Three

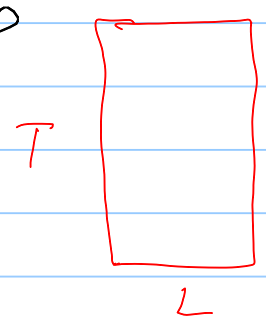
Note Title 10/27/2006

One intersting operator in gauge theory is Wilson loop. using AdS/CFT corresponcen one would expect to get some information about it using the gravity dual. In fact following our computation on how to get n-point function from gravity one would like to compute vaccum expection value of wilson loop using gravity dual.

In gauge theory the Wilson loop is defined by integral of $A_\mu$ over a loop C

$$W(C) = Tr\left(P\exp\left[\oint_C A \cdot dx\right]\right)$$

The trace is taking over some representation.
we take the fundamental rep.
This operator represents the YM
Contribution to the propagation of
a heavy quark in fundamental rep.
From expectation value of the wilson loop
$\langle W(C) \rangle$ we can calculate the $q-\bar{q}$
potential. The aim is to find this
potential using gravity dual.
For this purpose consider a rectangular
loop




For Larg $T$

For Larg $T$ one gets

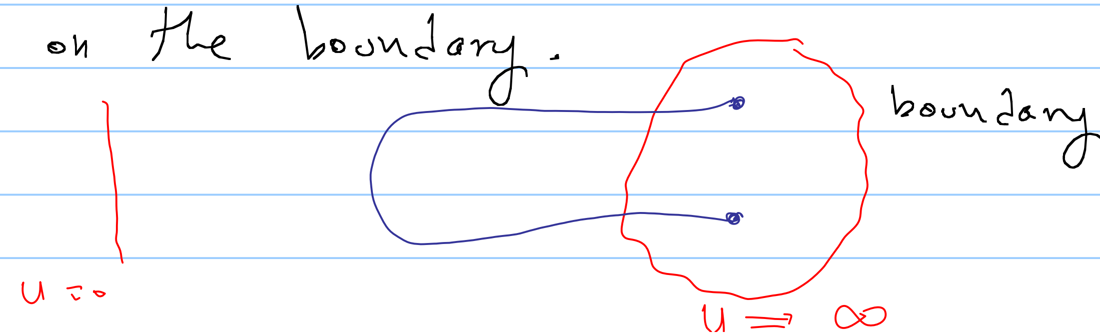
$$\langle W(C)\rangle \approx e^{-T(\text{energy})} \sim e^{-TV(L)}$$

lowest possible energy of $q\bar{q}$

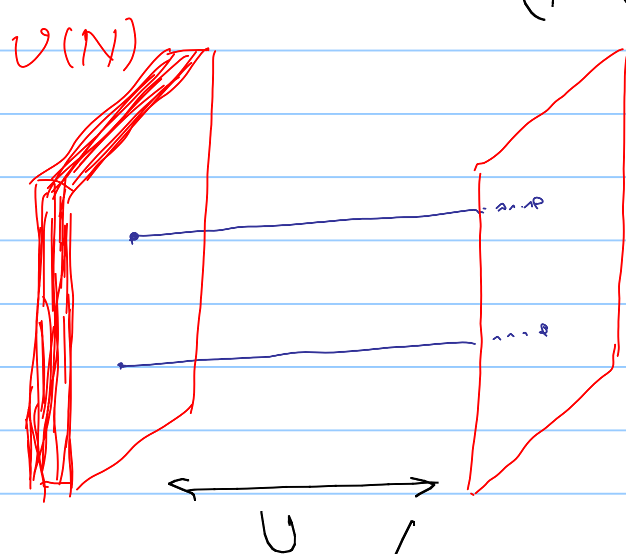
using AdS/CFT Correspondence we want to compute $V(L)$.

In QCD wilson loop is related to flux (string) connecting $q$ and $\bar{q}$.

In our Case we expect that the $q-\bar{q}$ is connected by Type IIB string live in 10-dimensions and Can end to two points on the boundary.



To get an insight how it works
Consider $U(N+1)$ gauge theory which can live on $(N+1)$ D3-branes




we break $U(N+1)$ to $U(N) \times U(1)$

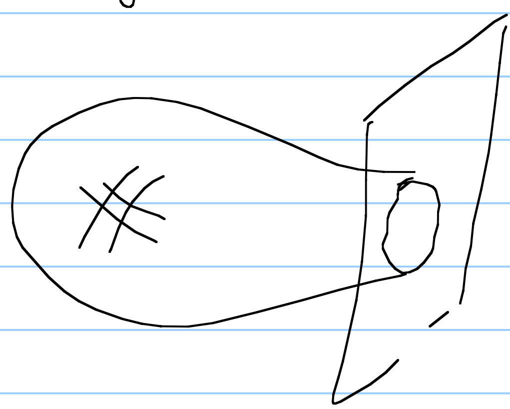
$$\left(\begin{array}{c|c} N\times N & N\times 1 \\ \hline 1\times N & 1\times 1 \end{array}\right)$$

rep. $N$ of $U(N)$

It is massive with mass $\sim U$

To get non-dynamical external quark one send $m \longrightarrow \infty \Rightarrow U \longrightarrow \infty$

so string should end on the boundary



one needs to compute the classical action of string with condition that the world sheet intersects the boundary on the loop "C".

