Lloyd, H. Pagels, "Complexity as Thermodynamic Depth" Ann. Phys. 1988.
K. Zyczkowski, H-J Sommers, J. Phys. A. 2001.
Average Entanglement:

$$\langle E \rangle = -\int d\lambda_1, .., d\lambda_{N_1} \sum_i \lambda_i \log(\lambda_i) P_2(\lambda_1, .., \lambda_{N_1}) = -N_1 \int \lambda \log(\lambda) f(\lambda) d$$

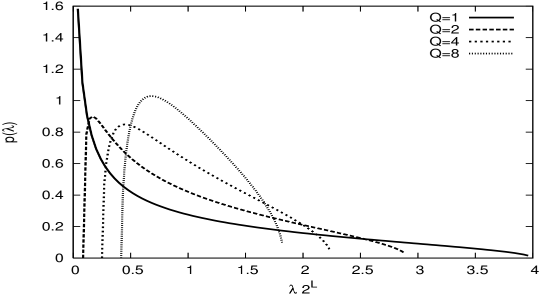
$$f(\lambda) = \int d\lambda_2 \cdots \int d\lambda_{N_1} P_2(\lambda, \lambda_2, \cdots, \lambda_{N_1})$$

Distribution of eigenvalues of RDM

$Q = N_2/N_1$. For large N_2 and N_1 and finite Q the distribution of $f(\lambda)$ is that of Marcenko and Pastur.

$$f(\lambda) = \frac{Q}{2\pi} \frac{\sqrt{(\lambda - \lambda_{min})(\lambda_{max} - \lambda)}}{\lambda}$$

$$\lambda_{max,min} = \frac{1}{N_1}(1 \pm \sqrt{Q})^2$$



Implications

L qubits in a typical pure state. What is the entanglement between two blocks having L_1 and L_2 number of qubits, when $L_1 + L_2 < L$?

- If $L_1 + L_2 < L/2$ then ρ_{12} has a minimum eigenvalue $\sim 1/N$.
- If $L_1 + L_2 = L/2$ the minimum eigenvalue $\sim 1/N^3$. (S. Majumdar, O. Bohigas, AL, JSP, 2009)
- If $L_1 + L_2 > L/2$ there are eigenvalues that are zero; RDM does not have full-rank. ($N = N_1 N_2 = 2^{L_1+L_2}$).

Partial Transpose: Reminders

- It preserves the first two moments: $\mathrm{tr}(\rho_{12}^{T_2}) = \mathrm{tr}(\rho_{12}) = 1$ and $\mathrm{tr}(\rho_{12}^{T_2})^2 = \mathrm{tr}(\rho_{12})^2 < 1$.
  That is if $spec(\rho_{12}^{T_2}) = \{\mu_i,\ i = 1, \ldots, N_1 N_2\}$, then $\sum_i \mu_i = 1$ and $\mu_i^2 < 1$.

Measure of bipartite entanglement in a density matrix:
Negativity:

$$\mathcal{N}(\rho_{12}) = \frac{\sum_i |\mu_i| - 1}{2}$$

Log-negativity:

$$E_{LN} = \log\left(||\rho_{12}^{T_2}||_1\right) = \log\left(\sum_i |\mu_i|\right)$$

Both are *Entanglement monotones* that vanish for separable states.

The third moment, $\langle \mathrm{tr}\big(\rho_{12}^{T_2}\big)^3 \rangle$, after PT

Recall that the first two moments are the same before and after PT.
An *exact* calculation yield ensemble averages:

$$\langle \mathrm{tr}\big(\rho_{12}^{T_2}\big)^3 \rangle = \frac{N_1^2 + N_2^2 + N_3^2 + 3N_1 N_2 N_3}{(N_1 N_2 N_3 + 1)(N_1 N_2 N_3 + 2)}$$

constrast $[N_1 \to N_1 N_2,\ N_2 \to 1]$

$$\langle \mathrm{tr}\big(\rho_{12}\big)^3 \rangle = \frac{N_1^2 N_2^2 + N_3^2 + 3\ N_1 N_2 N_3 + 1}{(N_1 N_2 N_3 + 1)(N_1 N_2 N_3 + 2)}$$

