The Schwarzchild solution

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Derivation of TOV equation:

Consider the general static, spherically symmetric metric [1]

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\phi^2 \tag{1}$$

Let's now take this metric and use Einstien's equation to solve the function $\alpha(r)$ and $\beta(r)$. We are looking for nonvacuum solutions, so we turn to the full Einstien equtaion,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \tag{2}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and Ricci scalar, respectively. $g_{\mu\nu}$ is the metric

$$g_{\mu\nu} = \begin{bmatrix} g_{tt} & g_{tr} & g_{t\theta} & g_{t\phi} \\ g_{rt} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta t} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi t} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} -e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2sin^2\theta \end{bmatrix}$$

We begin by evaluting the christoffel (or affine connection or Levi-Civita or Riemann connection) symbols. If we use lables (t, r, θ, ϕ).

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \tag{3}$$

$$\begin{aligned} \Gamma^t_{tr} &= \frac{1}{2}g^{t\sigma}(\partial_t g_{\sigma r} + \partial_r g_{t\sigma} - \partial_\sigma g_{tr}) \\ &= \frac{1}{2}g^{t\sigma}(0 + \partial_r g_{t\sigma} - 0)) \\ &= \frac{1}{2}g^{tt}(\partial_r g_{tt}) \\ &= \frac{1}{2}(-e^{-2\alpha(r)})(\partial_r(-e^{2\alpha(r)})) \\ &= \frac{1}{2}(-e^{-2\alpha(r)})(-2e^{2\alpha(r)}\partial_r\alpha) \\ \Gamma^t_{tr} &= \partial_r\alpha \end{aligned} \tag{4}$$

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\begin{aligned}
\Gamma^r_{tt} &= \frac{1}{2}g^{r\sigma}(\partial_t g_{\sigma t} + \partial_t g_{t\sigma} - \partial_\sigma g_{tt}) \\
&= \frac{1}{2}g^{rr}(\partial_t g_{rt} + \partial_t g_{tr} - \partial_r g_{tt}) \\
&= \frac{1}{2}(e^{-2\beta(r)})(0 + 0 - \partial_r(-e^{2\alpha(r)})) \\
&= \frac{1}{2}(e^{-2\beta(r)})(2e^{2\alpha(r)})\partial_r \alpha \\
\Gamma^r_{tt} &= e^{2(\alpha(r)-\beta(r))}\partial_r \alpha
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2}g^{r\sigma}(\partial_r g_{\sigma r} + \partial_r g_{r\sigma} - \partial_\sigma g_{rr}) \\
&= \frac{1}{2}g^{rr}(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}) \\
&= \frac{1}{2}g^{rr}(\partial_r g_{rr}) \\
&= \frac{1}{2}(e^{-2\beta(r)})\partial_r(e^{2\beta(r)}) \\
&= \frac{1}{2}(e^{-2\beta(r)})(2)e^{2\beta(r)}\partial_r \beta \\
\Gamma^r_{rr} &= \partial_r \beta
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\Gamma^\theta_{r\theta} &= \frac{1}{2}g^{\theta\sigma}(\partial_r g_{\sigma\theta} + \partial_\theta g_{r\sigma} - \partial_\sigma g_{r\theta}) \\
&= \frac{1}{2}g^{\theta\theta}(\partial_r g_{\theta\theta} + \partial_\theta g_{r\theta} - \partial_\theta g_{r\theta}) \\
&= \frac{1}{2}(r^{-2})\partial_r(r^2) = \frac{1}{2}(r^{-2})(2r) \\
\Gamma^\theta_{r\theta} &= \frac{1}{r}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\Gamma^r_{\theta\theta} &= \frac{1}{2}g^{r\sigma}(\partial_\theta g_{\sigma\theta} + \partial_\theta g_{\theta\sigma} - \partial_\sigma g_{\theta\theta}) \\
&= \frac{1}{2}g^{rr}(\partial_\theta g_{r\theta} + \partial_\theta g_{\theta r} - \partial_r g_{\theta\theta}) \\
&= \frac{1}{2}g^{rr}(0 + 0 - \partial_r g_{\theta\theta}) = \frac{1}{2}g^{rr}(-\partial_r g_{\theta\theta}) \\
&= \frac{1}{2}(e^{-2\beta(r)})(-\partial_r r^2) = -\frac{1}{2}(e^{-2\beta(r)})(2r) \\
\Gamma^r_{\theta\theta} &= -re^{-2\beta(r)}
\end{aligned} \tag{8}
$$

