Quantum Entanglement- Fundamental Aspects

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Abstract

Entanglement is one of the most useful resource in quantum information processing. It is effectively the quantum correlation between different subsystems of a composite system. Mathematically one of the most hard task in quantum mechanics is to characterize entanglement. However progress in this field is remarkable but not complete yet. There are many things to do with entanglement. In this lecture we want describe some of its fundamental aspects, viz., how far we are able to classify quantum states and what are the basic issues of quantifying entanglement.

1 Introduction

Characterization of quantum entanglement is closely related with the following three aspects in quantum information theory.

(i) The detection of quantum entanglement. i.e., whether a state is entangled or not?

(ii) Classification of states of a composite system.

(iii) Quantification of entanglement.

We would like to address the above three aspects of quantum entanglement in our lecture. However we restrict ourselves only with the composite systems consist of two parties, i.e., only bipartite systems. Now before going to discuss these matters we first describe some preliminary notions regarding quantum systems.

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System. Every quantum system is associated with a separable complex Hilbert space, say, H.

Observable. Observables are linear, self-adjoint operators (may be unbounded) acting on H.

State. States are represented by density operators acting on H.

By density operators we mean any self-adjoint(hermitian) linear operator ρ acting on H that satisfies the following two relations:

(a) ρ is non-negative definite, i.e., $\rho \geq 0$.

(b) $tr(\rho) = 1$.

Now for any density operator/matrix the following holds;

$$\rho^2 \leq \rho$$

It readily divides density operators/matrices into two distinct classes, viz.,

(i) $\rho^2 = \rho$, and (ii) $\rho^2 < \rho$.

Correspondingly the states are classified for any quantum system into two different categories;

(1) The states for which $\rho^2 = \rho$, are called *pure states*. In this case there is always a unique state vector $|\psi\rangle$ in H so that we can express ρ as projection operator on $|\psi\rangle$, i.e., $\rho = |\psi\rangle\langle\psi|$. So, pure states are identified with vectors/rays of the Hilbert space H.

(2) The states for which $\rho^2 < \rho$, are called *mixed states*. Mixed states have no such representations like pure states. There are infinitely many ways to represent a mixed state by mixture of pure states. They have no unique representation.

Actually in density matrix/operator theory the set of all states is a convex set and pure states are extreme points in that convex set.

Now for simplest quantum system, i.e., the system with dimension two, states are known as qubits (quantum bits) like classical bits. It is interesting to note that qubits has a unique geometric configuration in unit sphere (known as Bloch sphere). In Bloch sphere representation qubits have the following form by Pauli matrices $\sigma_x, \sigma_y, \sigma_z$;

$$\rho = \frac{1}{2}[I + \overrightarrow{n}.\overrightarrow{\sigma}],$$

where, $\overrightarrow{n} = (n_x, n_y, n_z)$ is called Bloch vector, $\overrightarrow{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $I =$ two dimensional identity operator. For pure qubits $|\overrightarrow{n}| = 1$ and for mixed qubits $|\overrightarrow{n}| < 1$.

However, for systems with dimension greater than two there are no such geometric configuration like qubits. The states in a d- dimensional system are called qudits.

2 Composite systems and Entanglement

Consider a quantum system consists of several number of parties, say, A, B, C, D, etc. Such systems are known as multipartite systems. Mathematically, the joint system H is described by the tensor product of individual systems $H_A, H_B, H_C, \ldots$; i.e., as $H = H_A \otimes H_B \otimes H_C \otimes \ldots$. The first classification of quantum states in such composite systems is as follows:

(i)**Separable states:** Any state $\rho_{ABCD\ldots}$ of a composite system consists of parties A, B, C, D, etc., is called a separable/classically correlated state iff we can represent the state in the following manner:

$$\rho_{ABCD\ldots} = \sum_i w_i \rho_A^i \otimes \rho_B^i \otimes \rho_C^i \otimes \rho_D^i \otimes \cdots$$

where $0 \leq w_i \leq 1, \quad \sum_i w_i = 1$, and $\rho_A^i, \rho_B^i, \rho_C^i, \rho_D^i, \ldots$ are states of different subsystems.

In other words separable states are those which can be prepared locally.

(ii)**Entangled states:** The states which cannot be represented as above considering all possible changes in local bases are known as entangled states.

Alternately, all the states other than the separable states are entangled.

It is really a hard task to characterize entangled states. Till now we have only partial answers and this domain is one of the fundamental task in quantum information theory. We concentrate here only with the bipartite systems where we would find so many problems to solve.