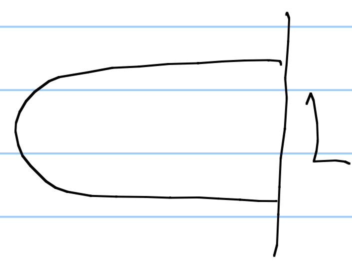
The string action is

$$S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det\left(G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu\right)}$$

in our case where we have $AdS_5 \times S^5$

The background metric $G_{\mu\nu}$ is given by

$$ds^2 = \frac{U^2}{R^2}\left(-dt^2 + \cdots + dx_3^2\right) + \frac{R^2}{U^2}dU^2 + R^2 d\Omega_5^2$$



The String is parametrized by

$t = \tau \qquad \sigma = x_1 \equiv x \qquad U(x)$

So that

$$G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu = \begin{pmatrix} G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu & G_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu \\ G_{\mu\nu}\partial_\sigma X^\mu \partial_\tau X^\nu & G_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu \end{pmatrix}$$

$$= \begin{pmatrix} G_{00} & 0 \\ 0 & G_{11} + G_{UU}\left(\frac{\partial U}{\partial x}\right)^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{U^2}{R^2} & 0 \\ 0 & \frac{U^2}{R^2} + \frac{R^2}{U^2} U'^2 \end{pmatrix}$$

$$U' = \frac{\partial U}{\partial x}$$

$$-\det\left(G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right) = \frac{U^4}{R^2}\left(1 + \frac{R^4}{U^4} U'^2\right)$$

$$S = \frac{T}{2\pi\alpha'} \int dx \, \frac{U^2}{R^2} \sqrt{1 + \frac{R^4}{U^4} U'^2}$$

$$P_U = \frac{\partial \mathcal{L}}{\partial U'} = \frac{R^2}{U^2} \frac{U'}{\sqrt{1 + \frac{R^4}{U^4} U'^2}}$$

$$H = P_U U' - \mathcal{L} = \frac{U^2/R^2}{\sqrt{1 + \frac{R^4}{U^4} U'^2}} = \text{cte} = \frac{U_0^2}{R^2}$$

where $U_0$ is the point in which

$$U'\big|_{U_0} = 0$$

So we get $U'^2 = \frac{U^4}{R^4}\left(\frac{U^4}{U_0^4} - 1\right) = \left(\frac{dU}{dx}\right)^2$

$$\Rightarrow dx = \frac{R^3}{U^2} \frac{dU}{\left(\frac{U^4}{U_0^4} - 1\right)^{1/2}}$$

$$x = \int_{U_0}^{U} \frac{R^2}{U^2} \frac{dU}{\left(\frac{U^4}{U_0^4} - 1\right)^{1/2}}$$

$$\frac{L}{2} = \frac{R^2}{U_0^2} \int_1^{\infty} \frac{dy}{y^2 \sqrt{y^4 - 1}} \qquad y = \frac{U}{U_0}$$

$$\frac{L}{2} = \frac{R^3}{U_0} \frac{\sqrt{2}\,\pi^{3/2}}{\Gamma(1/4)^2}$$

on the other hand the action reads

$$S = \frac{T}{2\pi\alpha'} \int_{U_0}^{\infty} \frac{R^3}{U^2} \frac{dU}{\left(\frac{U^4}{U_0^4} - 1\right)} \frac{U^3}{R^2} \frac{U^3}{U_0^2}$$

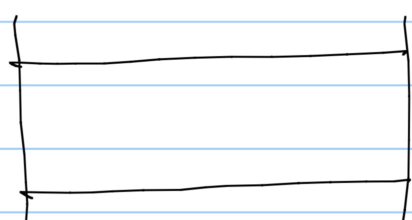
$$S = \frac{T}{2\pi\alpha'} U_0 \int_1^\infty \frac{y^2\,dy}{(y^4-1)^{1/2}}$$

Therefore the energy is

$$V(L) = \frac{N_0}{2\pi\alpha'} \int_1^\infty \frac{y^2\,dy}{(y^4-1)^{1/2}}$$

In general it diverges (it is infinity)

This is due to the self energy of dual corresponding to strings right away from boundary (infinity) to $U=0$