Remarkable **permuation symmetry in the PT**. Related to invariants.
In fact:

$$\sum_i \left(\mu_i^{(12)}\right)^3 = \sum_i \left(\mu_i^{(23)}\right)^3 = \sum_i \left(\mu_i^{(31)}\right)^3 .$$

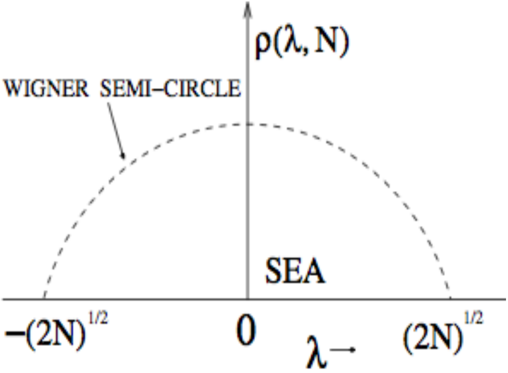
Gaussian Random Matrices

- $(N \times N)$ Gaussian random matrix: $X \equiv [x_{ij}]$
- $\text{Prob}[x_{ij}] = \exp\left[\frac{-\beta}{2}\text{Tr}(X, X)\right]$
- Dyson index $\beta = 1, 2, 4$ (GOrthogonalE,GUnitaryE,GSymplecticE).
- N real eignvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ are correlated random variables
- Joint distribution (Wigner, 1951)

$$P(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{Z_N} \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N} \lambda_i^2\right] \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

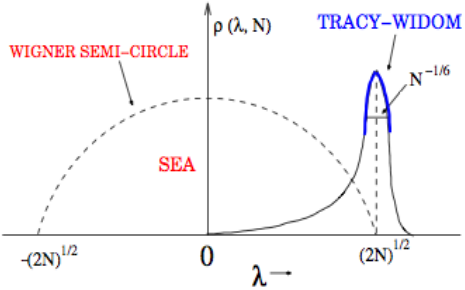
Spectral Density: Wigner's Semicircle law

- Average density of states: $\rho(\lambda, N) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta\left(\lambda - \lambda_i\right) \right\rangle$
- Wigner's Semicirlce: $\rho(\lambda, N) \to \sqrt{\frac{2}{N\pi^2}} \left[1 - \frac{\lambda^2}{2N}\right]^{1/2}$



- $\langle \lambda_{max} \rangle = \sqrt{2N}$ for large N.
- λ_{max} fluctuates. What is Prob$[\lambda_{max}, N]$?

Tracy-Widom distribution for extreme λ_{max}

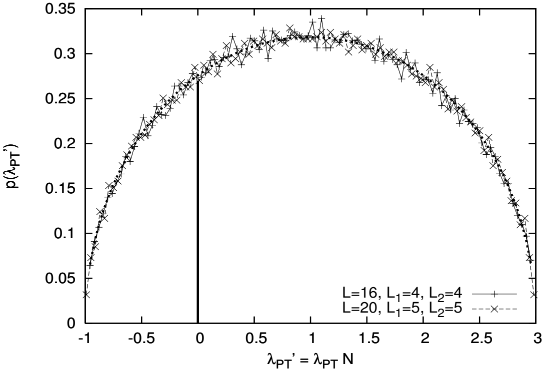


- $\langle \lambda_{max} \rangle = \sqrt{2N}$, typical fluctuations $|\lambda_{max} - \sqrt{2N}| \sim N^{-1/6}$.
- Typical fluctuations are distributed according to the *Tracy-Widom* law (1994).
- $\text{Prob}[\lambda_{max} \leq t, N] \to F_\beta\left(\sqrt{2}N^{1/6}(t - \sqrt{2N})\right)$
- $F_\beta(z)$ obtained from solutions of a Painleve-II equation

Wigner's semicircle in the PT

If $L_1 = L_2 = L/2$ the spectrum of the ρ_{12}^{PT} fits the Wigner semicircle law!
The Partial Transpose is NPT.

