Quantum Computation

Concepts and Prospects

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28 February 2008, IISER, Pune

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Status

Laws of quantum mechanics are precisely known.
Theoretical foundation of the subject is clear.
Elementary hardware components work as predicted.
Large scale integration (say 10 or more components) is a technological challenge. Noone knows when that will arrive, or what a quantum computer will be used for.

It is inevitable

"Because the nature isn't classical, damn it . . ."

—R.P. Feynman

Science: Observe and explain phenomena. Theorise!
Technology: Design and control phenomena. Optimise!

Yesterday's science becomes tomorrow's technology.

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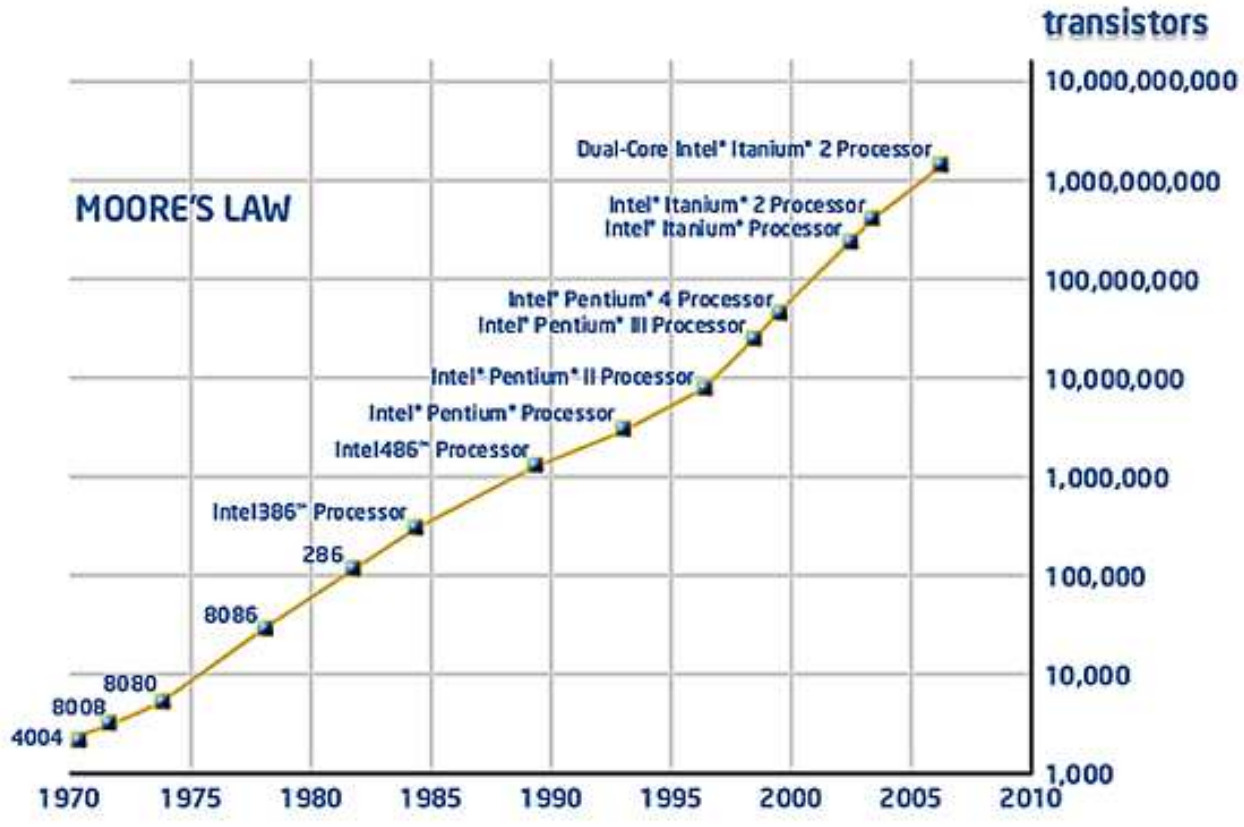
Laws of classical physics are convenient and useful (and yet) approximations to the laws of quantum physics.

Quantum effects (discreteness, dispersion, tunnelling etc.) have been considered "undesirable nuisance" in the classical computer design.
Why not go to the other extreme, where classical effects (decoherence, thermal fluctuations etc.) become "undersirable nuisance" in the quantum computer design?