The self energy is $\frac{U_0}{2\pi\alpha'} \int_0^\infty dy$

$$E = \frac{1}{2\pi\alpha'} U_0 \int_1^\infty \frac{y^2 dy}{(y^4-1)^{1/2}} - \frac{U_0}{2\pi\alpha'} \underbrace{\int_0^\infty dy}_{\int_0^1 dy + \int_1^\infty dy}$$

$$E = \frac{U_0}{2\pi\alpha'}\left[\int_1^\infty \left(\frac{y^2}{(y^4-1)^{1/2}} - 1\right) dy - 1\right]$$

$$E \sim -\frac{U_0}{2\pi\alpha'} \qquad U_0 = \frac{R^2}{L}$$

$$\text{So} \quad E \sim -\frac{R^2}{L} = \frac{\sqrt{g_{YM}^2 N}}{L}$$

As far as the L-dependent is concerend it is expected since it is conformal theory. $N^{1/2}$ behavior is not expect. From gauge theory we would expect to get $N$

This might be due to strong coupling limit. Since we used classical gravity the gauge theory results are expected to be in strong coupling limit.

In general one could have different string in the bulk which might correspond to some operator in gauge theory. So it would be intersting to know what is the corresponding operater given a classical string in in the bulk.

To proceed let's play a littel bit

with different string and see what kind of information one can get. To do this we will start from $AdS_5 \times S^5$ global coordinates

$$ds^2 = R^2 \Big[ -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \, d\Omega_3^2 + \cos^2\theta \, d\varphi^2 + d\theta^2 + \sin^2\theta \, d\tilde{\Omega}_3^2 \Big]$$

one would like to study a classical string in this background. As we have seen in Long $\lambda$ limit one may use saddel point approximation. Therefore we can read some information from the classical action which is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma\, \eta^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$$

for our background

$$S = \frac{R^2}{4\pi\alpha'} \int d\tau d\sigma\, \eta^{\alpha\beta} \Big[ -\cosh^2\rho\, \partial_\alpha t \partial_\beta t + \partial_\alpha \rho \partial_\beta \rho$$

$$+ \sinh^2\rho\, \partial_\alpha \Omega \partial_\beta \Omega + \cos^2\theta\, \partial_\alpha \varphi \partial_\beta \varphi + \partial_\alpha \theta \partial_\beta \theta$$

$$+ \sin^2\theta\, \partial_\alpha \tilde{\Omega} \partial_\beta \tilde{\Omega} \Big]$$

where

$$d\Omega_3^2 = d\psi_1^2 + \sin^2\psi_1\, d\psi_2^2 + \cos^2\psi_1\, d\psi_3^2$$

$$d\tilde{\Omega}_3^2 = d\beta_1^2 + \sin^2\beta_1\, d\beta_2^2 + \cos^2\beta_1\, d\beta_3^2$$

usin this parametrization one can have different Conserved charges corresponding to isometry of the metric.

$$t \rightarrow P_t \equiv \partial_t \longrightarrow E$$

$$\left.\begin{aligned} \psi_2 &\rightarrow P_{\psi_2} \equiv \partial_{\psi_2} \longrightarrow S_1 \\ \psi_2 &\rightarrow \qquad\qquad\quad\; \rightarrow S_2 \end{aligned}\right\} \text{ angular momenta in } S^3 \subset AdS_5$$

$$\left.\begin{aligned} \varphi &\rightarrow \qquad\qquad \rightarrow J_1 \\ \beta_2 &\rightarrow \qquad\qquad \rightarrow J_2 \\ \beta_3 &\rightarrow \qquad\qquad \rightarrow J_3 \end{aligned}\right\} \text{ angular momenta in } S^5$$

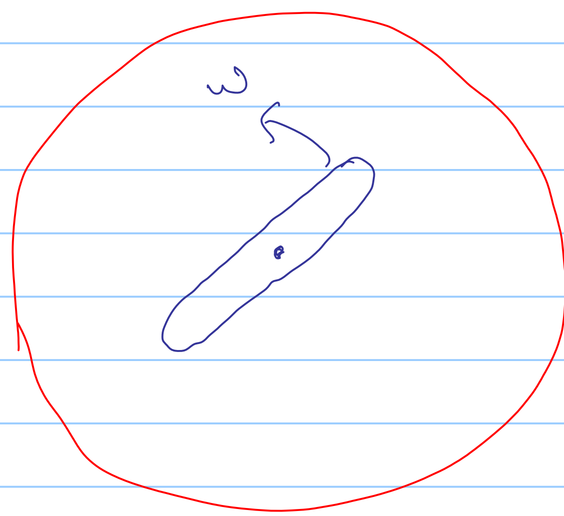
gravity $\qquad\qquad\qquad\qquad$ gauge theory

$$\left| E, S_1, S_2, J_1, J_2, J_3 \right\rangle \equiv \mathcal{O}_{\Delta, S_1, S_2, J_1, J_2, J_3}$$

For simplicity suppose only $S_1$ and $J_1$ are non-zero. So we are looking for a closed string which is extended along "$\rho$" and is rotating along $\psi_3$ and $\varphi$.

of course you might get diffrent string (closed or open) as well. For example consider following cases. In our notion a circle represents $AdS_5$.