$$x = \mu N, \quad p(x) = \frac{1}{2\pi}\sqrt{4-(x-1)^2}$$



The DoS before and after PT: Marcenko-Pastur to Wigner Semicircles

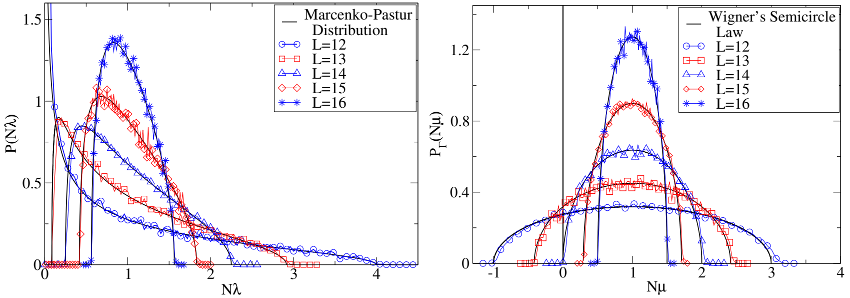


Figure: $L_1 = L_2 = 3$, $L = L_1 + L_2 + L_3$

Critical Dimensions: $L_1 + L_2 = L/2 - 1$ or $N_3 = 4N_1N_2$.
$N_3 > 4N_1N_2$, states are dominantly PPT, $N_3 < 4N_1N_2$ dominantly NPT

Non-symmetric cases

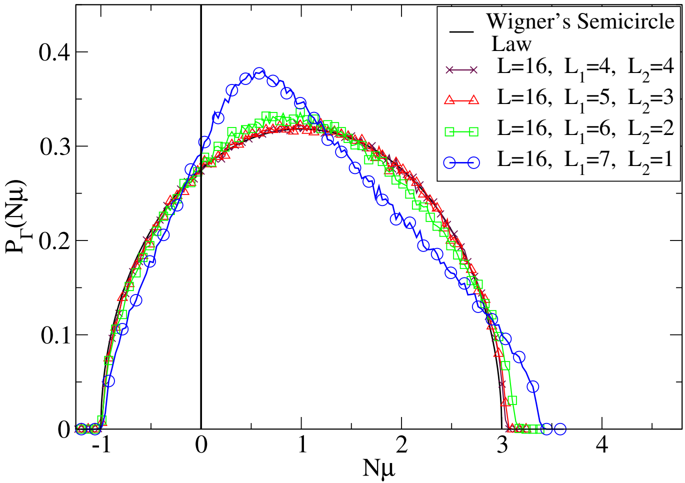


Figure: Fixed L and $L_1 + L_2$. Skewness is minimum for $L_1 = L_2 = 4$ and maximum for $L_1 = 1$ and $L_2 = 7$.

A simple random matrix model for the PT

$$\rho_{12}^{T_2} == A + \frac{I_N}{N}, \quad (N = N_1 N_2)$$

where A is a $N \times N$ GUE random matrix and I_N is the identity matrix. Find $\langle \mathrm{tr}(A^2) \rangle$ such that it gives Lubkin's 1978 formula for average purity $\langle (\rho_{12}^{T_2})^2 \rangle$, fixing the only scale in the GUE.

$$P(\mu) = \frac{2}{\pi R^2} \sqrt{R^2 - \left(\mu - \frac{1}{N}\right)^2}, \; -R + \frac{1}{N} < \mu < R + \frac{1}{N}$$

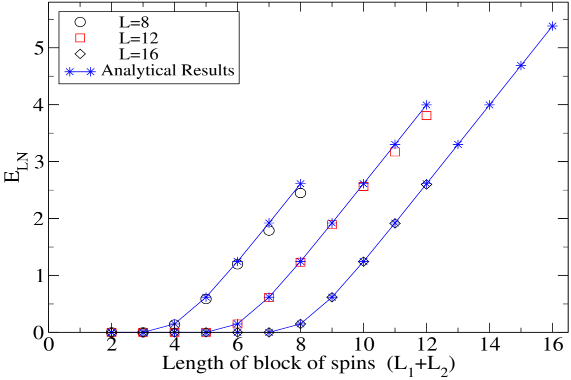
$$R = \frac{2}{\sqrt{N_1 N_2 N_3}} = 2^{-L/2+1}$$