$$\begin{aligned}
\Gamma^{\phi}_{r\phi} &= \frac{1}{2}g^{\phi\sigma}(\partial_r g_{\sigma\phi} + \partial_\phi g_{r\sigma} - \partial_\sigma g_{r\phi}) \\
&= \frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi} + \partial_\phi g_{r\phi} - \partial_\phi g_{r\phi}) \\
&= \frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi} + 0 - 0) = \frac{1}{2}g^{\phi\phi}(\partial_r g_{\phi\phi}) \\
&= \frac{1}{2}(r^2 sin^2\theta)^{-1}\partial_r(r^2 sin^2\theta) \\
&= \frac{1}{2}(r^2 sin^2\theta)^{-1}(2rsin^2\theta) = \frac{1}{r}
\end{aligned} \tag{9}$$

$$\begin{aligned}
\Gamma^{r}_{\phi\phi} &= \frac{1}{2}g^{r\sigma}(\partial_\phi g_{\sigma\phi} + \partial_\phi g_{\phi\sigma} - \partial_\sigma g_{\phi\phi}) \\
&= \frac{1}{2}g^{rr}(\partial_\phi g_{r\phi} + \partial_\phi g_{\phi r} - \partial_r g_{\phi\phi}) \\
&= \frac{1}{2}g^{rr}(0 + 0 - \partial_r g_{\phi\phi}) = \frac{1}{2}g^{rr}(\partial_r g_{\phi\phi}) \\
&= \frac{1}{2}(e^{-2\beta(r)})(-\partial_r(r^2 sin^2\theta)) \\
&= \frac{1}{2}(e^{-2\beta(r)})(-2rsin^2\theta) \\
\Gamma^{r}_{\phi\phi} &= -rsin^2\theta e^{-2\beta(r)}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}g^{\theta\sigma}(\partial_\phi g_{\sigma\phi} + \partial_\phi g_{\phi\sigma} - \partial_\sigma g_{\phi\phi}) \\
&= \frac{1}{2}g^{\theta\theta}(\partial_\phi g_{\theta\phi} + \partial_\phi g_{\phi\theta} - \partial_\theta g_{\phi\phi}) \\
&= \frac{1}{2}g^{\theta\theta}(0 + 0 - \partial_\theta g_{\phi\phi}) \\
&= \frac{1}{2}(r^2)^{-1}(-\partial_\theta(r^2 sin^2\theta)) \\
\Gamma^{\theta}_{\phi\phi} &= -sin\theta cos\theta
\end{aligned} \tag{11}$$

$$\begin{aligned}
\Gamma^{\phi}_{\theta\phi} &= \frac{1}{2}g^{\phi\sigma}(\partial_\theta g_{\sigma\phi} + \partial_\phi g_{\theta\sigma} - \partial_\sigma g_{\theta\phi}) \\
&= \frac{1}{2}g^{\phi\phi}(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\theta\phi} - \partial_\phi g_{\theta\phi}) \\
&= \frac{1}{2}g^{\phi\phi}(\partial_\theta g_{\phi\phi} + 0 - 0) \\
&= \frac{1}{2}(r^2 sin^2\theta)^{-1}\partial_\theta(r^2 sin^2\theta) \\
&= \frac{1}{2}(r^2 sin^2\theta)^{-1}(2r^2 sin\theta cos\theta) \\
\Gamma^{\phi}_{\theta\phi} &= \frac{cos\theta}{sin\theta}
\end{aligned} \tag{12}$$

Summary:

$$\Gamma^t_{tr} = \partial_r\alpha, \Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha, \Gamma^r_{rr} = \partial_r\beta$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r}, \Gamma^r_{\theta\theta} = -re^{-2\beta}, \Gamma^\phi_{r\phi} = \frac{1}{r}$$
$$\Gamma^r_{\phi\phi} = -re^{-2\beta}sin^2\theta, \Gamma^\theta_{\phi\phi} = -sin\theta cos\theta, \Gamma^\phi_{r\phi} = \frac{cos\theta}{sin\theta} \tag{13}$$

We know that Riemann tensor

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{14}$$

Now, We calculate Ricci tensor

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$$
$$R_{tt} = R^\lambda_{t\lambda t} = R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t} \tag{15}$$