Shrinking computer circuits



Number of transistors on a chip doubles every two years.

1948: First transistor, size 1 cm.

Today: VLSI circuits, size 45 nm.

Atomic size, 0.1 nm, is not very far!

It is a breakthrough

Computers are physical devices, not mere mathematical entities to implement algorithms. Quantum mechanics demonstrates that complex numbers are physical.
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Superposition allows multiple signals at the same point at the same time. All of them can be simultaneously processed, and any one of them can be selectively observed (e.g. radio or cell-phone transmissions).
This offers an SIMD parallel computing paradigm with no extra hardware. Which algorithms can exploit this?

Basics

A qubit is the simplest quantum system, with two basis vectors $|0\rangle$ and $|1\rangle$ (e.g. an electron spin). A generic qubit state is a complex unit vector in the 2-dim Hilbert space.

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

A quantum register is an ordered string of n qubits.
It is a complex unit vector in the 2^n-dim Hilbert space.

$$|x\rangle = \sum_{i_1,i_2\ldots i_n=0}^{1} c_{i_1 i_2 \ldots i_n} |x_{i_1}\rangle|x_{i_2}\rangle \ldots |x_{i_n}\rangle, \quad \sum_{i_1,i_2\ldots i_n=0}^{1} |c_{i_1 i_2 \ldots i_n}|^2 = 1.$$

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A generic instruction is a rotation of the quantum state vector in the Hilbert space. It is a unitary transformation. This evolution is deterministic and fully reversible.

A measurement is a projection. In the computational basis, it yields the state $|x_{i_1}\rangle|x_{i_2}\rangle \ldots |x_{i_n}\rangle$ with probability $|c_{i_1 i_2 \ldots i_n}|^2$. This operation is probabilistic and irreversible.

Applications

Complexity of a quantum algorithm is decided by the trade-off between the number of evolution steps and the number of states that can be coherently superposed.
The gain may be exponential or just marginal (factor of 2).

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Pattern recognition: Clever superposition and interference can amplify the desired feature. The gain depends on the structure of the pattern in the data.

Factorisation

No polynomial (in the number of digits) classical algorithm for factoring a number is known. Security of public key cryptography (e.g. RSA) relies on this fact.

The problem of factoring a number N can be reduced to finding the period of the function $f(x) = a^x \bmod N$.

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Period r: $f(0) = 1,\ f(1) = a, \ldots,\ f(r) = a^r \bmod N = 1.$

When r is even, $(a^{r/2} - 1)(a^{r/2} + 1) = 0 \bmod N$. So either $(a^{r/2} - 1)$ or $(a^{r/2} + 1)$ has a factor in common with N.

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Example: $N = 15$ and $a = 2$.

$2^x \bmod 15 = 1, 2, 4, 8, 16 \rightarrow 1, 32 \rightarrow 2, \ldots \ \Rightarrow\ r = 4, r/2 = 2.$

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Periodic patterns are easily detected by Fourier Transform, which is a unitary operation (common in quantum theory).

Quantum Fourier Transform

$$\sum_x f(x)|x\rangle = \sum_y \left(\frac{1}{\sqrt{N}} \sum_x e^{2\pi i x y/N} f(x) \right) |y\rangle$$

Let $N = 2^n$, and use the same tricks as in FFT.
In binary notation, $x = x_{n-1} \cdot 2^{n-1} + \ldots + x_1 \cdot 2 + x_0$.
$\mathrm{frac}(\frac{xy}{N}) = y_{n-1}(.x_0) + y_{n-2}(.x_1 x_0) + \ldots + y_0(.x_{n-1} \ldots x_0)$.

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Unitary rotation of QFT: $|x\rangle \to \frac{1}{\sqrt{N}} \sum_y e^{2\pi i x y/N} |y\rangle$

$$= \frac{\left(|0\rangle + e^{2\pi i (.x_0)} |1\rangle\right)}{\sqrt{2}} \frac{\left(|0\rangle + e^{2\pi i (.x_1 x_0)} |1\rangle\right)}{\sqrt{2}} \ldots \frac{\left(|0\rangle + e^{2\pi i (.x_{n-1} \ldots x_0)} |1\rangle\right)}{\sqrt{2}}$$

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Peter Shor gave this polynomial factorisation algorithm.
Fourier Transform is a multiplication by an $N \times N$ matrix.
FFT factorisation reduces the operations to $O(N \log N)$.
QFT parallelism cuts down the operations to $O((\log N)^2)$.

