Methods for multi-loop computations

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Lecture I

Reduction to Master Integrals

- **Introduction:**
  1. Multi-loop amplitudes
  2. **Tensor Reduction** $\rightarrow$ scalar integrals

- **Identities for reduction to MIs:**
  1. Integration-by-parts identities
  2. Lorentz identities
  3. Symmetry relations
  4. Schouten identities

- **The Laporta Algorithm**
  1. Reduze 2.

Introduction

Prologue - Perturbative calculations

For the sake of simplicity we work in (*massless* or *massive*) **QCD**
Cross section for N-particle **scattering process**:

$$\sigma_N = \sigma_N^{(0)} + \sigma_N^{(1)} \left(\frac{\alpha_S}{2\pi}\right) + \sigma_N^{(2)} \left(\frac{\alpha_S}{2\pi}\right)^2 + \dots$$

- **LO:**

$$\sigma_N^{(0)} \approx \int |\mathcal{M}_N^{(0)}|^2 \, d\Phi_N$$

- **NLO:**

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- **NNLO:**

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  all momenta $p_1, ..., p_N$ are **incoming**
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Example (in massless QCD): $q(p_1) + \bar{q}(p_2) \to Z(p_3) + Z(p_4)$

- $\mathcal{S}^{(0)}(p_1, ..., p_4) \approx$ **2** tree-level diagrams
- $\mathcal{S}^{(1)}(p_1, ..., p_4) \approx$ **10** one-loop diagrams
- $\mathcal{S}^{(2)}(p_1, ..., p_4) \approx$ **143** two-loop diagrams
- $\mathcal{S}^{(3)}(p_1, ..., p_4) \approx$ **2922** three-loop diagrams
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$q\bar{q} \to ZZ$ at l-**loop**, take sum of all Feynman Diagrams:

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with:

- $u(p_1)$, $\bar{u}(p_2)$ are the spinors of the **incoming quarks**.
- D_j are t different **propagators**.
- $T^{\mu\nu}(p_i; k_i)$ is a **rank two tensor** built out of $\{p_i^{\mu}, k_i^{\mu}, \gamma_i^{\mu}, g^{\mu\nu}\}$

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Exploiting **Lorentz invariance** we can **project out** all loop momenta from the tensor $T^{\mu\nu}$:

$$\int \mathfrak{D}^d k \frac{k^\mu k^\nu}{(k^2+m^2)^2} = C(m^2)\, g^{\mu\nu} \,,$$

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so that

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Tensor Reduction 2:

More interesting example

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multiplying this equation once by $g^{\mu\nu}$, once by $p^\mu p^\nu$

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and inverting for C_1 and C_2 we find:

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where now **all dependence** from **loop momenta** k_j is contained into the **scalar** coeffcients C_i.

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Through tensor reduction every coefficients is given by a linear combination of **scalar integrals** of the form

$$\mathcal{I}(p_j) = \int \prod_{i=1}^{l} \mathfrak{D}^d k_i \, \frac{S_1^{a_1} \dots S_\rho^{a_\rho}}{D_1^{b_1} \dots D_\tau^{b_\tau}}$$

where:

ρ scal. prod. $S_j = q_n \cdot q_m \,,$ with $q_i = p_1, \dots, p_N, \, k_1, \dots, k_l \,,$

τ different *(euclidean)* propagators $D_j = (q_j^2 + m_j^2) \,,$

and a_j, b_j are just **integer powers**.

Irreducible Scalar Products

Given N external momenta, l loop momenta

$$\rho = l\left(N + \frac{l}{2} - \frac{1}{2}\right)$$ scalar prod. with *1 loop momentum*

Given the τ different propagators, if $\rho > \tau \quad \longrightarrow \quad \sigma = \rho - \tau$ *irreducible scalar products*

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The others can be expressed as **linear combination of propagators!**

For example integral seen before:

$$\int \mathfrak{D}^d k \, \frac{k \cdot p}{D_1 \, D_2}\,, \qquad \text{with} \qquad D_1 = k^2 + m^2\,, \quad D_2 = ((k-p)^2 + m^2)\,,$$

