The purpose of these lectures is to show how one handles an interacting system in a mean field way. The same approximation will be derived in various ways illuminating different aspects of the approximation. All will be based on the Ising model. Certain generalizations will also be considered.

MEAN FIELD THEORIES

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I. MODEL AND DEFINITIONS

The model system is the Ising model described by the Hamiltonian

$$H = -J \sum_{<ij>} s_i s_j - h \sum_i s_i \tag{1.1}$$

where $s_i = \pm 1$ is a spin at the ith site of the lattice, and $< ij >$ represent nearest neighbour pairs. For simplicity we consider only ferromagnetic interaction ($J > 0$). The range of interaction will be relaxed in certain cases.

The quantities of interest are (i) magnetization $m = \langle s \rangle = \langle \sum s_i \rangle / N$, where N is the total number of spins, (ii) susceptibility $\chi = \partial m / \partial h \mid_{h=0}$, the response of the system to a small external magnetic field, (iii) specific heat $c_h = \partial u / \partial T$, where u is the average internal energy per spin, etc.

Convention: Total thermodynamic quantities to be denoted by capital letters and densities(i.e., per particle) by corresponding small letters.

II. AGE-OLD MEAN FIELD THEORY

Start with the Hamiltonian, Eq 1.1, and focus on one particular spin, say s_i. There are two fields acting on it, (i) the external one, h, and (ii) the field by the neighbouring spins. We can write the hamiltonian for the ith spin as

$$H_i = -s_i (J \sum s_j + h) \tag{2.1}$$

with the summation over the interacting spins only. The field due to the interacting spins is a fluctuating object - and is the source of difficulty. We replace this by the average field produced by them, i.e., replace s_j by $\langle s \rangle$, and forget about fluctuations. The name of the game is to decouple the spins to reduce the interacting system to a noninteracting one (the whole system is a uniform background, as in Hartree, Hartree-Fock, Thomas-Fermi etc).

A. Single spin case

Take $H = -hs$ for a single spin $s = \pm 1$. The partition function is $Z = 2\cosh(h/kT)$, and $\langle s \rangle = \tanh(h/kT)$.

`Problem` **II.1** *Derive these.*

Evaluate susceptibility for $h = 0$ and $h \neq 0$. Note the drastic difference in behaviour as $T \to 0$. Why? If you don't see why, try $H = -\Delta\ s - hs$, with $\Delta \neq 0$ fixed.

B. Effective field case

The mean field hamiltonian for any spin is now

$$H_i = -(qJm + h)s_i \tag{2.2}$$

where q is the coordination number of the lattice (i.e. the number of nearest neighbors). Note that all detailed structures of the lattice are lost - q alone cannot describe a lattice!

Now, selfconsistency requires that the average spin of the ith particle $\langle s_i \rangle$ evaluated with the meanfield hamiltonian of Eq. 2.2 be the same as $\langle s \rangle$ used there - after all, the spins are all equivalent.

Using the results of the single spin case, we have

$$\boxed{m = \tanh(qJm\beta + h\beta)} \tag{2.3}$$

where $\beta = (kT)^{-1}$. This is the famous meanfield equation.

C. Comments

(1) What is the approximation: s_i's are random variables, with more or less well behaved statistics. The effective field involves a sum of these random variables, and so we expect the sum to approach its average (Central Limit Theorem), if a large number of spins are involved. In other words q should be large ($\to \infty$). MF approximation is exact in infinite dimensions. what is CLT?

(2) Is tanh sacred?: NO! We got tanh because of Ising(i.e. two component) spins. For arbitrary spins, or with arbitrary probability distribution, we may replace tanh by an appropriate function f with $f(0) = 0$. The analog of Eq. 2.3 is $m = f(qJm\beta + h\beta)$. The "universal" features of the MF equation will turn out to be independent of the detailed nature of f, but certain features do matter. Incidentally, the most crucial property required is the nonlinearity of f. Guess why?

(3) Note that MF is just a feed back mechanism - and is of much wider applicability than stat mech. For example: Clausius-Mosotti, Lorentz-Lorenz equations in electrodynamics etc. Think of amplifiers/oscillators in electronics.