Quantum random walk

Efficient solutions of many practical problems require non-deterministic algorithms, which contain probabilistic branched evolution trees.
These problems are typically described using graphs, with vertices denoting the states and edges denoting the evolutionary routes.

A classical computer can explore only one branch at a time, and random numbers (or equivalently coin toss instructions) are used to explore different evolutionary branches.
A particular evolution corresponds to a specific walk on the graph. The final solution is obtained by combining the results for many random walks.

Quantum computers can explore multiple evolutionary branches of an algorithm—in a single attempt—by using clever superpositions of states. (Coin is unnecessary.)

Quantum diffusion

Random walks represent a diffusion process.

Classical diffusion operator is the Laplacian: $\frac{\partial P}{\partial t} = \nabla^2 P$.

A spatial mode with wave vector $\vec{k}$ evolves as $\exp(-E(\vec{k})t)$, with $E(\vec{k}) \propto |\vec{k}|^2$. The slowest propagating modes (small $\vec{k}$) produce the characteristic Brownian motion signature:

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But there is an alternative. Relativistic Dirac equation uses the diffusion operator $\vec{\alpha} \cdot \vec{\nabla}$ (α_i are anticommuting objects, e.g. Pauli matrices) with $E(\vec{k}) \propto |\vec{k}|$ and the signature:

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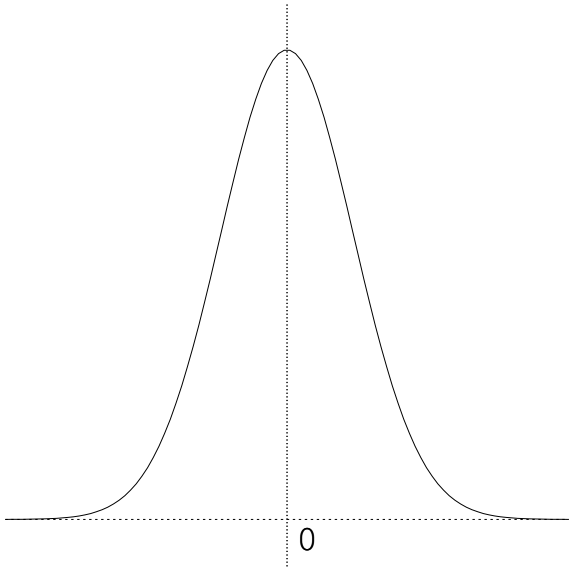
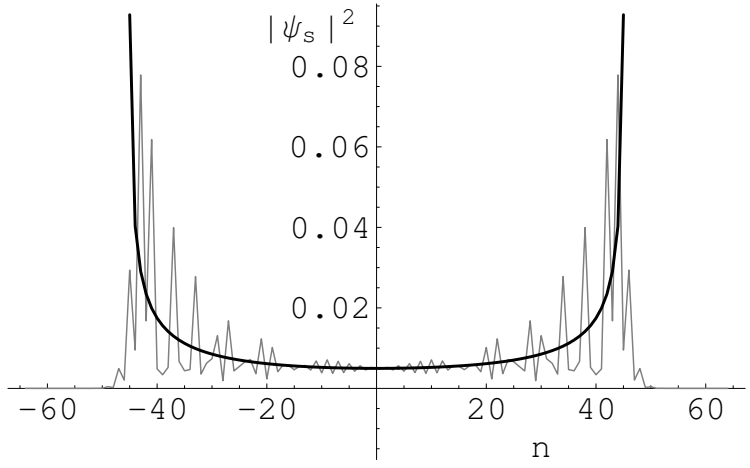
Any NP-complete problem speeds up at least quadratically.