1) folded rotating string

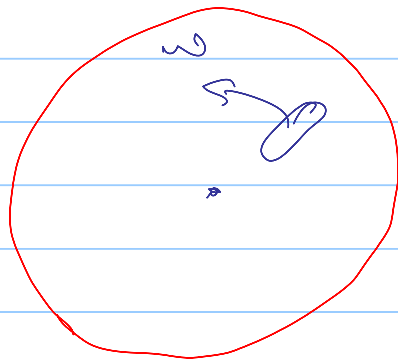



It may also have rotation in $S^5$ part.

$t = \kappa\tau$ , $\varphi_3 = \omega\tau$ $\varphi = \nu$

$\rho = \rho(\sigma) = \rho(\sigma + 2\pi)$

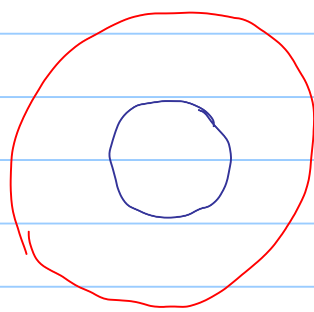
2) Spining string




$t = \kappa\tau$, $\varphi_3 = \omega\tau$

$\varphi = \nu\tau$ $\quad \rho = \rho(\sigma)$

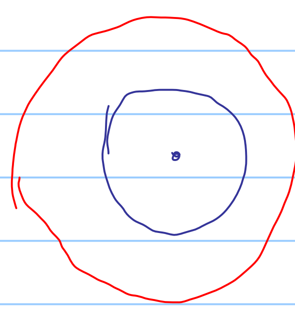
3) circular rotating string



$\rho = \rho_0$ $\quad \psi = n\sigma$

$t = \kappa\tau$ $\quad \varphi = \nu\tau$

4) circular pulsating string



$\rho = \rho(\tau)$ $\quad \psi = n\sigma$

$t = \kappa\tau$ $\quad \varphi = \nu\tau$

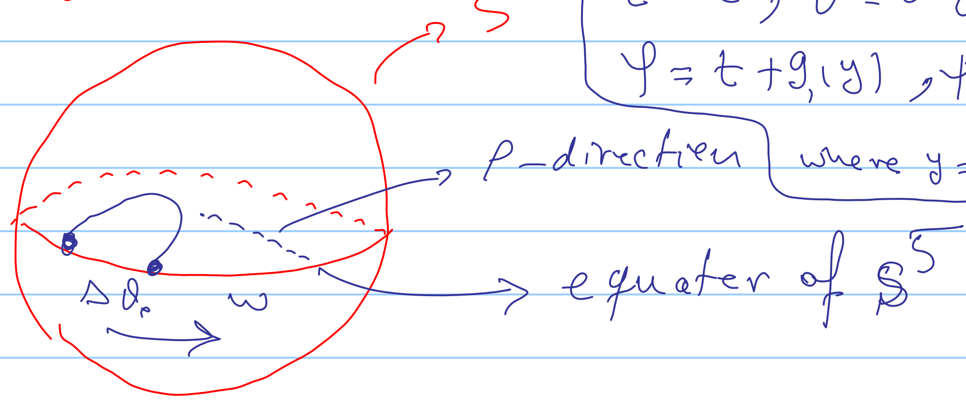
of course one may consider multi-spin

$\phi_2 = \omega_2 \tau \qquad \phi_3 = \omega_3 \tau$

or multi-J

$\varphi = \nu_1 \tau \; , \; \beta_2 = \nu_2 \tau \; , \; \beta_3 = \nu_3 \tau$

5) Magnon (open string)




$t = \tau \; , \; \theta = \theta(y)$

$\varphi = t + g_1(y) \; , \; \phi_2 = t + g_2(y)$

where $y = \tau - v\sigma$

We can simply use the classical action to find $E$ as a function of $S_1$, $S_2$, $J_2$, $S_2$, $J_3$ in general. To see the procedure we will give the detail of a classical folded rotating string. To do this consider the following classical configuration

$$t = \kappa\tau \quad , \psi_3 = \omega\tau , \varphi = \nu\tau ; \rho = \rho(\sigma) = \rho(\sigma + 2\pi)$$

others $= 0$ $\quad \kappa, \omega, \nu = \text{const.}$

To get a solution it needs to satisfy the Virasoro Constraints. which one

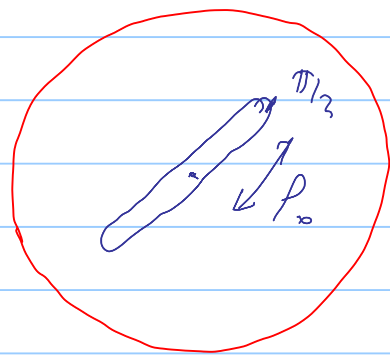
1) $G_{\mu\nu}\left(\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu\right) = 0$

