LOCAL UNITARY EQUIVALENT CLASSES OF SYMMETRIC *N*-QUBIT MIXED STATES

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Introduction

- ρ and ρ' are said to be LU equivalent if $\rho' = U\rho U^{\dagger}$: $U \in SU(2)^{\times N}$
- States belonging to the same LU equivalent class can be used for similar kind of quantum information processing tasks as they posses the same amount entanglement.

Symmetric States

- Set of N-qubit pure states that remain unchanged by permutations of individual particles are called symmetric states.
- Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to $(N+1)$.
- Such a Hilbert space is considered to be spanned by the eigen states $\{|j,m\rangle; -j \leq m \leq +j\}$ of angular momentum operators J^2 and J_z, where $j = \frac{N}{2}$.

Symmetric States

Examples

- Bell State $\longrightarrow \begin{cases} |\psi_1\rangle = \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}} \\ |\psi_2\rangle = \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}} \\ |\psi_3\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \end{cases}$

- $|\psi_W\rangle \equiv \frac{|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle}{\sqrt{3}}$.

- $|\psi_{GHZ}\rangle = \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$.

Majorana Representation of Pure Symmetric States[1]

The most general spin-j pure state $|\psi^j\rangle$ is given by

$$|\psi^j\rangle = \sum_{m=-j}^{+j} a_m \, |jm\rangle \tag{1}$$

Consider a rotation $R(\phi,\theta,0)$ of the frame of reference such that the expansion coefficient a_{-j} in the rotated frame vanishes i.e;

$$(a_{-j})^R = 0 = \langle j - j|R^{-1}(\phi,\theta,0)|\psi_j\rangle = \sum_m a_m \langle j - j|R^{-1}(\phi,\theta,0)|jm\rangle$$

$$= \sum_m a_m \, D^{*j}_{m-j}(\phi,\theta,0) = \sum_m a_m \, (-)^{-(j+m)} \, D^j_{-mj}(\phi,\theta,0), \tag{2}$$

[1] E. Majorana, *Nuovo Cimento* **9** (1932) 43.

Majorana Representation of Pure Symmetric States[2]

$\boxed{D^{j}_{-mj}(\phi, \theta, 0) \text{ are the matrix elements of Wigner rotation matrices}}$

$$D^{j}_{m'm}(\alpha\beta\gamma) = e^{-im'\alpha} e^{-im\gamma} \sum_{s} \frac{(-)^{s}\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{s!(j-s-m')!(j+m-s)!(m'+s-m)!}$$

$$\times \left(cos\frac{\beta}{2}\right)^{2j+m-m'-2s} \left(-sin\frac{\beta}{2}\right)^{m'-m+2s} .$$

where $s = j + m$.

[2]M. E. Rose *Elementary theory of angular momentum* (John Wiley, New York, 1957).

Majorana Representation of Pure Symmetric States

$$\mathcal{A} \sum_{m=-j}^{+j} (-1)^{j+m} \sqrt{^{2j}\, C_{j+m}}\, a_m\, z^{j+m} = 0 \;\; where \;\; \begin{cases} z = tan\left(\frac{\theta}{2}\right)\; e^{i\phi} \\ \mathcal{A} = cos^{2j}\left(\frac{\theta}{2}\right)\; e^{-i\phi j} \end{cases}$$

$$P(z) = \sum_{m=-j}^{+j} (-1)^{j+m} \sqrt{^{2j}\, C_{j+m}}\, a_m\, z^{j+m} = 0, \quad for\;\; \theta \neq \pi. \tag{3}$$

$$\mathcal{A}' \sum_{m=-j}^{+j} (-1)^{j+m} \sqrt{^{2j}\, C_{j+m}}\, a_m\, z'^{j-m} = 0 \;\; where \;\; \begin{cases} z' = \frac{1}{z} = cot\left(\frac{\theta}{2}\right)\; e^{-i\phi} \\ \mathcal{A}' = sin^{2j}\left(\frac{\theta}{2}\right)\; e^{i\phi j} \end{cases}$$

$$P(z') = \sum_{m=-j}^{+j} (-1)^{j-m} \sqrt{^{2j}\, C_{j+m}}\, a_m\, z'^{j-m} = 0, \quad for\;\; \theta \neq 0. \tag{4}$$

Solving either of the polynomial equation one gets $2j$ solutions namely $\{(\theta_1, \phi_1), (\theta_2, \phi_2), ..., (\theta_{2j}, \phi_{2j})\}$ in general. Thus every symmetric state $|\psi^j\rangle$ can be represented by a constellation of $2j$ points on the Block sphere or

$$|\psi^{2j}_{sym}\rangle = \mathcal{N} \sum_P \hat{P} |\epsilon_1, \epsilon_2, ..., \epsilon_{2j}\rangle, \tag{5}$$

where

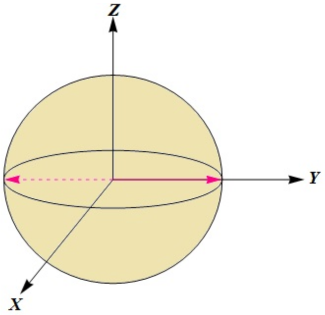
$$|\epsilon_k\rangle = cos(\theta_k/2) e^{-i\phi_k/2}\, |0\rangle + sin(\theta_k/2) e^{i\phi_k/2}\, |1\rangle, \quad k = 0, 1, ..., 2j. \tag{6}$$

refer to the $2j$ spinors constituting the symmetric state $|\psi^{2j}_{sym}\rangle$; $\hat{P}$ corresponds to the set of $(2j)!$ permutations of the spinors and $\mathcal{N}$ corresponds to an overall normalization factor.