Now we consider first the case of *pure bipartite states.* All the pure separable states have the product form, called usually pure product states. Symbolically, if $|\psi\rangle_{AB}$ is a pure product state, then we can write it always as $|\psi\rangle_{AB} = |\chi\rangle_A \otimes |\phi\rangle_B$.

For pure entangled states, if $|\psi\rangle_{AB}$ is a pure entangled state, then we can write it always as,

$$|\psi\rangle_{AB} = \sum_i^{k \leq \min\{dim(H_A), dim(H_B)\}} \sqrt{\lambda_i} |i\rangle_A \otimes |i\rangle_B$$

where $0 \leq \lambda_i$, $\sum_i \lambda_i = 1$ and $\{|i\rangle_A\}, \{|i\rangle_B\}$ are two orthonormal bases of systems A and B respectively. Here it should be noted that for pure entangled states the number of non-zero terms in the above sum must be at least two. The form what we have written above for pure entangled states is known as *Schmidt decomposition of pure states.* The non-zero elements λ_i are called Schmidt terms/coefficients. This is a consequence of singular value decomposition theorem in linear operator theory. Actually we can write any pure state in the above form whether product or entangled.

However for mixed states we have no such theorem by which we can find whether a state is entangled or separable. We need further tools to detect entangled states. We first describe some good detectors of entanglement/separability.

2.1 Bell Inequalities:

These are very good detectors of entanglement. Any state of composite system that violets some Bell type inequalities is known to be an entangled state. For separability, all separable states would satisfy Bell type inequalities. The most common Bell inequalities are known as Bell-CHSH inequalities, given by,

$$tr(\rho\mathcal{B}) \leq 2,$$

where $\mathcal{B}$ is the Bell operator,

$$\mathcal{B} = \overrightarrow{a}.\overrightarrow{\sigma} \otimes (\overrightarrow{b} + \overrightarrow{b'}).\overrightarrow{\sigma} + \overrightarrow{a'}.\overrightarrow{\sigma} \otimes (\overrightarrow{b} - \overrightarrow{b'}).\overrightarrow{\sigma}$$

with $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{a'}$, $\overrightarrow{b'}$ are three dimensional unit vectors.

For two qubit system an equivalent inequality is given by Horodecki et.al., as follows;

$$M(\rho) \leq 1$$

where $M(\rho)$ is constructed by the following prescription:

Write the two-qubit state ρ in the form:

$$\rho = \frac{1}{4}[I \otimes I + \overrightarrow{r}.\overrightarrow{\sigma} \otimes I + I \otimes \overrightarrow{s}.\overrightarrow{\sigma} + \sum_{i,j=x,y,z} t_{ij}\sigma_i \otimes \sigma_j]$$

Now form the real matrix $T = (t_{ij})$ and find the sum of the two greater eigenvalues of the matrix $T^{\dagger}T$. This sum is $M(\rho)$.

The Bell-inequalities are not sufficient to detect all entangled states. All pure entangled states violet some Bell-inequalities. However there are mixed entangled states that satisfy all the standard Bell inequalities. In the simplest dimension, i.e., in 2×2 the so-called Werner states provide us an example that satisfies Bell-CHSH inequalities but entangled for large region. The entanglement is found initially by the flip-operator. In 2×2, Werner states are given by,

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}I$$

where $\frac{-1}{3} \leq p \leq 1$ and $|\psi^-\rangle, I$ are the singlet state and the identity operator in 2×2. It is easy to check that, for $p > \frac{1}{\sqrt{2}}$, ρ_W violets Bell-inequalities. However, for $\frac{1}{3} < p \leq \frac{1}{\sqrt{2}}$, they remain entangled. We now present the most useful detector of entanglement via partial transposition.

2.2 Partial transposition:

Suppose ρ_{AB} be a bipartite state shared between two parties A and B and $\{|i\rangle_A\}, \{|i\rangle_B\}$ be two orthonormal bases of the corresponding systems. Then we can represent ρ_{AB} as follows;

$$\rho_{AB} = \sum_{i,j,k,l} \rho_{ij,kl} |ij\rangle_{AB}\langle kl|$$

where by $|ij\rangle_{AB}\langle kl|$ we mean $|i\rangle_A\langle k| \otimes |j\rangle_B\langle l|$. The *partial transposition* of ρ_{AB} with respect to system, say B, is then defined by;

$$\rho_{AB}^{T_B} = \sum_{i,j,k,l} \rho_{ij,kl} |il\rangle_{AB}\langle kj|$$

Or, in other words, the matrix elements of $\rho_{AB}^{T_B}$ is defined as;

$$\rho_{ij,kl}^{T_B} = \rho_{il,kj}$$

In linear operator theory the operation transposition is a positive operator but not completely positive(which are positive in all possible extended spaces). The consequence of not a completely positive map, partial transposition is unable to preserve positivity of density matrices/operators but preserves hermiticity. So after partial transposition a state cannot be remain in general a state, it may have negative eigenvalues. We then classify bipartite states in two distinct classes:

(i) *states with positive partial transposition* or, PPT states, i.e., after partial transposition they remain positive operator; and

(ii) *states with negative partial transposition* or, NPT states, i.e., after partial transposition they have at least one negative eigenvalues.