$$R^r_{trt} = \partial_r\Gamma^r_{tt} - \partial_t\Gamma^r_{rt} + \Gamma^r_{r\lambda}\Gamma^\lambda_{tt} - \Gamma^r_{t\lambda}\Gamma^\lambda_{rt}$$
$$= \partial_r\Gamma^r_{tt} - 0 + \Gamma^r_{rr}\Gamma^r_{tt} + \Gamma^r_{r\theta}\Gamma^\theta_{tt} + \Gamma^r_{r\phi}\Gamma^\phi_{tt} - \Gamma^r_{tt}\Gamma^t_{rt} - \Gamma^r_{tr}\Gamma^r_{rt} - \Gamma^r_{t\theta}\Gamma^\theta_{rt} - \Gamma^r_{t\phi}\Gamma^\phi_{rt}$$
$$= \partial_r(e^{2(\alpha-\beta)}\partial_r\alpha) + \partial_r\beta e^{2(\alpha-\beta)}\partial_r\alpha + 0 + 0 - e^{2(\alpha-\beta)}\partial_r\alpha\partial_r\alpha - 0 - 0 - 0$$
$$= \partial_r^2\alpha e^{2(\alpha-\beta)} + \partial_r\alpha\partial_r(e^{2(\alpha-\beta)}) + \partial_r\alpha\partial_r\beta e^{2(\alpha-\beta)} - e^{2(\alpha-\beta)}(\partial_r\alpha)^2$$
$$= \partial_r^2\alpha e^{2(\alpha-\beta)} + (2e^{2(\alpha-\beta)}\partial_r\alpha - 2e^{2(\alpha-\beta)}\partial_r\beta)\partial_r\alpha + \partial_r\alpha\partial_r\beta e^{2(\alpha-\beta)} - e^{2(\alpha-\beta)}(\partial_r\alpha)^2$$
$$= e^{2(\alpha-\beta)}[\partial_r^2\alpha + 2(\partial_r\alpha)^2 - 2\partial_r\alpha\partial_r\beta + \partial_r\alpha\partial_r\beta - (\partial_r\alpha)^2]$$
$$R^r_{trt} = e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta] \tag{16}$$

$$R^\theta_{t\theta t} = \partial_\theta\Gamma^\theta_{tt} - \partial_t\Gamma^\theta_{\theta t} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{tt} - \Gamma^\theta_{t\lambda}\Gamma^\lambda_{\theta t}$$
$$= 0 + 0 + \Gamma^\theta_{\theta t}\Gamma^t_{tt} + \Gamma^\theta_{\theta r}\Gamma^r_{tt} + \Gamma^\theta_{\theta\theta}\Gamma^\theta_{tt} + \Gamma^\theta_{\theta\phi}\Gamma^\phi_{tt} - \Gamma^\theta_{tt}\Gamma^t_{\theta t} - \Gamma^\theta_{tr}\Gamma^r_{\theta t} - \Gamma^\theta_{t\theta}\Gamma^\theta_{\theta t} - \Gamma^\theta_{t\phi}\Gamma^\phi_{\theta t}$$
$$= 0 + 0 + 0 + \Gamma^\theta_{\theta r}\Gamma^r_{tt} + 0 + 0 - 0 - 0 - 0 - 0$$
$$R^\theta_{t\theta t} = \frac{1}{r}e^{2(\alpha-\beta)}\partial_r\alpha \tag{17}$$

$$R^\phi_{t\phi t} = \partial_\phi\Gamma^\phi_{tt} - \partial_t\Gamma^\phi_{\phi t} + \Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{tt} - \Gamma^\phi_{t\lambda}\Gamma^\lambda_{\phi t}$$
$$= 0 + 0 + \Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{tt} - \Gamma^\phi_{t\lambda}\Gamma^\lambda_{\phi t}$$
$$= \Gamma^\phi_{\phi t}\Gamma^t_{tt} + \Gamma^\phi_{\phi r}\Gamma^r_{tt} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{tt} + \Gamma^\phi_{\phi\phi}\Gamma^\phi_{tt} - \Gamma^\phi_{tt}\Gamma^t_{\phi t} - \Gamma^\phi_{tr}\Gamma^r_{\phi t} - \Gamma^\phi_{t\theta}\Gamma^\theta_{\phi t} - \Gamma^\phi_{t\phi}\Gamma^\phi_{\phi t}$$
$$= 0 + \Gamma^\phi_{\phi r}\Gamma^r_{tt} + 0 + 0 - 0 - 0 - 0 - 0$$
$$R^\phi_{t\phi t} = \frac{1}{r}e^{2(\alpha-\beta)}\partial_r\alpha \tag{18}$$

From eq.(15) can be written as:

$$R_{tt} = R^\lambda_{t\lambda t} = R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t} = e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta] + \frac{2}{r}e^{2(\alpha-\beta)}\partial_r\alpha$$
$$R_{tt} = e^{2(\alpha-\beta)}[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha] \tag{19}$$

$$R_{rr} = R^{\lambda}_{r\lambda r} = R^{t}_{rtr} + R^{\theta}_{r\theta r} + R^{\phi}_{r\phi r}$$