Random walk on a line





$$P(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

$$\int P(x,t)dx = 1$$

$$\int |x| \cdot P(x,t)dx = \sqrt{\frac{2t}{\pi}}$$

$$\int x^2 P(x,t)dx = t$$

$$|\psi_s|^2 = \frac{4t^2}{\pi\sqrt{4t^2-2n^2}\,(4t^2-n^2)}$$

$$\int_{n=-\sqrt{2}t}^{\sqrt{2}t} |\psi_s|^2 dn = 1$$

$$\int_{n=-\sqrt{2}t}^{\sqrt{2}t} |n| \cdot |\psi_s|^2 dn = t$$

$$\int_{n=-\sqrt{2}t}^{\sqrt{2}t} n^2 |\psi_s|^2 dn = 2(2-\sqrt{2})t^2$$

Probability distributions for symmetric random walks:

Left: The classical one is a Gaussian.

Right: The quantum one is double peaked.

Quantum database search

Consider an unsorted database with N items.
Starting from an unbiased state, the desired item is to be found using the smallest number of binary oracle calls.

States: $|i\rangle$ any item, $|s\rangle$ starting state, $|t\rangle$ target state.

$$|\langle i|s\rangle|^2 = 1/N, \quad |\langle i|t\rangle|^2 = \delta_{it}.$$

Operators: Reflections along $|t\rangle$ and $|s\rangle$ directions.

$$U_t = 1 - 2|t\rangle\langle t| \quad \text{(Potential energy attraction)}$$

$$U_s = 1 - 2|s\rangle\langle s| \quad \text{(Kinetic energy diffusion)}$$

Algorithm: $(-U_s U_t)^Q |s\rangle = |t\rangle$

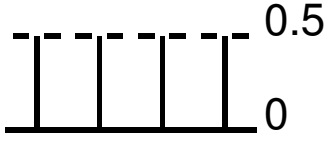
Solution: $(2Q+1)\sin^{-1}(1/\sqrt{N}) = \pi/2 \Longrightarrow Q = \pi\sqrt{N}/4$

The algorithm is optimal, evolving the starting state $|s\rangle$ to the target state $|t\rangle$ along the shortest geodesic route.

Lov Grover gave this $O(\sqrt{N})$ algorithm.
(Any Boolean logic algorithm needs $O(N)$ oracle calls.)

An example

The steps of the algorithm for the simplest case of 4 items in the database. Let the first item be desired by the oracle.

	Amplitudes	Algorithmic Steps	Physical Implementation
(1)		Uniform distribution	Equilibrium configuration

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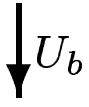
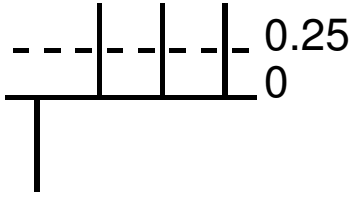
		Quantum oracle	Binary question
(2)		Amplitude of desired state flipped in sign	Sudden perturbation

(Dashed line denotes the average amplitude.)