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$$k \cdot p = \frac{1}{2}\left[(k^2 + m^2) - ((k-p)^2 + m^2) + p^2\right] = \frac{1}{2}\left[D_1 - D_2 + p^2\right]$$

and so finally:

$$\int \mathfrak{D}^d k \, \frac{k \cdot p}{D_1 \, D_2} = \frac{1}{2}\left(\int \mathfrak{D}^d k \, \frac{1}{D_1} - \int \mathfrak{D}^d k \, \frac{1}{D_2} + p^2 \int \mathfrak{D}^d k \, \frac{1}{D_1 D_2}\right)$$

$$= \frac{p^2}{2} \int \mathfrak{D}^d k \, \frac{1}{D_1 D_2}$$

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Note that:

- at **1 loop** all scalar products are always reducible !
  1. **2 legs**: 2 denominators, and 2 scalar products $k \cdot k$ and $k \cdot p$
  2. **3 legs**: 3 denominators, and 3 scalar products $k \cdot k$, $k \cdot p_1$, $k \cdot p_2$
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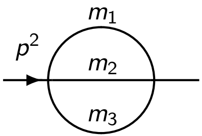
Example: massive two-loop Sunrise



3 Denominators: $D_1 = k^2 + m_1^2\,, \quad D_2 = l^2 + m_2^2\,, \quad D_3 = (k - l - p)^2 + m_3^2$

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2 scalar products are **irreducible**! $\rightarrow \quad \{S_4 = k \cdot p\,,\; S_5 = l \cdot p\}$

So the **most general** integral in **two-loop sunrise graph** is

$$\mathcal{I}(n_1, n_2, n_3; n_4, n_5) = \int \mathfrak{D}^d k\, \mathfrak{D}^d l\, \frac{S_4^{n_4}\, S_5^{n_5}}{D_1^{n_1}\, D_2^{n_2}\, D_3^{n_3}} \qquad \text{with} \qquad n_1, n_2, n_3, n_4, n_5 \geq 0\,.$$

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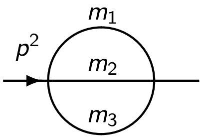

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Alternative approach $\to$ **integral families** *(see Reduze 2)*

- Instead of **irreducible scalar products** we introduce **auxiliary denominators**
- For example, two-loop sunrise again. Instead of taking

$$\{ D_1\,,\ D_2\,,\ D_3\,,\ k\cdot p\,,\ l\cdot p\}$$

- We can take two new **denominators**

$$\left\{ D_1\,,\ D_2\,,\ D_3\,,\ D_4=(k-p)^2\,,\ D_5=(l-p)^2\right\}$$

- **Integral Family** for reduction of the two-loop massive sunrise becomes:

$$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) = \int \frac{\mathfrak{D}^d k \, \mathfrak{D}^d l}{D_1^{n_1} \, D_2^{n_2} \, D_3^{n_3} \, D_4^{n_4} \, D_5^{n_5}} \,, \quad n_1, n_2, n_3 \geq 0 \,, \quad n_4, n_5 \in \mathbb{Z} \,.$$

- In this way **all scalar products** can be expressed as linear combinations of the 5 denominators !

- The two approaches are **completely equivalent!**

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$$\mathcal{I}(p_j) = \int \prod_{i=1}^{l} \mathfrak{D}^d k_i \, \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_\tau^{b_\tau}}$$

where $a_j, b_j \geq 0$.

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The topology is defined only by the propagators, regardless of their powers and of any scalar products !

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Example: the **sub-topology tree** of the two-loop Sunrise is:

1. $\{ D_1, D_2, D_3 \} \quad \rightarrow \quad \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \frac{S_4^{n_4} \, S_5^{n_5}}{D_1^{n_1} \, D_2^{n_2} \, D_3^{n_3}} \quad \text{with} \quad n_1, n_2, n_3 > 0 \,, \quad n_4, n_5 \geq 0$

2. $\{ D_1, D_2 \} \quad \rightarrow \quad \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \frac{S_4^{n_4} \, S_5^{n_5}}{D_1^{n_1} \, D_2^{n_2}} \quad \text{with} \quad n_1, n_2 > 0 \,, \quad n_4, n_5 \geq 0$

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In short...:

- Every multi-loop amplitude can be reduced to **scalar integrals**
- The scalar integrals can be organised into **topologies**

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- We've showed the amplitude can be *reduced* to scalar coefficients

$$\mathcal{F}_f^{(l)}(p_1, \ldots, p_4) = \epsilon_3^{\mu}(p_3)\, \epsilon_4^{\nu}(p_4)\; \bar{u}(p_2) \left(\sum_{i=1}^{m} C_i(p_1, \ldots, p_4)\; T_i^{\mu\nu}(p_j) \right) u(p_1)\,,$$

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- Integration-by-parts identities **(IBPs)**
- Lorentz-invariance identities **(LIs)**
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- The most important class of identities.
  Generalisation of **Gauss's theorem** in d dimensions

- Any d-dimensional integral is **convergent** !