$$\begin{aligned} R^{t}_{rtr} &= \partial_t \Gamma^{t}_{rr} - \partial_r \Gamma^{t}_{tr} + \Gamma^{t}_{t\lambda}\Gamma^{\lambda}_{rr} - \Gamma^{t}_{r\lambda}\Gamma^{\lambda}_{tr} \\ = 0 - \partial_r \Gamma^{t}_{tr} + \Gamma^{t}_{tt}\Gamma^{t}_{rr} + \Gamma^{t}_{tr}\Gamma^{r}_{rr} + \Gamma^{t}_{t\theta}\Gamma^{\theta}_{rr} + \Gamma^{t}_{t\phi}\Gamma^{\phi}_{rr} &- \Gamma^{t}_{rt}\Gamma^{t}_{tr} - \Gamma^{t}_{rr}\Gamma^{r}_{tr} - \Gamma^{t}_{r\theta}\Gamma^{\theta}_{tr} - \Gamma^{t}_{r\phi}\Gamma^{\phi}_{tr} \\ &= -\partial_r(\partial_r \alpha) + 0 + \partial_r \alpha \partial_r \beta + 0 + 0 - \partial_r \alpha \partial_r \alpha - 0 - 0 - 0 \\ R^{t}_{rtr} &= \partial_r \alpha \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2 \end{aligned} \quad (20)$$

$$\begin{aligned} R^{\theta}_{r\theta r} &= \partial_\theta \Gamma^{\theta}_{rr} - \partial_r \Gamma^{\theta}_{\theta r} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{rr} - \Gamma^{\theta}_{r\lambda}\Gamma^{\lambda}_{\theta r} \\ = 0 - \partial_r(\frac{1}{r}) + \Gamma^{\theta}_{\theta t}\Gamma^{t}_{rr} + \Gamma^{\theta}_{\theta r}\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{rr} + \Gamma^{\theta}_{\theta\phi}\Gamma^{\phi}_{rr} &- \Gamma^{\theta}_{rt}\Gamma^{t}_{\theta r} - \Gamma^{\theta}_{rr}\Gamma^{r}_{\theta r} - \Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{\theta r} - \Gamma^{\theta}_{r\phi}\Gamma^{\phi}_{\theta r} \\ &= \frac{1}{r^2} + 0 + \frac{1}{r}\partial_r \beta + 0 - \frac{1}{r}\frac{1}{r} - 0 - 0 - 0 \\ R^{\theta}_{r\theta r} &= \frac{1}{r}\partial_r \beta \end{aligned} \quad (21)$$

$$\begin{aligned} R^{\phi}_{r\phi r} &= \partial_\phi \Gamma^{\phi}_{rr} - \partial_r \Gamma^{\phi}_{\phi r} + \Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{rr} - \Gamma^{\phi}_{r\lambda}\Gamma^{\lambda}_{\phi r} \\ = 0 - \partial_r(\frac{1}{r}) + \Gamma^{\phi}_{\phi t}\Gamma^{t}_{rr} + \Gamma^{\phi}_{\phi r}\Gamma^{r}_{rr} + \Gamma^{\phi}_{\phi\theta}\Gamma^{\theta}_{rr} + \Gamma^{\phi}_{\phi\phi}\Gamma^{\phi}_{rr} &- \Gamma^{\phi}_{rt}\Gamma^{t}_{\phi r} - \Gamma^{\phi}_{rr}\Gamma^{r}_{\phi r} - \Gamma^{\phi}_{r\theta}\Gamma^{\theta}_{\phi r} - \Gamma^{\phi}_{r\phi}\Gamma^{\phi}_{\phi r} \\ &= \frac{1}{r^2} + 0 + \frac{1}{r}\partial_r \beta + 0 - 0 - 0 - 0 - \frac{1}{r}\frac{1}{r} \\ R^{\phi}_{r\phi r} &= \frac{1}{r}\partial_r \beta \end{aligned} \quad (22)$$

$$\begin{aligned} R_{rr} &= \partial_r \alpha \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2 + \frac{1}{r}\partial_r \beta + \frac{1}{r}\partial_r \beta \\ R_{rr} &= \partial_r \alpha \partial_r \beta - \partial_r^2 \alpha - (\partial_r \alpha)^2 + \frac{2}{r}\partial_r \beta \end{aligned} \quad (23)$$

$$R_{\theta\theta} = R^{\lambda}_{\theta\lambda\theta} = R^{t}_{\theta t\theta} + R^{r}_{\theta r\theta} + R^{\phi}_{\theta\phi\theta}$$

$$\begin{aligned} R^{t}_{\theta t\theta} &= \partial_t \Gamma^{t}_{\theta\theta} - \partial_\theta \Gamma^{t}_{t\theta} + \Gamma^{t}_{t\lambda}\Gamma^{\lambda}_{\theta\theta} - \Gamma^{t}_{\theta\lambda}\Gamma^{\lambda}_{t\theta} \\ = 0 - 0 + \Gamma^{t}_{tt}\Gamma^{t}_{\theta\theta} + \Gamma^{t}_{tr}\Gamma^{r}_{\theta\theta} + \Gamma^{t}_{t\theta}\Gamma^{\theta}_{\theta\theta} + \Gamma^{t}_{t\phi}\Gamma^{\phi}_{\theta\theta} &- \Gamma^{t}_{\theta t}\Gamma^{t}_{t\theta} - \Gamma^{t}_{\theta r}\Gamma^{r}_{t\theta} - \Gamma^{t}_{\theta\theta}\Gamma^{\theta}_{t\theta} - \Gamma^{t}_{\theta\phi}\Gamma^{\phi}_{t\theta} \\ &= 0 + \partial_r \alpha(-re^{-2\beta}) + 0 + 0 - 0 - 0 - 0 - 0 \\ R^{t}_{\theta t\theta} &= -re^{-2\beta}\partial_r \alpha \end{aligned} \quad (24)$$