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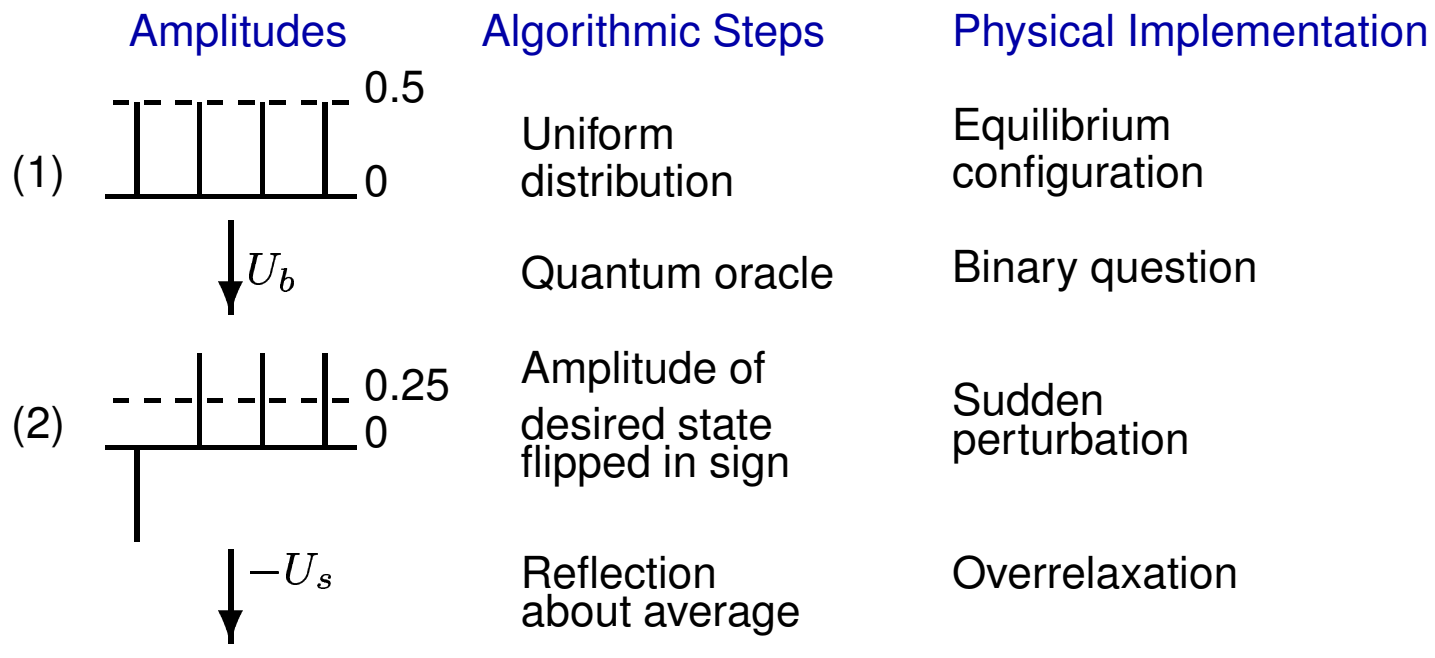

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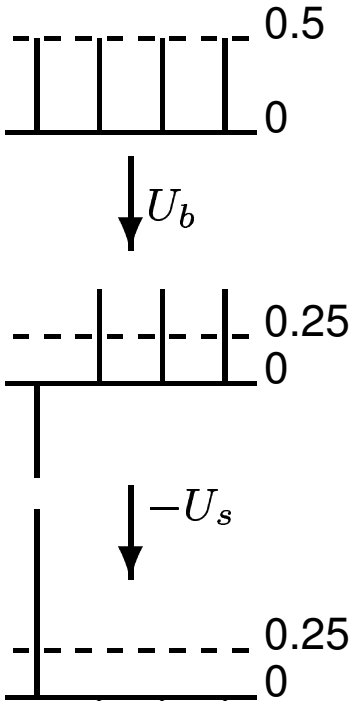

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A mechanical model

Grover's algorithm is an amplitude amplification process. A system of coupled wave modes can execute it, provided (1) Superposition of modes maintains phase coherence. (2) The two reflection operations (phase changes of π for the appropriate mode) can be efficiently implemented.

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Consider N identical coupled harmonic oscillators.
Identical coupling between them is arranged by attaching them to a big oscillator through the centre-of-mass mode.
Elastic reflection of an oscillator implements the binary oracle in velocity space. Evolution by half an oscillation period implements the reflection about average operation.

Possible uses

Decoherence of quantum behaviour is extremely fast, but vibrational systems with small damping can be made easily.

Focusing of energy:

Concentration of total energy of a coupled oscillator system into a specific oscillator can have potential applications in processes that are highly sensitive to energy availability.

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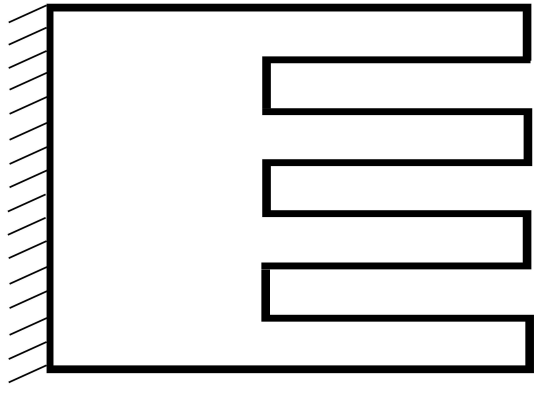
Nanomechanical devices: A coupled oscillator system can provide efficient focusing of energy at a specific location, when one cannot directly control the component concerned.