Majorana Representation of Pure Symmetric States (Examples)

Bell State

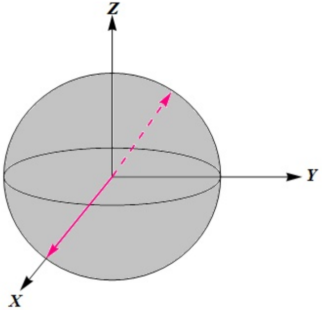
- $|\psi\rangle = \frac{|11\rangle + |1-1\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$
- $z = e^{\pm i\frac{\pi}{2}}$
- $(\frac{\pi}{2}, \frac{\pi}{2}),\ (\frac{\pi}{2}, \frac{3\pi}{2})$



Majorana Representation of Pure Symmetric States (Examples)

Bell State

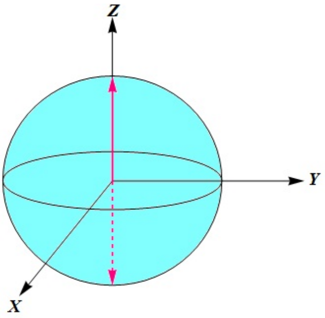
- $|\psi\rangle = \frac{|11\rangle - |1-1\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}$
- $z = \pm 1$
- $(\frac{\pi}{2}, 0), \ (\frac{\pi}{2}, \pi)$



Majorana Representation of Pure Symmetric States (Examples)

Bell State

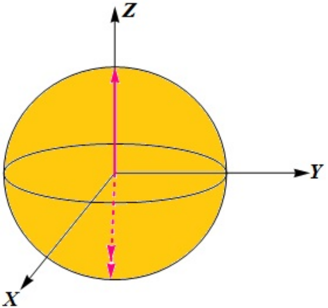
- $|\psi\rangle = |10\rangle \equiv \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$
- $z = 0$
- $\theta = 0, \;\; \theta = \pi$



Majorana Representation of Pure Symmetric States (Examples)

W State

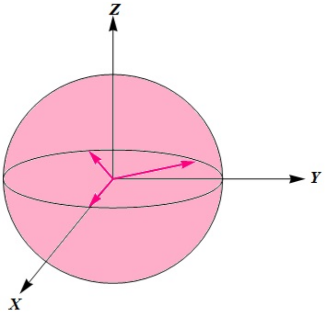
- $|\psi_W\rangle = |3/2 \;\; -1/2\rangle \equiv \frac{|\uparrow\downarrow\downarrow\rangle+|\downarrow\uparrow\downarrow\rangle+|\downarrow\downarrow\uparrow\rangle}{\sqrt{3}}.$
- $Z_{1,2,3} = 0$
- $\theta = 0, \;\; \theta = \pi$



Majorana Representation of Pure Symmetric States (Examples)

GHZ State

- $|\psi_{GHZ}\rangle = \frac{|\frac{3}{2},\frac{3}{2}\rangle + |\frac{3}{2},-\frac{3}{2}\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$.
- $Z = e^{\frac{2\pi i r}{3}} \quad r = 0, 1, 2$
- $(\frac{\pi}{2}, 0), \ (\frac{\pi}{2}, \frac{2\pi}{3}), \ (\frac{\pi}{2}, \frac{4\pi}{3})$



Classification of Pure Symmetric States[3]

- *Degeneracy Number*: Number of identical spinors $|\epsilon_i\rangle$
- *Degeneracy Configuration* $\mathcal{D}_{\{n_i\}}$: $\{n_i\}$ is the set of degeneracy numbers ordered in decreasing order by convention.
- Number of $n_i{}'s$ defines the diversity degree of the symmetric state.
- Diversity Degree: $[d]$

[3] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata and E. Solano, *Phys. Rev. Lett.* **103** (2009) 070503.

Classification of Pure Symmetric States

Example (1)

$|\psi\rangle \in \mathcal{D}_N$ with $d=1$; . Separable Class: $|\psi\rangle = \mathcal{N}|\epsilon\epsilon\epsilon...\rangle$

Example (2)

$|\psi\rangle \in \mathcal{D}_{N-2,2}$ with $d=2$; $|\psi\rangle = \mathcal{N}\sum_P \hat{P}|\epsilon'\epsilon'\epsilon\epsilon...\rangle$.
Or
$|\psi'\rangle \in \mathcal{D}_{N-2,1,1}$ with $d=3$; $|\psi'\rangle = \mathcal{N}\sum_P \hat{P}|\epsilon'\epsilon''\epsilon\epsilon...\rangle$

Classification of Pure Symmetric States

Such a classification based on Majorana representation is valid for symmetric pure states only. Therefore we propose a novel scheme for the most general symmetric pure as well as mixed state based on an equally elegant multiaxial representation of the density matrix.

Multiaxial Representation of Pure and Mixed Symmetric States

A standard expression[4] for the most general spin-j density matrix in terms of Fano statistical tensor parameters $t^k_q\,'s$

$$\rho(\vec{J}) = \frac{Tr(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} \; t^k_q\, \tau^{k^\dagger}_q(\vec{J}) \;, \qquad (7)$$

- $\vec{J}$ is the angular momentum operator with components J_x, J_y, J_z
- $\tau^k_q\,'s$ (with $\tau^0_0 = I$,the identity operator) are irreducible tensor operators of rank k in the $2j+1$ dimensional spin space with projection q along the axis of quantization in the real 3-dimensional space.

[4] U. Fano, *Phys. Rev.* **90** (1953) 577.

Multiaxial Representation of Pure and Mixed Symmetric States

Properties of τ^k_q

- The matrix elements of τ^k_q : $\langle jm' | \tau^k_q(\vec{J}) | jm\rangle = [k]\ C(jkj; mqm')$

$\boxed{C(jkj; mqm') \text{ are the Clebsch-Gordan coefficients}}$

Orthogonality relations: $Tr(\tau^{k^\dagger}_q \tau^{k'}_{q'}) = (2j+1)\,\delta_{kk'}\delta_{qq'}$.