- **Necessary** condition for convergence: the integrand be **zero on the boundary**

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- In order to deal only with **scalar quantities**

$$\int \prod_{i=1}^{l} \mathfrak{D}^d k_i \frac{\partial}{\partial k_j^{\mu}} \left(v_n^{\mu} \frac{S_1^{a_1} \dots S_\sigma^{a_\sigma}}{D_1^{b_1} \dots D_\tau^{b_\tau}} \right) = 0 \,, \qquad v_n^{\mu} = \{p_1, \dots, p_N; k_1, \dots, k_l\}$$

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Example IBPs: The 1loop tadpole

$$\mathcal{I}(n) = \int \frac{\mathfrak{D}^d k}{(k^2 + m^2)^n}$$

$$0 = \int \mathfrak{D}^d k \left(\frac{\partial}{\partial k_\mu} k_\mu \right) \frac{1}{(k^2 + m^2)^n} = (d - 2\, n)\mathcal{I}(n) + 2\, n\, m^2\, \mathcal{I}(n+1)$$

Recursive relation for reduction to a **single Master Integral** which gives for example:

- $$(d-2)\mathcal{I}(1) + 2m^2\mathcal{I}(2) = 0 \quad \rightarrow \quad \mathcal{I}(2) = -\frac{(d-2)}{2m^2}\mathcal{I}(1)$$
- $$(d-4)\mathcal{I}(2) + 4m^2\mathcal{I}(3) = 0 \quad \rightarrow \quad \mathcal{I}(3) = +\frac{(d-2)(d-4)}{8m^4}\mathcal{I}(1)$$
- $$(d-6)\mathcal{I}(3) + 6m^2\mathcal{I}(4) = 0 \quad \rightarrow \quad \mathcal{I}(4) = -\frac{(d-2)(d-4)(d-6)}{48m^6}\mathcal{I}(1)$$

The *topology* of the Tadpole has the **Master Integrals** $\mathcal{I}(1)$.

Lorentz invariance identities (LIs) [Gehrmann, Remiddi]

- Integrals are **Lorentz scalars**:

$$p_i^\mu \to p_i^\mu + \delta p_i^\mu = p_i^\mu + \omega_{\mu\nu} p_i^\nu \,, \qquad \text{with} \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

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- which in turn gives

$$\sum_j \left(p_{j,\mu} \frac{\partial}{\partial p_j^\nu} - p_{j,\nu} \frac{\partial}{\partial p_j^\mu} \right) \mathcal{I}(p_i) = 0 \,.$$

- This can be multiplied by any **antisymmetric** combination of $p_i^\mu \, p_j^\nu$ to give further **scalar** relations among the integrals $\mathcal{I}(p_i)$

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$$p_i^\mu \to p_i^\mu + \delta p_i^\mu = p_i^\mu + \omega_{\mu\nu} p_i^\nu\,, \qquad \text{with} \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

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$$\sum_j \left(p_{j,\mu} \frac{\partial}{\partial p_j^\nu} - p_{j,\nu} \frac{\partial}{\partial p_j^\mu} \right) \mathcal{I}(p_i) = 0\,.$$

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Symmetry relations (SRs)

- Sometimes are needed to ensure **complete reduction** to a minimal set of MIs.
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(Trivial) example on SRs: Two-loop massive sunrise with equal masses

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1. At the beginning IBPs were solved **by hand** for generic powers n_j of the denominators
2. Laporta realised that increasing number of scalar products and powers of denominators the system of IBPs becomes **apparently overconstraint**.
3. $\rightarrow$ Large **redoundancy**!
   With **ordering** the equations can be inverted one after the other!
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- Realistic cases systems of $\approx$ **100000** / **1000000** equations
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