For example, a comb-shaped cantilever beam can be used as a selective switch.



Catalysis: There exist many processes that need crossing of an energy threshold for completion. Their rates are typically governed by the Boltzmann factor for the energy barrier, $\exp(-E_{\rm barrier}/kT)$. Energy amplification can speed up the rates of such processes by large factors.

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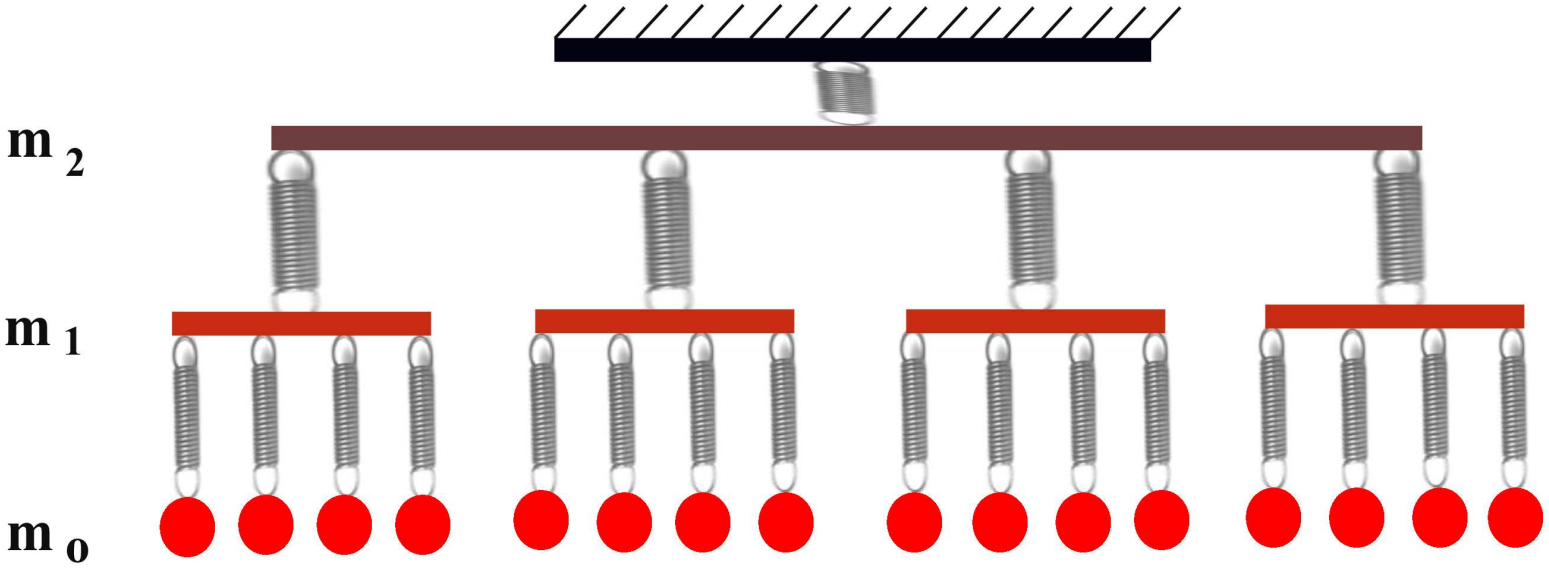
Shock absorbers and vibrational isolation: Instead of damping a single perturbed oscillator, it is much more efficient to disperse the energy into several oscillators while damping them together.

A hierarchical system of oscillators—four small ones coupled to a big one at every level with appropriate mass, spring and damping parameters—can provide a practical realisation of this idea.



(The initial impulse is taken to be a local disturbance, which subsequently spreads out.)

Genetic languages

1. What is the information processing task carried out by the genetic code?
   Assembling molecules by picking up components from an unsorted database.
2. What is the optimal way of carrying out this task?
   Lov Grover's quantum search algorithm.
   (Requires wave dynamics.)
3. What is the signature of this algorithm?

$$(2Q+1)\sin^{-1}\frac{1}{\sqrt{N}} = \frac{\pi}{2} \implies \begin{cases} Q=1, & \text{N=4} \\ Q=2, & \text{N=10.5} \\ Q=3, & \text{N=20.2} \end{cases}$$

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