$\tau^k_q\,'s$ in the rotated frame: $(\tau^k_q)^R = \sum_{q'=-k}^{+k} D^k_{q'q}(\phi,\theta,\psi)\,\tau^k_{q'}$

Multiaxial Representation of Pure and Mixed Symmetric States

Properties of t^k_q

- Average Expectation Values : $t^k_q = Tr(\rho\,\tau^k_q)$.
- Since ρ is Hermitian and $\tau^{k^\dagger}_q = (-1)^q \tau^k_{-q} \Longrightarrow \quad t^{k^*}_q = (-1)^q\, t^k_{-q}$
- $t^k_q\,'s$ in the rotated frame: $(t^k_q)^R = \sum_{q'=-k}^{+k} D^k_{q'\,q}(\phi, \theta, \psi)\, t^k_{q'}$

Multiaxial Representation of Pure and Mixed Symmetric States[5]

Consider a rotation $R(\phi,\theta,0)$ of the frame of reference such that t^k_k in the rotated frame vanishes.

$$(t^k_k)^R = 0 = \sum_q \; D^k_{qk}(\phi,\theta,0)\, t^k_q \; . \qquad (8)$$

$$\mathcal{A}.\sum_q (-1)^{2(k-q)}\,\sqrt{{}^{2k}\,C_{k+q}}\;t^k_q\,z^{k-q} = 0 \;\; where \;\; \begin{cases} z = tan\left(\frac{\theta}{2}\right)\; e^{i\phi} \\ \mathcal{A} = cos^{2k}\left(\frac{\theta}{2}\right)\; e^{-ik\phi} \end{cases}$$

$$\boxed{P(z) = \textstyle\sum_q (-1)^{2(k-q)}\sqrt{{}^{2k}\,C_{k+q}}\;t^k_q\;z^{k-q} = 0}$$

[5] G. Ramachandran and V. Ravishankar, *J. Phys. G: Nucl. Phys.* **12** (1986).

Multiaxial Representation of Pure and Mixed Symmetric States

$$\mathcal{A}'.\sum_q (-1)^{2(k-q)}\sqrt{^{2k}\,C_{k+q}}\; t^k_q\; {z'}^{k+q} = 0 \; where \; \begin{cases} \mathcal{A}' = sin^{2k}\left(\frac{\theta}{2}\right)\; e^{ik\phi} \\ z' = \frac{1}{z} = cot\left(\frac{\theta}{2}\right)\; e^{-i\phi} \end{cases}$$

$$\boxed{P(z') = \sum_q (-1)^{2(k-q)}\sqrt{^{2k}\,C_{k+q}}\; t^k_q\; {z'}^{k+q} = 0}$$

Since $(t^k_k)^* = (-1)^k t^k_{-k}$

$$(t^k_k)^R = 0 \Longrightarrow (t^k_{-k})^R = 0$$

Multiaxial Representation of Pure and Mixed Symmetric States

Which for every k leads to $2k$ solutions namely $\{(\theta_1, \phi_1), (\theta_2, \phi_2), ..., (\theta_k, \phi_k), (\pi - \theta_1, \pi + \phi_1), ..., (\pi - \theta_k, \pi + \phi_k)\}$.

Thus the $2k$ solutions constitute k axes or k double headed arrows.

- $t^k_q = r_k(...((\hat{Q}(\theta_1,\phi_1)\otimes\hat{Q}(\theta_2,\phi_2))^2\otimes\hat{Q}(\theta_3,\phi_3))^3\otimes...)^{k-1}\otimes\hat{Q}(\theta_k,\phi_k))^k_q$
- $(\hat{Q}(\theta_1,\phi_1)\otimes\hat{Q}(\theta_2,\phi_2))^2_q = \sum_{q_1} C(11k;q_1q_2q)(\hat{Q}(\theta_1,\phi_1))^1_{q_1}(\hat{Q}(\theta_2,\phi_2))^1_{q_2}$
- $(\hat{Q}(\theta,\phi))^1_q = \sqrt{\frac{4\pi}{3}}\, Y_{1q}(\theta,\phi)$

Multiaxial Representation of Pure and Mixed Symmetric States

Thus, one can say that there exist two sets of k-coordinate frames in which $(t^k_{\pm k}) = 0$.

Consequently, the symmetric state of N-qubit assembly can be represented geometrically by a set of N spheres of different radii $r_1, r_2, ..., r_k$ corresponding to each value of k. The k^{th} sphere in general consists of a constellation of $2k$ points on its surface specified by $\hat{Q}(\theta_i, \phi_i)$ and $\hat{Q}(\pi - \theta_i, \pi + \phi_i)$; $i = 1, 2, ..., k$. In other words, every t^k is specified by k double headed arrows or axes.

Local Unitary Invariants (LUI)[6]

Since $(\hat{Q}(\theta_i, \phi_i) \otimes \hat{Q}(\theta_j, \phi_j))^0_0$ is an invariant under rotation, one can construct in general $C_2^{j(2j+1)}$ invariants out of $j(2j+1)$ axes together with $2j$ real positive scalars specifying a spin-j density matrix. Here $C_2^{j(2j+1)}$ denotes binomial coefficient.

- spin-1 or symmetric two qubit state is in general parametrized in terms of 3 axes and 2 real scalars and has $C_2^3+2=5$ invariants.
- spin-3/2 or symmetric three qubit state is represented by 6 axes and 3 real scalars and has $C_2^6+3 = 18$ invariants
- spin-2 or symmetric four qubit state is characterized by 10 axes and 4 real scalars and has $C_2^{10}+4=49$ invariants.

[6] S. Sirsi and V. Adiga, 2011 *Journal of Russian Laser Research* **5** (2011) No. 32

Classification of Pure and Mixed Symmetric States

- Degeneracy number represents the number of identical axes characterizing the given spherical tensor parameters t^k.
- Degeneracy configuration $\mathcal{D}_{\{n_i\}}$ of t^k are the set of degeneracy numbers $\{n_i\}$ ordered by convention in the decreasing order.
- Number of $n_i\,'s$ define the diversity degree of the given t^k such that $\sum_i n_i = k$ and $k = 0, 1, ..., 2j$.
- In addition to the above, the rank k refers to the rank of t^k.
- Thus the notation for the degeneracy configuration of t^k becomes $\mathcal{D}^k_{\{n_i\}}$.
- Every Spin-j state is in general characterized by $2j$ configurations.