Immediate consequence of the above classification is as follows:

(a) all NPT states are entangled;

(b) all separable states are PPT states.

However there are entangled states which are PPT states and also they have another fundamental property of bound entanglement, for which we require some notions of quantification that are necessary for information processing. One can now check that Werner states in 2×2 are entangled if $p > \frac{1}{3}$ and they are separable for $p \leq \frac{1}{3}$. For 2×2 and 2×3 systems partial transposition is a necessary and sufficient condition for separability, however for higher dimensional systems it is only a necessary condition for separability, i.e., all separable states are necessarily PPT.

Now to characterize further and also to use entanglement as a resource for information processing, we have to describe some quantification schemes for entanglement. After reviewing some notions of measure of entanglement and the problem of quantification, we again describe some other useful detector like, reduction criterion, maximally entangled fraction, etc. Before going to the concept of measure of entanglement, we first describe some notions of physical operations and local operations with classical communications (in short, LOCC).

2.3 Physical operations and LOCC

Suppose a physical system is described by a state ρ. By a physical operation on ρ we mean a completely positive map $\mathcal{E}$ acting on the system and described by

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger$$

where each A_k is positive linear operator that satisfies the relation $\sum_k A_k^\dagger A_k \leq I$. If $\sum_k A_k^\dagger A_k = I$, then the operation is trace preserving. When the state is shared between a number of parties, say, A, B, C, D,. .... and each A_k has the form $A_k = L_k^A \otimes L_k^B \otimes L_k^C \otimes L_k^D \otimes \cdots$ with all the $L_k^A, L_k^B, L_k^C, L_k^D, \cdots$ are linear positive operators, the operator is then called a separable superoperator. Now every LOCC is a separable superoperator but it is unknown to us whether the converse is also true or not.

2.4 Measure of entanglement

Entanglement can be used to perform various tasks which are otherwise impossible. To apply this resource perfectly the quantification is very much necessary. However there are no measure of entanglement which satisfies all the properties of a good measure of entanglement. Now we will discuss about some important measures of entanglement.

2.4.1 Von-Neumann entropy and Pure state entanglement

The Von-Neumann entropy for any state whose density matrix is ρ, is given by

$$S(\rho) \; = \; tr(\rho \log \rho)$$

Now for a pure bipartite state $|\Psi\rangle_{AB}$, the entanglement of the state is defined by the Von-Neumann entropy of any of its reduced density matrices. Interestingly, this is the unique measure of pure state entanglement.

2.4.2 Entanglement of formation

Entanglement of Formation E_F of a bipartite mixed state ρ_{AB} is defined to be the minimum value of convex sum of pure state entanglements $E(\xi)$ over all possible ensembles of pure states $\xi \equiv \{p_i, |\Psi_i\rangle\}$ which realizes the mixed state ρ_{AB}, i.e., ρ_{AB} can be prepared from a mixture of that ensemble as $\rho_{AB} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ and $E_F(\rho_{AB}) = \min_\xi [p_i E(|\Psi_i\rangle)]$.

2.4.3 Distillable entanglement

The distillable entanglement $E_D(\rho_{AB})$ is defined formally as

$$E_D(\rho_{AB}) = \sup_m \zeta_m \equiv \lim_{n\to\infty} \frac{m}{n} \tag{1}$$

where m copies of Bell state $|\Phi^+\rangle \equiv \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ can be extracted from n copies of ρ_{AB}.

It is interesting to note that distillable entanglement of any PPT state is zero. The entangled states which have zero distillable entanglement are called bound entangled states.

There are some other well known measures like, Relative entropy of Entanglement, Logarithmic Negativity, Squashed Entanglement, Concurrence, etc. For some specific tasks they are very much usable.

Now we describe our last detection criterion, known as reduction criterion.

2.5 Reduction criteria:

Separable states must satisfy the following two inequalities

$$I \otimes \varrho_B - \varrho_{AB} \geq 0 , \quad \varrho_A \otimes I - \varrho_{AB} \geq 0$$

If any one of the above two conditions is violated then the state ϱ_{AB} would be entangled.