2) $G_{\mu\nu}\, \partial_\tau X^\mu \partial_\sigma X^\nu = 0$

the "2" condition is trivial which from the first we get:

$$-\cosh^2\rho\, \kappa^2 + \sinh^2\rho\, \omega^2 + \nu^2 + \rho'^2 = 0$$

$$\Rightarrow \rho'^2 + \cosh^2\rho\,(\nu^2-\kappa^2) + \sinh^2\rho\,(\omega^2-\nu^2) = 0$$




$$\rho'\big|_{\rho_0} = 0$$

$$\cosh^2\rho_0\,(\nu^2-\kappa^2) + \sinh^2\rho_0\,(\omega^2-\nu^2) = 0$$

$$\coth^2\rho_0 = \frac{\omega^2-\nu^2}{\kappa^2-\nu^2} = 1+\eta \qquad \eta>0$$

since it is a closed string one has

$$2\pi = \int_0^{2\pi} d\sigma = 4\int_0^{\rho_0} \frac{d\rho}{\sqrt{(\kappa^2-\nu^2)\cosh^2\rho-(\omega^2-\nu^2)\sinh^2\rho}}$$

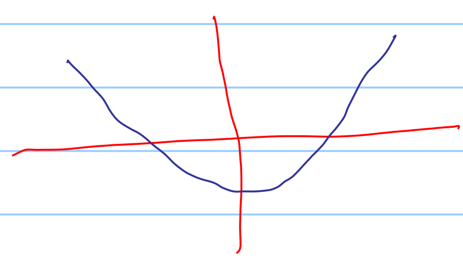
Consider the V.C. coming from the frist

condition as a one dimensional system therefore we look at the potential of the system to study different situations.

$$\rho'^2 + (-\varkappa^2 \operatorname{ch}^2\rho + \omega^2 \operatorname{sh}^2\rho + \nu^2) = 0$$

to get a solution one needs to look at zero energy configuration of the one-dimensional system when we get a periodic solution.

There are several possibilities

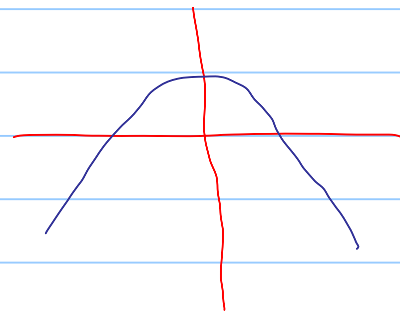
1) 

$\rho \to 0 \quad -\varkappa^2 + \nu^2$

$\rho \to \infty \quad -\varkappa^2 + \omega^2$

we have solution if $\kappa > \nu$ and $\omega > \kappa$.

2)

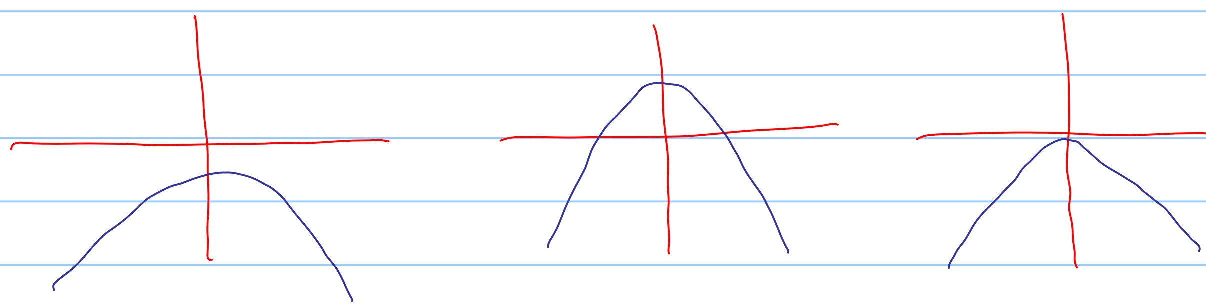


no solution in general

In this case for $\omega = 0$ we have

$\rho \to 0 \qquad -\kappa^2 + \nu^2$

$\rho \to \infty \qquad -\kappa^2$



$\kappa > \nu \qquad \nu > \kappa \qquad \kappa = \nu$

$\downarrow$

only acceptable solution

For $\omega=0$ and $\kappa=\nu$ we have

$$\coth^2\rho_0 = \frac{-\nu^2}{\kappa^2-\nu^2} \to \infty \Rightarrow \rho_0 \to 0$$

Therefore we get a closed string shrunk to a point at the center of $AdS_5$ and rotates along a circle of $S^5$ with speed of light.