III. VARIATIONAL PRINCIPLE

A general result: We want to estimate the free energy of a system described by a Hamiltonian H, $F = -kT \ln Z$

where $Z = \text{Tr } e^{-\beta H}$, Tr being the trace taken over the states of H. We are writing "Tr" for convenience. For a classical case, it would mean a sum over all configurations. Now, choose another Hamiltonian H_0, with free energy $F_0 = -kT \ln Z_0$, where $Z_0 = \text{Tr} \exp(-\beta H_0)$. Using the convexity relation $\langle \exp(x) \rangle \geq \exp\langle x \rangle$, one can show that

convexity?

$$F \leq F_0 + \langle H - H_0 \rangle_0 \tag{3.1}$$

where the average on the right hand side is done with the Boltzmann weight $\exp(-\beta H_0)$.

`Problem` **III.1** *Derive this. Guess why convexity is important.*

As in any variational calculation, one generally chooses H_0 with some adjustable parameter so that the right hand side can be minimized to get the best estimate.

Let us now use this variational principle. Since, a non-interacting hamiltonian is the one we can solve best, we choose

$$H_0 = -\lambda \sum s_i. \tag{3.2}$$

where λ is the effective field we like to determine. Calculating averages with this hamiltonian is simple. The final answer is

$$F \leq -NkT \ln(2 \cosh \beta\lambda) - \tfrac{1}{2} NqJm^2 - Nhm + N\lambda m \tag{3.3}$$

with $m = \tanh(\beta\lambda)$. We now minimize the RHS of Eq. 3.3 with respect to λ. Use $Nm = -\partial F_0/\partial\lambda$ to get

$$\frac{\partial}{\partial\lambda}\text{RHS} = N\frac{\partial m}{\partial\lambda}\ (qJm + h - \lambda) = 0. \tag{3.4}$$

This identifies $\lambda = h + qJm$, as in Eq. 2.2. As before, one ends up with the MF equation of Eq. 2.3.

`Problem` **III.2** *Derive Eqs. 3.3, 3.4.*

IV. INFINITE RANGE MODEL

We have already said that the MF approximation works best if the number of neighbours is large. What we did was to replace the fluctuating spins by their averages, $\langle s \rangle = \sum_j s_j/(N-1)$, excluding the ith spin from the sum. We now make the following replacement:

$$Js_i \sum s_j \to qJ\langle s \rangle s_i \to \frac{qJ}{N-1} \sum_{i \neq j} s_i s_j \tag{4.1}$$

where the first sum involves only nearest neighbours while the last one involves all. The Ising Hamiltonian of Eq. 1.1 can then be replaced by

$$H = -\frac{J}{N} \sum_{(ij)} s_i s_j. \tag{4.2}$$

The summation now extends over all possible pairs of the system. This is the infinite range (but infinitely weak(!) as N goes to ∞) model. The folklore is that all weak long range interactions lead to a mean field type description.

Question: Is $1/N$ necessary? Yes! Because, we have to ensure a proper thermodynamic limit ($N \to \infty$). Look at the ground state. For ferromagnetic interactions ($J > 0$), the ground state has all parallel spins. There are $N(N-1)/2$ pairs. So the energy is proportional to $N(N-1)$. However, we want (or rather demand) that the energy be proportional to N (extensivity of thermodynamic quantities). This is ensured by the $1/N$ in the coupling constant.

A. Solution I: Maximum term method

Since all pairs are interacting, we can write the interaction term, upto a trivial constant, as

why $\frac{1}{2}$

$$H = -\frac{J}{2N}(\sum s_i)^2 = -\frac{J}{2N} M^2 \tag{4.3}$$

where, as before, M is the total magnetization (no averaging yet). Since the energy of a configuration is completely determined by M (NOT for Eq. 1.1), doing the configurational sum is easy. We however need the degeneracy of a state of magnetization M. Suppose N_+ spins are up. Then $N_+ = (M+N)/2$. The degeneracy is then given by $N!/(N_+!(N-N_+)!)$.