Rescaled radius: $\tilde{R} = NR = 2^{L_1 + L_2 - (L/2-1)}$. $x = \mu N$:

$$P_\Gamma(x) = \frac{2}{\pi \tilde{R}^2} \sqrt{\tilde{R}^2 - (x-1)^2}, \;\; 1 - \tilde{R} < x < 1 + \tilde{R}.$$

Average entanglement in a pure tripartite state $(N_1 \otimes N_2) \otimes N_3$

$$\langle E_{LN}^{12} \rangle = \log \left[\frac{2}{\pi} \sin^{-1}\left(\frac{1}{\tilde{R}}\right) + \frac{2}{3\pi \tilde{R}} \sqrt{1 - \frac{1}{\tilde{R}^2}} \left(1 + 2\tilde{R}^2\right) \right], \quad \tilde{R} = 2\sqrt{\frac{N_1 N_2}{N_3}}$$



Average Log-negativity $(N_1 \otimes N_2) \otimes N_3$

$\tilde{R} \gg 1$, $N_1 N_2 \gg N_3$, deep in the NPT regime, this gives

$$\langle E_{LN} \rangle \approx \log \left(\frac{8}{3\pi} \sqrt{\frac{N_1 N_2}{N_3}} \right).$$

When $N_3 = 1$ the state ρ_{12} is pure.

$$\langle E_{LN} \rangle = \left\langle \log \left(\sum_{i=1}^{N_1 N_2} |\mu_i| \right) \right\rangle = \left\langle \log \left(\sum_{i=1}^{N_1} \sqrt{\lambda_i} \right)^2 \right\rangle \approx \log(\kappa^2 N_1).$$

$$\kappa = \left(\frac{8}{3\pi} \right) \text{ when } N_1 = N_2.$$

Slightly different (more analytic & correct) *c.f.* A. Datta, Phys. Rev. A, **81**, 052312 (2010).

Entanglement at Criticality and Extreme Eigenvalues

For critical dimensions $\tilde{R} = 1$ and the semicircle gives zero entanglement. This is **not true** due to eigenvalues in the **tail of the semicircle**

Table: Percentage of NPT states for $L_1 = L_2$ and various L for the critical case when $L_1 + L_2 = L/2 - 1$.

L_1	L	% NPT (Complex states)	% NPT (Real states)
1	6	0.06 ± 0.008	3.18 ± 0.017
2	10	1.40 ± 0.036	7.82 ± 0.085
3	14	1.92 ± 0.065	11.18 ± 0.121
4	18	2.40 ± 0.077	13.43 ± 0.161
5	22	2.60 ± 0.145	15.17 ± 0.35

The fraction of NPT states = fraction whose μ_{min}, the **min. eigenvalue after PT < 0 : A problem in the theory of extreme value statistics**.

Away from criticality

Table: Percentage of NPT states for $L_1 = L_2$ and various L (Real states).

L_1	$L = 4L_1 + 1$	% NPT	$L = 4L_1 + 3$	% NPT
1	5	25.39	7	4.4×10^{-2}
2	9	96.82	11	8.3×10^{-5}
3	13	≈ 100	15	$< 10^{-5}$
4	17	≈ 100	19	≈ 0
5	21	≈ 100	23	≈ 0