(localized at $\rho=0$)

This is one of the intersting case which we will back to it later

For the moment consider the general case with $\omega\neq 0$ $\nu\neq 0$

In this case setting $\sqrt{\lambda}=\frac{R^2}{\alpha'}$

we get

$$E = P_t = \sqrt{\lambda}\,\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \operatorname{ch}^2\rho \equiv \sqrt{\lambda}\,\mathcal{E}$$

$$S = P_{\phi_3} = \sqrt{\lambda}\,\omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \operatorname{sh}^2\rho \equiv \sqrt{\lambda}\,\mathcal{S}$$

$$J = P_\varphi = \sqrt{\lambda}\,\nu \int_0^{2\pi} \frac{d\sigma}{2\pi} = \sqrt{\ }\,\nu$$

one can see

$$\mathcal{E} = \kappa + \frac{\kappa}{\omega}\,\mathcal{S}$$

From the above expression we find

$$\sqrt{\kappa^2-\nu^2} = \frac{1}{\sqrt{4}}\; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{4}\right)$$

$$\mathcal{E} = \frac{\kappa}{\sqrt{\kappa^2-\nu^2}}\; \frac{1}{\sqrt{4}}\; {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; -\frac{1}{4}\right)$$

$$S = \frac{\omega}{\sqrt{\varkappa^2 - \nu^2}} \frac{1}{2\eta\sqrt{\eta}} \, {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; -\frac{1}{\eta}\right)$$

We will assume that $\lambda$ is Long but $\omega, k, \nu$ are fixed (doing semi-classical)

The aim is to find $\mathcal{E}$ as a function of $\mathcal{J}$ and $S$. or $E(J, S)$. To do this one needs to eleminate $\varkappa, v, \omega, \eta$ from the above equations for $J$, $S$, and $E$. In general it is hard to do that but in some limit it is possible to do.

For example when the string is short $\eta \gg 1$ one finds

$$x^2 \sim v^2 + \frac{1}{\eta} \qquad \omega^2 \sim x^2 + 1 \sim v^2 + 1 + \frac{1}{\eta}$$

$$\frac{1}{\eta} \sim \frac{2\Delta}{\sqrt{1+v^2}} \ll 1$$

using $\mathcal{E} = x + \frac{x}{\omega}\Delta$ one gets

$$\mathcal{E} = \sqrt{v^2 + \frac{2\Delta}{\sqrt{1+v^2}}} + \sqrt{\frac{v^2 + \frac{2\Delta}{\sqrt{1+v^2}}}{1+v^2+\frac{2\Delta}{\sqrt{1+v^2}}}}\,\Delta$$

- for $v \ll 1 \qquad \Delta \ll 1$

$$\mathcal{E} \sim \sqrt{v^2 + 2\Delta} \Rightarrow E^2 = J^2 + 2\sqrt{\lambda}\, S$$

- for $v^2 \ll \Delta \qquad \mathcal{E} = \sqrt{2\Delta} + \frac{v^2}{2\sqrt{2\Delta}}$

Regge trajectory in flat space.

- For $\nu \gg 1$ at $\nu \gg 2\Delta$

$$\mathcal{E} = \nu + \Delta + \frac{\lambda s}{2\nu^2}$$

$$\Rightarrow E = J + S + \frac{\lambda S}{2J^2}$$

In long string limit $\eta \ll 1$

and therefore we get

$$\kappa^2 \sim \nu^2 + \frac{1}{\pi^2} \ln^2 \frac{1}{\eta}$$

$$\omega^2 \sim \nu^2 + \frac{1}{\pi^2} (1+\eta) \ln^2 \frac{1}{\eta}$$

$$\Delta = \frac{2\omega}{\eta \ln 1/\eta}$$

- For small $\nu$

$$\mathcal{E} = \Delta + \frac{1}{\pi} \ln \mathcal{S} + \frac{\pi \nu^2}{2 \ln \mathcal{S}}$$

$$\Rightarrow \quad E \sim S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}}$$

As conclution one may say that in each case there is an operator with dimension $\Delta = E$ and R-chagre $J$ and spin $S$ such that in each case we get a relation between $\Delta$ and $J$ and $S$.

Note that the solutions we consider do not correspond to a BPS state since there is no a simple $E = J$ relation. In each case this relation gets correction.

Let's go to a more interesting case namely $w=0$ where we get solution only the closed string is localized at $\rho_0=0$ and $\kappa=\nu$.

localized string with speed of light.