$$R^r_{\theta r\theta} = \partial_r\Gamma^r_{\theta\theta} - \partial_\theta\Gamma^r_{r\theta} + \Gamma^r_{r\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^r_{\theta\lambda}\Gamma^\lambda_{r\theta}$$
$$= \partial_r(-re^{-2\beta}) - 0 + \Gamma^r_{rt}\Gamma^t_{\theta\theta} + \Gamma^r_{rr}\Gamma^r_{\theta\theta} + \Gamma^r_{r\theta}\Gamma^\theta_{\theta\theta} + \Gamma^r_{r\phi}\Gamma^\phi_{\theta\theta} - \Gamma^r_{\theta t}\Gamma^t_{r\theta} - \Gamma^r_{\theta r}\Gamma^r_{r\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta} - \Gamma^r_{\theta\phi}\Gamma^\phi_{r\theta}$$
$$= 2re^{-2\beta}\partial_r\beta - e^{-2\beta} + 0 + \partial_r\beta(-re^{-2\beta}) + 0 + 0 - 0 - 0 - (-re^{-2\beta})\frac{1}{r} - 0$$
$$= 2re^{-2\beta}\partial_r\beta - e^{-2\beta} + \partial_r\beta(-re^{-2\beta}) + e^{-2\beta}$$
$$R^r_{\theta r\theta} = re^{-2\beta}\partial_r\beta \ldots\ldots (25)$$

$$R^\phi_{\theta\phi\theta} = \partial_\phi\Gamma^\phi_{\theta\theta} - \partial_\theta\Gamma^\phi_{\phi\theta} + \Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{\theta\theta} - \Gamma^\phi_{\theta\lambda}\Gamma^\lambda_{\phi\theta}$$
$$= 0 - \partial_\theta\Gamma^\phi_{\phi\theta} + \Gamma^\phi_{\phi t}\Gamma^t_{\theta\theta} + \Gamma^\phi_{\phi r}\Gamma^r_{\theta\theta} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\theta\theta} + \Gamma^\phi_{\phi\phi}\Gamma^\phi_{\theta\theta} - \Gamma^\phi_{\theta t}\Gamma^t_{\phi\theta} - \Gamma^\phi_{\theta r}\Gamma^r_{\phi\theta} - \Gamma^\phi_{\theta\theta}\Gamma^\theta_{\phi\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi\theta}$$
$$= -\partial_\theta(cot\theta) + 0 + \frac{1}{r}(-re^{-2\beta}) + 0 - 0 - 0 - 0 - cot^2\theta$$
$$R^\phi_{\theta\phi\theta} = (1 - e^{-2\beta}) \ldots\ldots (26)$$

$$R_{\theta\theta} = -re^{-2\beta}\partial_r\alpha + re^{-2\beta}\partial_r\beta + (1 - e^{-2\beta})$$
$$R_{\theta\theta} = e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1 \qquad (27)$$

$$R_{\phi\phi} = R^\lambda_{\phi\lambda\phi} = R^t_{\phi t\phi} + R^r_{\phi r\phi} + R^\theta_{\phi\theta\phi}$$
$$= -re^{-2\beta}sin^2\theta\partial_r\alpha + re^{-2\beta}sin^2\theta\partial_r\beta + (1 - e^{-2\beta})sin^2\theta$$
$$= \{e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1] + 1\}sin^2\theta$$
$$R_{\phi\phi} = sin^2\theta R_{\theta\theta} \qquad (28)$$