Classification of Pure and Mixed Symmetric States

Classes of Symmetric Two Qubit Systems

(i) $\{\mathcal{D}^1_1\} \longrightarrow \rho$ is pure vector polarized (Uniaxial with $t^1_q \neq 0$).

(ii) $\{\mathcal{D}^2_2\} \longrightarrow \rho$ is pure tensor polarized (Biaxial with $t^2_q \neq 0$).

(iii) $\{\mathcal{D}^1_1, \mathcal{D}^2_2\} \longrightarrow$ (Triaxial with $t^1_q \neq 0$, $t^2_0 \neq 0$) .

(iv) $\{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}\} \longrightarrow$ (Triaxial with $t^1_q \neq 0$, $t^2_q \neq 0$).

Classes of Symmetric Three Qubit Systems

$$\{\mathcal{D}^1_1, \mathcal{D}^2_2, \mathcal{D}^3_3\}, \{\mathcal{D}^1_1, \mathcal{D}^2_2, \mathcal{D}^3_{2,1}\},$$

$$\{\mathcal{D}^1_1, \mathcal{D}^2_2, \mathcal{D}^3_{1,1,1}\}, \{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}, \mathcal{D}^3_3\},$$

$$\{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}, \mathcal{D}^3_{2,1}\}, \{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}, \mathcal{D}^3_{1,1,1}\}.$$

Examples (Separable State)

- Let $|\psi^j\rangle = |\epsilon, \epsilon, ..., \epsilon\rangle$ be a separable state with diversity degree $d = 1$.

- Canonical form of the most general separable state $|S_n\rangle \equiv |\uparrow\uparrow\uparrow\ldots\uparrow\rangle \equiv |jj\rangle$ in the some rotated frame of reference.

- $\langle jj|\rho|jj\rangle = \frac{1}{2j+1}\sum_{k=0}^{2j} \; t_0^k \; C(jkj; q0q)\sqrt{2k+1} = 1.$

- In the density matrix language,

$$\rho = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{9}$$

Examples (Separable State)

Equivalently, the state has to be constructed out of $j(2j+1)$ axes which are collinear and not necessarily parallel to Z axis. Therefore the degeneracy configuration of $t^1, t^2, ..., t^k$ of ρ must be $\{\mathcal{D}_1^1, \mathcal{D}_2^2, \mathcal{D}_3^3, ...\mathcal{D}_k^k\}$ respectively with the understanding that all the axes are collinear.

Examples (Separable State)

Local Unitary Invariants (LUI) $r_k{}'s$

$$r_k = \frac{t_0^k}{(\hat{Q}(\theta,\varphi)\otimes\hat{Q}(\theta,\varphi)\otimes\ldots\hat{Q}(\theta,\varphi))_0^k} = \frac{[k]\, C(jkj;j0j)}{C(112;000).\, C(213;000)\ldots C(k-11k;000)}$$

$$\boxed{= \frac{[k]\,(2j)!\left(\frac{2j+1}{(2j-1)!(2j+k+1)!}\right)^{\frac{1}{2}}}{\prod_{n=1,.,k}\frac{n!}{(n-1)!}\left(\frac{2!\,2(n-1)!}{(2n)!}\right)^{\frac{1}{2}}} \qquad (10)}$$

Two Qubit Separable State

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow$$

$$r_0 = 1,\;\; r_1 = \sqrt{\frac{3}{2}},\;\; r_2 = \frac{\sqrt{3}}{2}.$$

Three Qubit Separable State

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow r_0 =$$

$$1,\;\; r_1 = \frac{3}{\sqrt{5}},\;\; r_2 = \sqrt{\frac{3}{2}},\;\; r_3 = \frac{1}{\sqrt{2}}.$$

Examples (Separable State)

- If $Tr(\rho^2) = 1$, compute all the $t_q^k{}'s$.
- Obtain the axes characterizing each t_q^k. $\longrightarrow$

  (*i*) Even if one of the axes is different from the rest, then the state is not separable.

  (*ii*) If all the axes are collinear, the state may be separable.

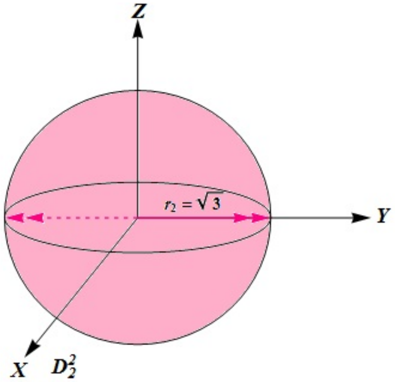
  (*iii*) If all the axes are collinear, then compute the values of $r_k{}'s$.

  (*IV*) The given state is separable iff the set of $r_k{}'s$ so obtained is identical to the set given by (10).
- If $Tr(\rho^2) < 1$, it is not clear as to the procedure to be followed to test the separability of a given N-qubit state as the definition of entanglement for a mixed state itself is problematic.