Remember that we assume $\lambda$ is large while $w, \nu, k$ are kept fixed.

$$E = \sqrt{\lambda}\,\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2\rho \sim \sqrt{\lambda}\,\kappa$$

$$J = \sqrt{\lambda}\,\nu \int_0^{2\pi} \frac{d\sigma}{2\pi} \sim \sqrt{\lambda}\,\nu$$

So we have

$$\frac{E}{\sqrt{\lambda}} \sim \kappa \qquad\qquad \frac{J}{\sqrt{\lambda}} \sim \nu$$

Remember $\lambda = g_{YM}^2 N$ so since $x$ and $\nu$ are fixed this means $E$ and $J$ are lang and if fact should go

$$E \sim N^{1/2} \qquad J \sim N^{1/2}$$

but $x = \nu$ so

$$E - J = 0$$

Taking higher order correction one get

$$E - J = \mathcal{O}\left(\frac{1}{J^2}\right)$$

Therefore they are not BPS states.

So in general we have

$E \sim N^{1/2}$, $J \sim N^{1/2}$ $\qquad$ $E - J =$ finite

$N \longrightarrow \infty$

on the other hand due to AdS/CFT correspondence one should have operators with dimension $\Delta$ and R-Charge $J$ such that both $\Delta$ and $J$ are long

$$\Delta, J \longrightarrow N^{1/2}$$

while $\Delta - J =$ finite.

These operators are called

BMN operators.

one may also study small fluctuations around this classical solution to see what kind of geometry the fluctuations see. In fact it will be a plane wave in which the string can be exactly solved on it. So we can exactly find

$$E - J = \sum_{n=-\infty}^{\infty} \sqrt{1 + \frac{\lambda n^2}{J^2}}\, N_n$$

occupation number of $\gamma$ - direction.

The classical solution was

$t = \kappa \tau$ $\quad \varphi = \kappa \tau$ $\quad \rho = 0$ $\quad$ others $= 0$

setting $\quad 2\rho = \ln \dfrac{1+\xi}{1-\xi}$

we consider the following fluctations

$$t = \nu\tau + \frac{1}{\lambda^{1/4}}\tilde{t}, \quad \xi_k = \frac{1}{\lambda^{1/4}}\tilde{\xi}_k$$

$$\varphi = \nu\tau + \frac{1}{\lambda^{1/4}}\tilde{\varphi} \qquad \text{others} \equiv \psi_k = \frac{1}{\lambda^{1/4}}\tilde{\psi}_k$$

so the action in second order reads

$$I^2 = -\frac{1}{4\pi}\int d^2\sigma \Big[ -\partial_\alpha \tilde{t}\, \partial^\alpha \tilde{t} + \partial_\alpha \tilde{\varphi}\, \partial^\alpha \tilde{\varphi}$$

$$+ \nu^2 (\tilde{\xi}^2 + \tilde{\psi}^2) + \partial_\alpha \tilde{\xi}_k\, \partial^\alpha \tilde{\xi}^k$$

$$+ \partial_\alpha \tilde{\psi}_k\, \partial \tilde{\psi}^k \Big]$$

define $x^\pm = \varphi \pm t$

$\Rightarrow$ $x^+ = 2\nu\tau \qquad x^- = 0 \qquad \xi_k = \psi_k = 0$

one gets

$$I = -\frac{1}{4\pi}\int d^2\xi \Big[ \partial_a X^+ \partial^a X^- - \frac{1}{4}(\xi_k^2 + \tilde{\psi}_k^2)$$
$$\partial_a X^+ \partial^a X^+ + (\partial \tilde{\xi}_k)^2 + (\partial \tilde{\psi}_k)^2 \Big]$$

This is string action in pp-wave background.

From Virasoro Const. one has:

$$1)\ 2\lambda^{1/4}\nu\,\partial_0 x^- - \nu^2(\tilde{\xi}_k^2 + \tilde{\psi}_k^2)$$
$$+ \partial_0 \tilde{\xi}_k\, \partial_0 \tilde{\xi} + \partial_1 \tilde{\xi}_k\, \partial_1 \tilde{\xi}_k$$
$$+ \partial_0 \tilde{\psi}_k\, \partial_0 \tilde{\psi}_k + \partial_1 \tilde{\psi}_k\, \partial_1 \tilde{\psi}_k$$
$$+ \partial_0 \tilde{x}^+ \partial_0 \tilde{x}^+ + \partial_1 \tilde{x}^+ \partial_1 \tilde{x}^+ + O\left(\frac{1}{\lambda}\right) = 0$$

$$2)\; 2\lambda^{1/4}\nu\,\partial_1 \tilde{u}^- + \partial_0 \tilde{\xi}_n\,\partial_1 \tilde{\xi}_n + \partial_0 \tilde{\varphi}\,\partial_1 \tilde{\varphi}$$

$$+ \frac{1}{2}\partial_0 \tilde{u}^+\,\partial_1 \tilde{u}^- + \frac{1}{2}\partial_0 \tilde{u}^-\,\partial_1 \tilde{u}^+$$

$$+ \mathcal{O}\left(\frac{1}{\lambda}\right) = 0$$

on the other hand we have

$$E = \int_0^{2\pi} \frac{d\sigma}{2\pi}\left(\sqrt{\lambda}\,\nu + \lambda^{1/4}\partial_0 \tilde{t} + \nu\,\tilde{\xi}_k^2 + \cdots\right)$$