Note: $\partial_r^2\alpha = \alpha'', \partial_r\alpha = \alpha', \partial_r\beta = \beta'$
Curvature scalar or Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}$$
$$= -e^{-2\alpha}e^{2(\alpha-\beta)}[\alpha'' - \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha'] + e^{-2\beta}[-\alpha'' - \alpha'^2 + \alpha'\beta'$$
$$+\frac{2}{r}\beta'] + \frac{1}{r^2}\{e^{-2\beta}[r(\beta' - \alpha') - 1] + 1\} + \frac{1}{r^2 sin^2\theta}\{e^{-2\beta}[r(\beta' - \alpha') - 1] + 1\}sin^2\theta$$
$$= e^{-2\beta}[-\alpha'' - \alpha'^2 + \alpha'\beta' - \frac{2}{r}\alpha' - \alpha'' - \alpha'^2 + \alpha'\beta' + \frac{2}{r}\beta'] + \frac{2}{r^2}\{e^{-2\beta}[r(\beta' - \alpha') - 1] + 1\}$$
$$= 2e^{-2\beta}[-\alpha'' - \alpha'^2 + \alpha'\beta' - \frac{1}{r}(\alpha' - \beta')] + 2e^{-2\beta}[\frac{1}{r}(\beta' - \alpha') - \frac{1}{r^2}] + \frac{2}{r^2}$$
$$= -2e^{-2\beta}[-\alpha'' - \alpha'^2 + \alpha'\beta' - \frac{2}{r}(\alpha' - \beta') - \frac{1}{r^2}] + \frac{2}{r^2}$$
$$= -2e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta})]$$
$$R = 2e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta})] \ldots\ldots\ldots (29)$$

Now, calculate full Einstien's equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \tag{30}$$

tt-component:

$$G_{tt} = R_{tt} - \frac{1}{2}g_{tt}R$$

$$= e^{2(\alpha-\beta)}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha'] - \frac{1}{2}(-2e^{2\alpha})(-2e^{-2\beta})[\alpha'' + \alpha' - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta}]$$

$$= e^{2(\alpha-\beta)}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha' - \alpha'' - \alpha'^2 + \alpha'\beta' - \frac{2}{r}(\alpha' - \beta') - \frac{1}{r^2}(1 - e^{2\beta})]$$

$$= e^{2(\alpha-\beta)}[\frac{2}{r}\beta' - \frac{1}{r^2}(1 - e^{2\beta})]$$

$$= \frac{1}{r^2}e^{2(\alpha-\beta)}[2\beta' r - (1 - e^{2\beta})]$$

$$G_{tt} = \frac{1}{r^2}e^{2(\alpha-\beta)}[2r\beta' - 1 + e^{2\beta}]............(31)$$

rr-component:

$$G_{rr} = R_{rr} - \frac{1}{2}g_{rr}R$$

$$= -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{2}{r}\beta' - \frac{1}{2}e^{2\beta}(-2e^{-2\beta})[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta})]$$

$$= \frac{2}{r}\alpha' + \frac{1}{r^2}(1 - e^{2\beta})$$

$$G_{rr} = \frac{1}{r^2}[2r\alpha' + 1 - e^{2\beta}]............. (32)$$

$$G_{\theta\theta} = R_{\theta\theta} - \frac{1}{2}g_{\theta\theta}R$$

$$= e^{-2\beta}[r(\beta' - \alpha') - 1] + 1 - \frac{1}{r^2}(-2e^{-2\beta})[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta})]$$

$$G_{\theta\theta} = r^2e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')]..... \quad (33)$$

$$G_{\phi\phi} = R_{\phi\phi} - \frac{1}{2}g_{\phi\phi}R$$

$$= sin^2\theta\{e^{-2\beta}[r(\beta' - \alpha') - 1] + 1\} - \frac{1}{2}r^2sin^2\theta(-2e^{-2\beta})[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta}]$$

$$= e^{-2\beta}sin^2\theta[r(\beta' - \alpha') - 1 + e^{2\beta}] + r^2sin^2\theta e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}(\alpha' - \beta') + \frac{1}{r^2}(1 - e^{2\beta})]$$

$$= e^{-2\beta}sin^2\theta[r\beta' - r\alpha' - 1 + e^{2\beta} + r^2\alpha'' + r^2\alpha'^2 - r^2\alpha'\beta' + 2r\alpha' - 2r\beta' + 1 - e^{2\beta}]$$

$$= e^{-2\beta}sin^2\theta[r\alpha' - r\beta' + r^2\alpha'' + r^2\alpha'^2 - r^2\alpha'\beta']$$

$$G_{\phi\phi} = sin^2\theta G_{\theta\theta}....(34)$$

Summary:

$$
\begin{aligned}
G_{tt} &= \frac{1}{r^2}e^{2(\alpha-\beta)}(2r\beta' - 1 + e^{2\beta} \\
G_{rr} &= \frac{1}{r^2}(2r\alpha' + 1 - e^{2\beta} \\
G_{\theta\theta} &= r^2 e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')] \\
G_{\phi\phi} &= sin^2\theta G_{\theta\theta}
\end{aligned} \tag{35}
$$