Examples (Bell State ($\rho \in \mathcal{D}_2^2$))

1. $|\psi\rangle = \frac{|11\rangle + |1-1\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}$
2. $\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.
3. $\begin{cases} t_0^2 = \frac{1}{\sqrt{2}} \\ t_{\pm 2}^2 = \frac{\sqrt{3}}{2} \end{cases}$
4. $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.
5. $t^2 \in \mathcal{D}_2^2$
6. $r_2 = \sqrt{3}$

$$r_k = \frac{[k]\,(2j)!\left(\frac{2j+1}{(2j-1)!(2j+k+1)!}\right)^{\frac{1}{2}}}{\prod_{n=1,..,k} \frac{n!}{(n-1)!}\left(\frac{2!\,2(n-1)!}{(2n)!}\right)^{\frac{1}{2}}}$$



Examples (Triaxial Pure State $\rho \in \{\mathcal{D}_1^1, \mathcal{D}_{1,1}^2\}$ or $\{\mathcal{D}_1^1, \mathcal{D}_2^2\}$)

This state can be generated by the time evolution operator $U(t, t_0)$ commonly found in quantum optics processes involving two intense laser beams with the same arbitrary amplitude modulation.

$$U(t, t_0) \;=\; 1 - i\frac{1}{\nu} H_0 sin\left(\nu \int_{t_0}^{t} f(\tau) d\tau\right) + \frac{1}{\nu^2} H_0^2 cos\left(\nu \int_{t_0}^{t} f(\tau) d\tau) - 1\right)$$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \frac{4a^2b^2}{\nu^4} & 0 & \frac{2ab}{\nu^4}(a^2 - b^2) \\ 0 & 0 & 0 \\ \frac{2ab}{\nu^4}(a^2 - b^2) & 0 & \frac{1}{\nu^4}(a^2 - b^2)^2 \end{pmatrix}$$

$$t_0^1 = -\sqrt{\frac{3}{2}} + \frac{4\sqrt{6}a^2b^2}{\nu^4}\ , \quad t_0^2 = \frac{1}{\sqrt{2}}\ , \quad t_{\pm 2}^2 = \frac{2\sqrt{3}ab}{\nu^4}(a^2 - b^2)\,.$$

Examples ($\rho \in \{\mathcal{D}_1^1, \mathcal{D}_2^2, \mathcal{D}_3^3\}$)

- $|\psi_W\rangle \equiv= |3/2 \ \ -1/2\rangle$.

- $\rho_W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- $t^1 \in \mathcal{D}_1^1$, $t^2 \in \mathcal{D}_2^2$ and $t^3 \in \mathcal{D}_3^3$

- $t_0^1 = \frac{-1}{\sqrt{5}}, \quad t_0^2 = -1, \quad t_0^3 = \frac{3}{\sqrt{5}}$.

$$\boxed{r_k = \frac{[k]\,(2j)!\left(\frac{2j+1}{(2j-1)!(2j+k+1)!}\right)^{\frac{1}{2}}}{\prod_{n=1,..,k} \frac{n!}{(n-1)!}\left(\frac{2!\,2(n-1)!}{(2n)!}\right)^{\frac{1}{2}}}}$$

- $r_1 = \frac{1}{\sqrt{2}}, \ r_2 = \sqrt{\frac{3}{2}}, \ r_3 = \frac{3}{\sqrt{5}}$.

- all the 6 axes are collinear and parallel to Z axis

W State

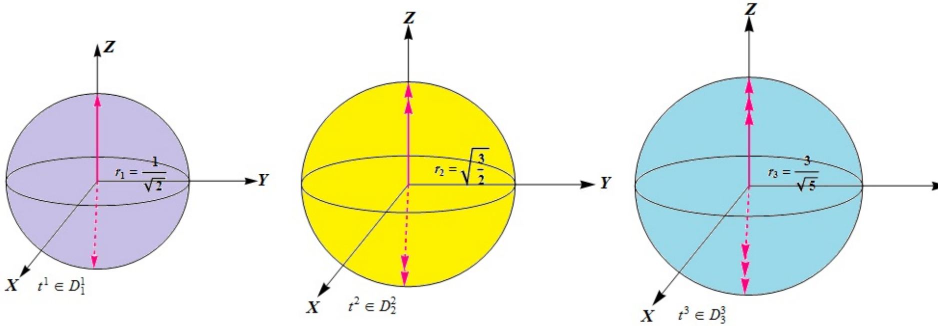


Figure: Multiaxial representation of t^1, t^2 and t^3 characterizing the W state.

Examples (GHZ State ($\rho \in \{\mathcal{D}^2_2, \mathcal{D}^3_{1,1,1}\}$))

- $|\psi_{GHZ}\rangle = \frac{|\frac{3}{2},\frac{3}{2}\rangle + |\frac{3}{2},-\frac{3}{2}\rangle}{\sqrt{2}}$.
- $t^2_0 = 1\ , \quad t^3_3 = -1\ , \quad t^3_{-3} = 1.$
- $t^2 \in \mathcal{D}^2_2$ and $t^3 \in \mathcal{D}^3_{1,1,1}$
- $\rho_{GHZ} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.
- $r_2 = \sqrt{\frac{3}{2}}, \quad r_3 = 2\sqrt{2}$
- $\boxed{(\frac{\pi}{2},0),\ (\frac{\pi}{2},\pi),\ (\frac{\pi}{2},\frac{\pi}{3}),\ (\frac{\pi}{2},\frac{4\pi}{3}),\ (\frac{\pi}{2},\frac{2\pi}{3}),\ (\frac{\pi}{2},\frac{5\pi}{3})}$

$$\boxed{r_k = \frac{[k]\,(2j)!\left(\frac{2j+1}{(2j-1)!(2j+k+1)!}\right)^{\frac{1}{2}}}{\prod_{n=1,.,k}\frac{n!}{(n-1)!}\left(\frac{2!\,2(n-1)!}{(2n)!}\right)^{\frac{1}{2}}}}$$

GHZ State

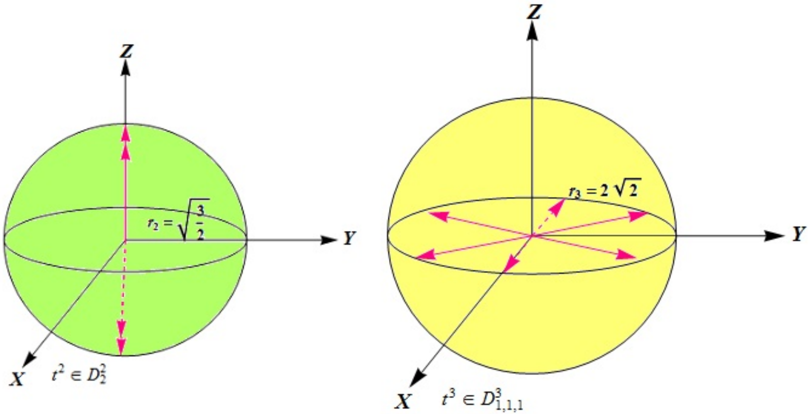


Figure: Multiaxial representation of t^2 and t^3 characterizing the GHZ state.