$$J = \int_0^{2\pi} \frac{d\sigma}{2\pi}\left(\sqrt{\lambda}\,\nu + \lambda^{1/4}\partial_0 \tilde{\varphi} - \nu\,\tilde{\varphi}_k^2 + \cdots\right)$$

$$\Rightarrow E - J = \int_0^{2\pi} \frac{d\sigma}{2\pi}\left[\lambda^{1/4}\,\overset{\partial_0 \tilde{u}^-}{\partial_0(\tilde{t} - \tilde{\varphi})} + \nu\left(\tilde{\xi}_k^2 + \tilde{\varphi}_k^2\right)\right]$$

Using the constraints one gets.

$E - J =$ Transvers Hamiltonian $= -p^-$

$$= \frac{1}{\nu} \sum_{n=-\infty}^{\infty} \sqrt{n^2 + \nu^2}\, N_n + \mathcal{O}\left(\frac{1}{J}\right)$$

$$E - J = \sum_{n=\infty}^{\infty} \sqrt{1 + \frac{\lambda n^2}{J^2}}\, N_n$$

The corresponding operators are those with $\Delta, J \to N^{1/2}$ while

$$(\Delta - J)_n = \sqrt{1 + \frac{\lambda n^2}{J^2}}$$

These are not BPS operators but their corrections are under control. In fact both $N$ and $J$ are long but $\frac{N}{J^2}$ which is going to be the effective coupling for these operators could be small.

The pp-wave background can also be obtained from $AdS_5 \times S^5$ by taking Penrose limit.

Taking Penrose limit in gauge theory corresponds to taking BMN limit and therefore string theory on pp-wave is dual to BMN sector of $\mathcal{N}=4$ SYM theory

$$\begin{tikzcd}
\mathcal{N}=4 \text{ SYM} \arrow[r, equal, "\text{dual}"] \arrow[d, "\text{BMN limit}"'] & AdS_5 \times S^5 \arrow[d, "\text{penrose limit}"] \\
\begin{array}{c} \Delta - J = \text{finite} \\ \Delta, J \sim N^{1/2} \end{array} \arrow[r, equal, "\text{dual}"] & \text{pp-wave background}
\end{tikzcd}$$

To get pp-wave from $AdS_5 \times S^5$ by penrose limit one start from global coordinates

$$ds^2 = R^2\Big[-\mathrm{ch}^2\rho\, dt^2 + d\rho^2 + \mathrm{sh}^2\rho\, d\Omega_3^2 + \cos^2\theta\, d\varphi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_3^2\Big]$$

define

$\varphi = x^+ + \frac{x^-}{R^2} \qquad t = x^+ - \frac{x^-}{R^2}$

$\rho = \frac{y}{R} \qquad \theta = \frac{z}{R}$

at long R limit we get pp-wave solution

$$ds^2 = -4\, dx^+ dx^- + (y^2 + z^2)\, dx^{+2} + d\vec{y}^{\,2} + d\vec{z}^{\,2}$$

$$F_{+1234} = F_{+5678} \sim 1$$

one would also like to write the corresponding operators in gauge theory

Consider $\phi_i$ $i = 1, \dots, 6$ to be The scalars of $N=4$ multiplet.

setting $Z_1 = \phi_1 + i\phi_2 \longrightarrow J_1$

$$Z_2 = \phi_3 + i\phi_4 \longrightarrow J_2$$

$$Z_3 = \phi_5 + i\phi_6 \longrightarrow J_3$$

$(J_1, J_2, J_3) \longrightarrow U(1)^3 \subset SO(6)$

R-charges

$Z_1 \equiv Z \quad J_1 \equiv J$

$$\mathcal{O}_1 = \mathrm{Tr}(Z^J) \Longrightarrow \Delta = J$$

$$\mathcal{O}_2 = \mathrm{Tr}(\phi_i Z^J) \qquad \Delta - J = 1$$

$i = 3,4,5,6$

$$\mathcal{O}_3 = \sum_{l=1}^{J} \mathrm{Tr}\left[\phi_i Z^l \phi_j Z^{J-l}\right] \qquad \Delta - J = 2$$

one may also put some phase for each position of $l$. Therefore for example we have

$$\mathcal{O} = \sum \mathrm{Tr}\left(\phi_3 Z^l \phi_4 Z^{J-l}\right) e^{\frac{2\pi i n l}{J}}$$

which corresponds to the following string state in string theory on pp-wave background

$$a_n^{\dagger(3)} \, a_{-n}^{\dagger(4)} \, |0, P_+\rangle$$

where $|0, P_+\rangle$ is the vacuum corresponding to $\mathrm{Tr}(Z^J)$

(for more detail see hep-th/0202021)