Will constitute a **problem of large deviation.**

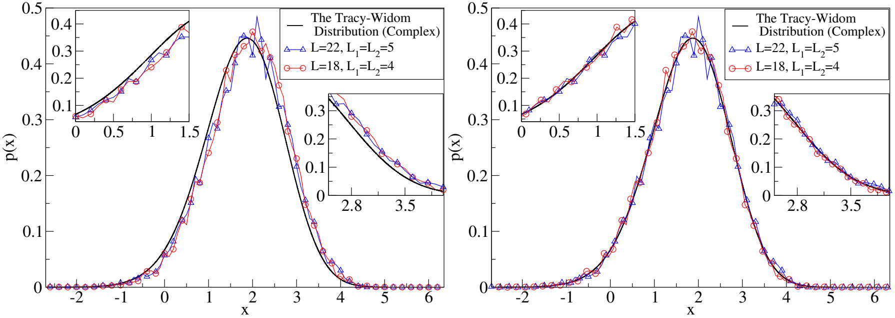
Tracy-Widom and fraction of NPT states

$x = 2\,N^{5/3}\,\mu_{min}$ is asymptotically distributed according to TW

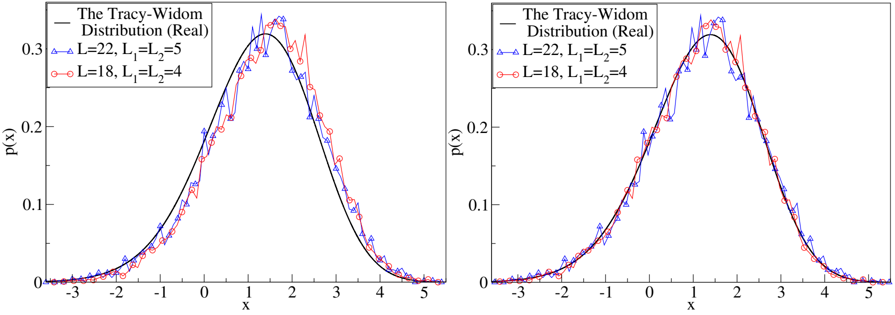
$$f_{NPT} = 1 - F_2(0) \approx .03$$

where $F_2(x)$ is related to a solution of the Painlevé-II equation

$q'' = xq + 2q^3$ with $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$.



The Real case, ≈ GOE. Log-Neg. at criticality



$$f_{NPT} \approx 0.17$$

$$E_{LN} = \log\Big(\sum_i |\mu_i|\Big) = \log\Big(1 - 2\sum_{i;\mu_i<0} \mu_i\Big) \approx -2\mu_{min}\Theta(\mu_{min})$$

$$\langle E_{LN}\rangle \approx -2\langle \mu_{min}\Theta(-\mu_{min})\rangle = \frac{2}{\sqrt{N_3}N^{7/6}}\int_{-\infty}^{-s} -(x+s)p(x)dx \sim N^{-5/3},$$

Log-Neg. at criticality

Table: Average log-negativity for $L_1 = L_2$ and various L for the critical case (complex).

L_1	$L = 4L_1 + 2$	Numerical $\langle E_{LN} \rangle$	$\langle E_{LN} \rangle$ using TW
3	14	7.28×10^{-6}	8.39×10^{-6}
4	18	9.28×10^{-7}	8.95×10^{-7}
5	22	9.47×10^{-8}	9.79×10^{-8}

Table: Average log-negativity for $L_1 = L_2$ and various L for the critical case (real).

L_1	$L = 4L_1 + 2$	Numerical $\langle E_{LN} \rangle$	$\langle E_{LN} \rangle$ using TW
3	14	7.62×10^{-5}	8.26×10^{-5}
4	18	9.41×10^{-6}	9.51×10^{-6}
5	22	1.13×10^{-6}	1.06×10^{-6}

Discussions and Summary

- Statistics of the PT of tripartite pure states give rise to Wigner semicircles.
- A simple RMT model captures the NPT-PPT transition
- At critical dimensions extreme value statistics and the Tracy Widom distribution gives the fraction of NPT states
- The average third moment of the PT and the skewness have been calculated exactly
- Three coupled standard maps show slight, but systematic deviations, from random states. Especially at criticality. RMT seems applicable strictly only asymptotically.

This work:

Udaysinh Bhosale, S. Tomsovic, AL: *In preparation*.

Related Works:

1. G. Aubrun, ArXiv:1011.0275v2 [mathPR]. [Discusses the emergence of shifted semicircles, using binary correlations]
2. A. Datta, Phys. Rev. A, **81**, 052312 (2010) [Average Log-negativity when 1+2 is pure.]

The End. Really. Thanks.