Examples (Mixed States)
Uniaxial Systems ($\rho \in \mathcal{D}_1^1$)

$$\rho = \frac{1}{3}\begin{pmatrix} 1+\sqrt{\frac{3}{2}}\, r_1 cos\theta_1 & \frac{\sqrt{3}}{2} r_1 sin\theta_1 e^{-i\varphi_1} & 0 \\ \frac{\sqrt{3}}{2} r_1 sin\theta_1 e^{i\varphi_1} & 1 & \frac{\sqrt{3}}{2} r_1 sin\theta_1 e^{-i\varphi_1} \\ 0 & \frac{\sqrt{3}}{2} r_1 sin\theta_1 e^{i\varphi_1} & 1-\sqrt{\frac{3}{2}}\, r_1 cos\theta_1 \end{pmatrix}.$$

$\rho \in \mathcal{D}_1^1 \quad (t_q^1 \neq 0,\ t_q^2 = 0)$

- $t_0^1 = r_1\, cos\theta_1$, $t_{\pm 1}^1 = \frac{r_1}{\sqrt{2}}\, sin\theta_1 e^{\mp i\varphi_1}$
- ρ is positive semi-definite iff $0 < r_1 \leq \sqrt{\frac{2}{3}}$
- ρ is entangled for $\frac{1}{\sqrt{2}} \leq r_1 \leq \sqrt{\frac{2}{3}}$ for all values of θ $(0 \leq \theta \leq \pi)$
- $Tr(\rho^2) = \frac{1}{3}[1 + r_1^2] < 1$, hence this class consists of mixed states only.

Examples (Mixed States)
Biaxial Systems ($\rho \in \mathcal{D}^2_{1,1}$ or $\mathcal{D}^2_2$)

$$\rho = \frac{1}{3}\begin{pmatrix} 1+\frac{1}{2\sqrt{3}}\,r_2(1+cos^2\theta) & 0 & \frac{-\sqrt{3}}{2}r_2 sin^2\theta \\ 0 & 1-\frac{1}{\sqrt{3}}\,r_2(1+cos^2\theta) & 0 \\ \frac{-\sqrt{3}}{2}r_2 sin^2\theta & 0 & 1+\frac{1}{2\sqrt{3}}\,r_2 \end{pmatrix}.$$

$\rho \in \mathcal{D}^2_{1,1}$ or $\mathcal{D}^2_2$) $(t^1_q = 0, \quad t^2_q \neq 0)$

- $t^2_0 = \frac{r_2}{\sqrt{6}}(1+cos^2\theta)$, $t^2_{\pm 2} = \frac{-r_2}{2}\,sin^2\theta$
- $\rho \in \mathcal{D}^2_{1,1}$ for $0 < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$
- $\rho \in \mathcal{D}^2_2$ for $\theta = 0, \frac{\pi}{2}, \pi$
- ρ is positive semi-definite iff $0 < r_2 \leq \sqrt{3}$ and the range of θ then depends on r_2.
- ρ is positive semi-definite and separable iff $0 < r_2 \leq \frac{\sqrt{3}}{4}$ and $0 \leq \theta \leq \pi$.
- For $r_2 = \sqrt{3}$ and $\theta = \frac{\pi}{2}$, ρ is pure as well as entangled.

Examples (Mixed States)
Triaxial Systems ($\rho \in \{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}\}$ *or* $\{\mathcal{D}^1_1, \mathcal{D}^2_2\}$)

$$\rho = \frac{1}{3}\begin{pmatrix} 1+\sqrt{\frac{3}{2}}\, r_1+\frac{1}{2\sqrt{3}}\, r_2(1+cos^2\theta) & 0 & \frac{-\sqrt{3}}{2} r_2 sin^2\theta \\ 0 & 1-\frac{1}{\sqrt{3}}\, r_2(1+cos^2\theta) & 0 \\ \frac{-\sqrt{3}}{2} r_2 sin^2\theta & 0 & 1-\sqrt{\frac{3}{2}}\, r_1+\frac{1}{2\sqrt{3}}\, r_2(1+cos^2\theta) \end{pmatrix}$$

$\rho \in \{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}\}$ *or* $\{\mathcal{D}^1_1, \mathcal{D}^2_2\}$ $\quad (t^1_q \neq 0, \quad t^2_q \neq 0)$

- $t^1_0 = r_1$, $t^2_0 = \frac{r_2}{\sqrt{6}}(1 + cos^2\theta)$, $t^2_{\pm 2} = \frac{-r_2}{2} sin^2\theta$
- $\rho \in \left\{\mathcal{D}^1_1, \mathcal{D}^2_{1,1}\right\}$ for $0 < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta < \pi$.
- $\rho \in \left\{\mathcal{D}^1_1, \mathcal{D}^2_2\right\}$ for $\theta = 0, \frac{\pi}{2}, \pi$.
- For $r_1 = \sqrt{\frac{3}{2}}$ and $r_2 = \frac{\sqrt{3}}{2}$ this class is pure as well as separable only for $\theta = 0, \pi$.