The energy-momentum tensor of the star itself as a perfect fluid is given by

$$T_{\mu nu} = (\epsilon + p)u_\mu u_\nu + pg_{\mu\nu} \tag{36}$$

Here, the energy density (ϵ) and pressure density (p) is a function of r alone. Since, we pursue static solution, we can take the four velocity to be pointing in timelike direction. Normalized to $u_\mu u_\nu = -1$, it becomes

$$u_\mu = (e^\alpha, 0, 0, 0) \tag{37}$$

So, the component of energy momentum tensor will be

$$T = \begin{bmatrix} e^{2\alpha(r)}\epsilon & 0 & 0 & 0 \\ 0 & e^{2\beta(r)}p & 0 & 0 \\ 0 & 0 & r^2 p & 0 \\ 0 & 0 & 0 & r^2 p sin^2\theta \end{bmatrix}$$

Einstien tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{38}$$

Using all previous results we can fins that the components of the Einstein tensor are:
The tt-component:

$$
\begin{aligned}
\frac{1}{r^2}e^{2(\alpha-\beta)}(2r\beta' - 1 + e^{2\beta}) &= 8\pi G e^{2\alpha}\epsilon \\
\frac{1}{r^2}e^{-2\beta}(2r\beta' - 1 + e^{2\beta}) &= 8\pi G\epsilon
\end{aligned} \tag{39}
$$

The rr-component:

$$
\begin{aligned}
\frac{1}{r^2}(2r\alpha' + 1 - e^{2\beta}) &= 8\pi G e^{2\beta}p \\
\frac{1}{r^2}e^{-2\beta}(2r\alpha' + 1 - e^{2\beta}) &= 8\pi G p
\end{aligned} \tag{40}
$$

The $\theta\theta$-component:

$$
\begin{aligned}
r^2 e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')] &= 8\pi G r^2 p \\
e^{-2\beta}[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')] &= 8\pi G p
\end{aligned} \tag{41}
$$

The $\phi\phi$-component is proportional to the $\theta\theta$-equation, so there is no need to consider seperately. So, the tt-component of the Einstien equation gives:

$$\begin{aligned}
\frac{1}{r^2}e^{-2\beta}(2r\beta' - 1 + e^{2\beta}) &= 8\pi G\epsilon \\
e^{-2\beta}(2r\beta' - 1) + 1 &= 8\pi G\epsilon r^2 \\
-\frac{d}{dr}\{r(e^{-2\beta} - 1)\} &= 8\pi G\epsilon r^2 \\
\frac{d}{dr}\{r(e^{-2\beta} - 1)\} &= -k\epsilon r^2 \\
d\{r(e^{-2\beta} - 1)\} &= -k\epsilon r^2 dr
\end{aligned} \tag{42}$$

Where, $k = -8\pi G$ and previous equation can be integrated

$$e^{-2\beta} = 1 - \frac{k}{r}\int_0^R \epsilon(r) r^2 dr \tag{43}$$

Let us define

$$\begin{aligned}
m(r) &= 4\pi \int_0^R \epsilon(r) r^2 dr \\
e^{-2\beta} &= 1 - \frac{8\pi G}{4\pi r} m(r) \\
e^{-2\beta} &= \big(1 - \frac{2Gm(r)}{r}\big) \\
e^{2\beta} &= \big(1 - \frac{2Gm(r)}{r}\big)^{-1}
\end{aligned} \tag{44}$$

So that, the metric will be

$$ds^2 = -e^{2\alpha(r)}dt^2 + \big(1 - \frac{2Gm(r)}{r}\big)^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{45}$$

M_G= graviational mass

$$\begin{aligned}
M_G &= 4\pi \int_0^R \epsilon(r) r^2 e^{2\beta} dr \\
M_G &= 4\pi \int_0^R \frac{\epsilon(r) r^2}{(1 - \frac{2Gm(r)}{r})} dr
\end{aligned} \tag{46}$$

The binding energy due to the internal gravitational attraction of the fluid elements in the star, which is given by

$$E_B = M_G - M > 0 \tag{47}$$

The binding energy is the amount of energy that would be required to disperse the matter in the star to be infinity. The rr-component can be written as

$$
\begin{aligned}
\frac{1}{r^2}e^{-2\beta}(2r\frac{d\alpha}{dr} + 1 - e^{-2\beta} &= 8\pi Gp \\
2r\frac{d\alpha}{dr} + 1 - e^{2\beta} &= 8\pi Gpr^2e^{2\beta} \\
2r\frac{d\alpha}{dr} &= 8\pi Gpr^2e^{2\beta} + e^{2\beta} - 1 \\
&= \frac{(8\pi Gpr^2+1)}{(1-\frac{2Gm(r)}{r})} - 1 \\
&= \frac{(8\pi Gpr^3+r)}{(r-2Gm(r))} - 1 \\
= \frac{8\pi Gpr^3 + r - r + 2Gm(r)}{(r-2Gm(r))} \\
&= \frac{8\pi Gpr^3 + 2Gm(r)}{(r-2Gm(r))} \\
\frac{d\alpha}{dr} &= \frac{(4\pi Gpr^3 + Gm(r))}{(r-2Gm(r))} \qquad (48)
\end{aligned}
$$