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

In order to bring out the similarities and the differences between MR and MAR, we take up the N-qubit GHZ state for a detailed investigation. Consider symmetric N-qubit GHZ state

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\Big[|\frac{N}{2}\frac{N}{2}\rangle + |\frac{N}{2}\frac{-N}{2}\rangle\Big] \equiv \frac{1}{\sqrt{2}}\Big[|jj\rangle + |j-j\rangle\Big].$$

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MR of GHZ State:

$$(-1)^{2j} Z^{2j} + 1 = 0$$

Depending on whether N is odd or even we have the following solutions:

- Odd N(Half odd integral j) :

$$Z = e^{\frac{2\pi i}{2j} r}; \quad r = 0, 1, 2, ..., 2j - 1$$

.

The $2j$ distinct spinors characterizing $N\,(odd)$-qubit GHZ state are

$$(\frac{\pi}{2}, 0), (\frac{\pi}{2}, \frac{2\pi}{2j}), (\frac{\pi}{2}, \frac{4\pi}{2j}), ..., (\frac{\pi}{2}, \frac{2(2j-1)\pi}{2j}).$$

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MR of GHZ State:

- Even N(integral j):

$$Z = e^{\frac{2\pi i}{2j}(r-\frac{1}{2})}; \quad r = 0, 1, 2, ..., 2j-1$$

.

The $2j$ distinct spinors characterizing $N\,(even)-$ qubit GHZ state are

$$(\frac{\pi}{2}, \frac{\pi}{2j}),\ (\frac{\pi}{2}, \frac{3\pi}{2j}),\ (\frac{\pi}{2}, \frac{5\pi}{2j}), ..., (\frac{\pi}{2}, \frac{(4j-1)\pi}{2j})$$

or equivalently j distinct axes.

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

According to Bastin et, al.[3], N-qubit GHZ state belong to

$$\mathcal{D}^{N}_{\underbrace{1,1,1\ldots 1}_{N}}$$

or equivalently

$$\mathcal{D}^{2j}_{\underbrace{1,1,1\ldots 1}_{2j}}$$

for both odd and even N's.

[3] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata and E. Solano, *Phys. Rev. Lett.* **103** (2009) 070503.

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MAR of GHZ State:

To find out the axes, consider the density matrix of N-qubit GHZ state in the $|jm\rangle$ basis; $m = +j ... -j$

$$\rho_{GHZ} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 1 \end{pmatrix}.$$

As in the case of MR, here also we take up the case of odd N and even N separately.

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

- Odd N (half odd integral j):

Since $t^{2j}_{\pm 2j}$ is the only non-zero parameter

$$t^2 \in \mathcal{D}^2_2,\ t^4 \in \mathcal{D}^4_4 \ldots t^{2j-1} \in \mathcal{D}^{2j-1}_{2j-1}.$$

There exist $4j$ solutions or $2j$ axes namely

$$(\frac{\pi}{2}, 0),\ (\frac{\pi}{2}, \frac{\pi}{2j}),\ (\frac{\pi}{2}, \frac{2\pi}{2j}) \ldots (\frac{\pi}{2}, \frac{(4j-1)\pi}{2j}).$$

Thus

$$t^{2j} \in \mathcal{D}^{2j}_{\underbrace{1,1,1\ldots 1}_{2j}}$$

The degeneracy configuration of N-qubit GHZ state for odd N is

$$\{\mathcal{D}^2_2, \mathcal{D}^4_4, \mathcal{D}^6_6, ..., \mathcal{D}^{2j-1}_{2j-1}, \mathcal{D}^{2j}_{\underbrace{1,1,1\ldots 1}_{2j}}\}.$$

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

- Even N (integral j):

Since $t^{2j}_0 \neq 0$ and $t^{2j}_{\pm 2j} \neq 0$

$$t^2 \in \mathcal{D}^2_2,\ \ t^4 \in \mathcal{D}^4_4 \ldots t^{2j-2} \in \mathcal{D}^{2j-2}_{2j-2}.$$

There exist two identical sets of solutions or j axes namely

$$(\frac{\pi}{2}, \frac{\pi}{2j}),\ (\frac{\pi}{2}, \frac{3\pi}{2j}),\ (\frac{\pi}{2}, \frac{5\pi}{2j})...(\frac{\pi}{2}, \frac{(4j-1)\pi}{2j}).$$

Thus

$$t^{2j} \in \mathcal{D}^{2j}_{\underbrace{2,2\ldots 2}_{j}}$$

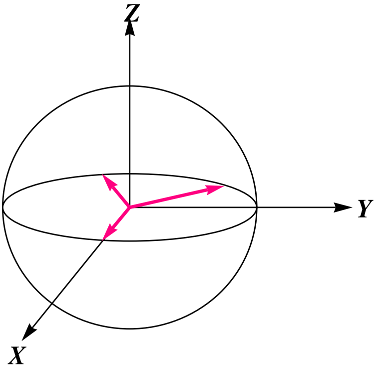
The degeneracy configuration of N-qubit GHZ state for even N is

$$\{\mathcal{D}^2_2,\ \mathcal{D}^4_4,\ \mathcal{D}^6_6, ..., \mathcal{D}^{2j-2}_{2j-2},\ \mathcal{D}^{2j}_{\underbrace{2,2\ldots 2}_{j}}\}.$$

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MR of 3-qubit GHZ state:

1. $|\psi_{GHZ}\rangle = \frac{|\frac{3}{2},\frac{3}{2}\rangle + |\frac{3}{2},-\frac{3}{2}\rangle}{\sqrt{2}} \equiv \frac{|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle}{\sqrt{2}}$
2. $Z^3 = 1.$
3. $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, \frac{2\pi}{3})$, $(\frac{\pi}{2}, \frac{4\pi}{3})$.
4. $|\psi_{GHZ}\rangle \in \mathcal{D}^3_{1,1,1}$.



Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MAR of 3-qubit GHZ state:

- $\rho_{GHZ} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$
- $t^2_0 = 1\,, \quad t^3_3 = -1\,, \quad t^3_{-3} = 1\,.$
- $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, \pi)$, $(\frac{\pi}{2}, \frac{\pi}{3})$, $(\frac{\pi}{2}, \frac{4\pi}{3})$, $(\frac{\pi}{2}, \frac{2\pi}{3})$, $(\frac{\pi}{2}, \frac{5\pi}{3})$.
- $t^2 \in \mathcal{D}^2_2$, $t^3 \in \mathcal{D}^3_{1,1,1}$
- $\rho \in \{\mathcal{D}^2_2, \mathcal{D}^3_{1,1,1}\}$.

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

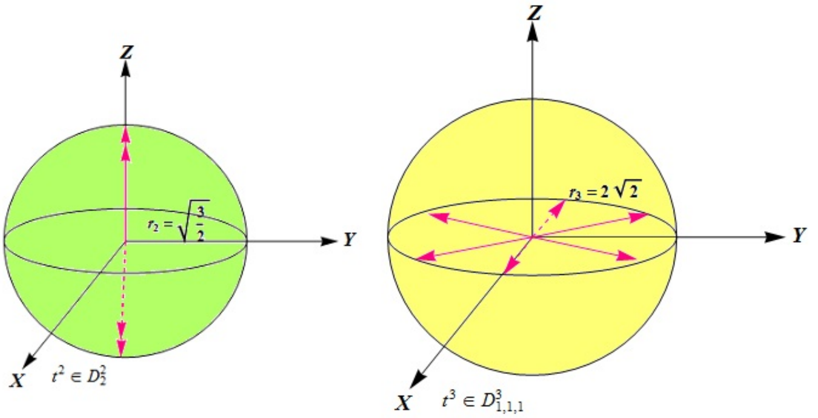
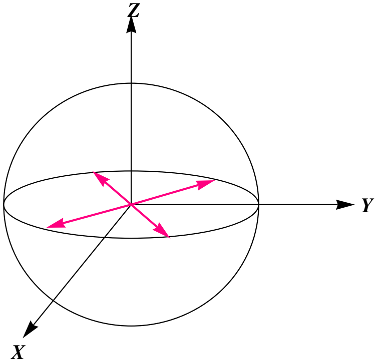


Figure: Multiaxial representation of t^2 and t^3 characterizing the GHZ state.

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MR of 4-qubit GHZ state:

1. $|\psi_{GHZ}\rangle = \frac{|2,2\rangle+|2,-2\rangle}{\sqrt{2}}$.
2. $Z^4 = -1$
3. $(\frac{\pi}{2}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{3\pi}{4})$, $(\frac{\pi}{2}, \frac{5\pi}{4})$, $(\frac{\pi}{2}, \frac{7\pi}{4})$.
4. $|\psi_{GHZ}\rangle \in \mathcal{D}^4_{1,1,1,1}$.



Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MAR of 4-qubit GHZ state:

- $\rho_{GHZ} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$
- $t_0^2 = \sqrt{\frac{10}{7}}, \;\; t_0^4 = \frac{1}{\sqrt{14}}, \;\; t_4^4 = \frac{\sqrt{5}}{2}, \;\; t_{-4}^4 = \frac{\sqrt{5}}{2}.$
- $(\frac{\pi}{2}, \frac{\pi}{4}), (\frac{\pi}{2}, \frac{3\pi}{4}), (\frac{\pi}{2}, \frac{5\pi}{4}), (\frac{\pi}{2}, \frac{7\pi}{4}.$
- $t^2 \in \mathcal{D}_2^2\,, \;\; t^4 \in \mathcal{D}_{2,2}^4$
- $\rho \in \{\mathcal{D}_2^2, \, \mathcal{D}_{2,2}^4\}\,.$

Comparison between Majorana Representation and Multiaxial Representation of the GHZ State

MAR of 4-qubit GHZ state:

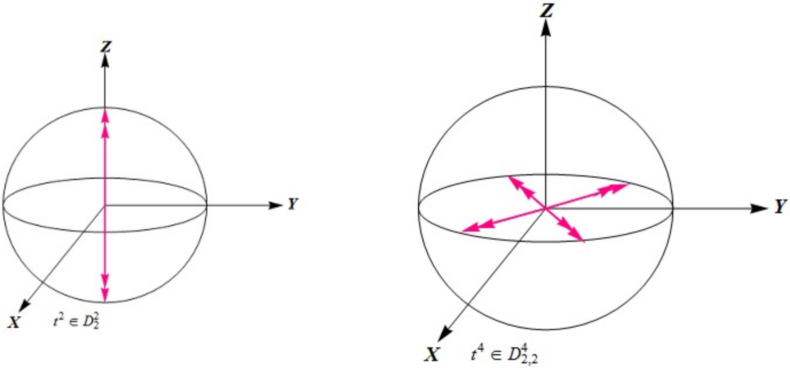


Figure: Multiaxial representation of t^2 and t^4 characterizing the GHZ state.

Conclusion

- We have developed a method of classifying LU equivalent classes of symmetric N-qubit mixed states based on the little known Multiaxial representation of the density matrix.
- Multiaxial representation is more general than the Majorana representation as it can be applied to pure as well as mixed states.
- Our classification is characterized by three parameters namely diversity degree, degeneracy configuration and rank.
- A comparative study of Majorana representation and Multiaxial representation for the N-qubit GHZ state has been carried out to bring out the differences and similarities between the two representations. We have shown that Majorana representation is not a special case of the Multiaxial representation.
- Recipe for identifying N-qubit pure separable state is described in International Journal of Quantum I

Conclusion

- Recipe for identifying N-qubit pure separable state is described in detail and the method is tested for some well known examples of symmetric two and three qubit pure states.

- We illustrate with examples, the classification of uniaxial, Biaxial and triaxial mixed states which can be produced in the laboratory. It is not clear as to why for certain configuration of the axes and certain values of r_k, the mixed states exhibit entanglement.

- An indepth study of the onset of entanglement as a function of some suitable combination of LUI is needed and will be taken up in the near future.

Thank You