Now, we have to calculate $\frac{dp}{dr}$ =?. Again, solve the tt-component equation

$$
\begin{aligned}
(2r\beta' - 1 + e^{2\beta}) &= 8\pi G\epsilon r^2 e^{2\beta} \\
2r\beta' &= (8\pi G\epsilon r^2 e^{2\beta} - 1)e^{2\beta} + 1 \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
(2r\alpha' + 1 - e^{2\beta}) &= 8\pi Gpr^2e^{2\beta} \\
2r\alpha' &= (1 + 8\pi Gpr^2e^{2\beta})e^{2\beta} - 1 \qquad (50)
\end{aligned}
$$

Take the derivative of the previous equation then

$$
\begin{aligned}
2\alpha' + 2r\alpha'' &= (8\pi Gp'r^2 + 16\pi Gpr)e^{2\beta} + 2\beta'(1 + 8\pi Gpr^2)e^{2\beta} \\
2r\alpha' + 2r^2\alpha'' &= [2r\beta'(1+8\pi Gr^2p) + (16\pi Gr^2p + 8\pi Gr^3p')]e^{2\beta} \\
2r^2\alpha'' &= [2r\beta'(1+8\pi Gr^2p) + (16\pi Gr^2p + 8\pi Gr^3p')]e^{2\beta} - 2r\alpha' \qquad (51)
\end{aligned}
$$

Put the values of $2r\alpha'$ and $2r\beta'$ from previous equations then

$$
2r^2\alpha'' = 1 + (16\pi Gr^2p + 8\pi Gr^3p')e^{2\beta} - (1 + 8\pi r^2p)(1 - 8\pi Gr^2\epsilon)e^{4\beta} \qquad (52)
$$

Square eq.(50) to obtain the result

$$
2r^2\alpha'^2 = \frac{1}{2}(1 + 8\pi Gr^2p)^2e^{4\beta} - (1 + 8\pi Gr^2p)e^{2\beta} - \frac{1}{2} \qquad (53)
$$

Now, we have the expressions for $\alpha', \alpha'', \alpha'^2$, and β' in terms of p, p', ϵ, and $e^{2\beta}$. Hence, equation (52) can be written as:

$$
\begin{aligned}
(\epsilon + p)\frac{d\alpha}{dr} &= -\frac{dp}{dr} \\
\frac{dp}{dr} &= -\frac{(\epsilon+p)(Gm(r) + 4\pi Gr^3p)}{r(r-2Gm(r))} \qquad (54)
\end{aligned}
$$

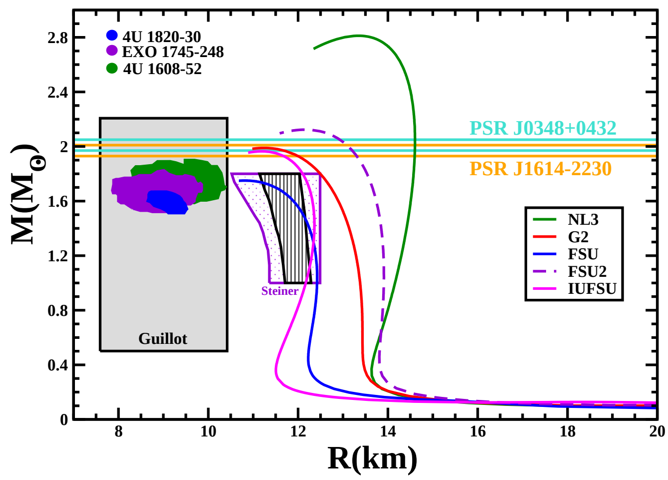

FIG. 1: The mass-radius profile for the force parameters.

Summary: For G=c=1, the TOV equations are written as [2]

$$\frac{dp}{dr} = -\frac{(\epsilon + p)(m(r) + 4\pi r^3 p)}{r(r - 2m(r))} \tag{55}$$

$$\frac{dm}{dr} = 4\pi r^2 \epsilon(r) \tag{56}$$

For a given EOS [3], the Tolmann-Oppenheimer-Volkov (TOV) equations must be integrated from the boundary conditions $P(0) = P_c$, and $M(0) = 0$, where P_c and $M(0)$ are the pressure and mass of the star at $r = 0$ and the value of $r(= R)$, where the pressure vanish defines the surface of the star. Thus, at each central density we can uniquely determine a mass M and a radius R of the static neutron and hyperon stars using the four chosen EOSs. The calculated results shown in Fig.1.

References

[1] Sean M. Carroll, Spacetime and Geometry, An introduction to general relativity.
[2] Compact stars by Glendenning, 1996.
[3] Bharat Kumar, S. K. Biswal, and S. K. Patra, Phys. Rev. C **95** (2017) 015801.