The partition function is given by

$$Z = \sum_{M=-N}^{M=+N} \frac{N!}{N_+!(N-N_+)!} \exp(-\beta J M^2/2N). \tag{4.4}$$

An exact evaluation, for finite N is not possible. For large N, we use the maximum term method where it is assumed that the partition function is dominated by the maximum term of the summand. To determine the maximum term, use the Stirling approximation ($\ln N! = N(\ln N - 1)$, good even for small N), and extremize

$$\frac{\beta J}{2N} M^2 - [\ln N! - \ln \tfrac{1}{2}(M+N)! - \ln \tfrac{1}{2}(N-M)!] \tag{4.5}$$

treating M as a continuous variable. Do it and you will get back the MF equation Eq. 2.3.

`Problem` **IV.1** *Do it. For practice, also keep the magnetic field term.*

1. Comments

(1) Have we made any approximation in keeping only the maximum term? Of course not, but why not? Look, we are interested in the free energy per particle, not the partition function *per se*. So let's consider a case $Z = \sum_{m=1}^{m=\mathcal{N}} z_m$ for N particles. $\mathcal{N}$ is a measure of the number of distinct states. Suppose z_m is maximum for $m = max$. Note that z_m is not just the Boltzmann factor, it includes "entropy", AND it is strictly nonnegative. We have the following trivial inequality,

$$\mathcal{N} z_{max} \geq Z \geq z_{max}. \tag{4.6}$$

which implies,

$$\frac{\ln \mathcal{N}}{N} + \frac{\ln z_{max}}{N} \geq \frac{1}{N} \ln Z \geq \frac{1}{N} \ln z_{max}. \tag{4.7}$$

Now if the growth of $\mathcal{N}$ is some power of N, and NOT EXPONENTIAL, then, in the limit $N \to \infty$, the two bounds above become equal. Hence, in the thermodynamic limit (i.e. $N \to \infty$) the free energy per particle is given by the free energy of the maximum term. So, Mean field is exact only in the thermodynamic limit, not for finite N.

Question: Why doesn't it work for the nearest neighbour problem?

(2) Why is it solvable: Let's think of it in a different language, namely the lattice gas. The two states for each site can be thought of as occupied ($s = +1$) or vacant ($s = -1$) . Then the interaction depends only if particles are nearest neighbours (NN). One therefore needs to know the short distance behaviour of the spins, i.e., for a given configuration how many form NN pairs. This is a complicated issue. At the simplest level one would argue that the probability of getting two particles as NN is just (density)^2, density being the probability. What is needed is the conditional probability - this is a product of two unconditional ones only if there is no correlation. This turns out to be the case for the infinite range model.

(3) The Stirling approximation gives the log of the degeneracy factor as $N[\rho \ln \rho + (1-\rho)\ln(1-\rho)]$ where $\rho = N_+/N$ is the particle density in the lattice gas picture. $1-\rho$ is the density of vacant sites or of particles of a second kind. This is the approximate form of entropy that's used everywhere from binary alloys (Bragg-Williams) to polymer solutions (Flory/deGennes). It is boring but useful!

(4) The "mod" approach is this: instead of approximating a realistic Hamiltonian, start with the crazy infinite range model and solve it exactly - the answer is the same in any case. (paraphrasing Mermin)

B. Solution II: Saddle point

We now solve the infinite range model in a different way. This is a rather standard (sorry, sophisticated) procedure for any field theoretic approach. Use the Gaussian identity

$$\exp(as^2) = \text{const} \int_{-\infty}^{\infty} dx \, \exp\left(-\frac{x^2}{a} + 2sx\right). \tag{4.8}$$

`Problem` **IV.2** *Find const.*

The Boltzmann factor for Eq 4.3 with a magnetic field can be written as

$$\exp[\beta h \sum s_i + \frac{\beta J}{2N}(\sum s_i)^2] =$$
$$\text{const} \int_{-\infty}^{\infty} dx \, \exp\left(-\frac{2N}{\beta J}x^2 + (2x+\beta h)\sum s_i\right). \tag{4.9}$$

We change the integration variable to $\lambda = 2x/(\beta J)$, and ignore the constant to write guess why?

$$Z =$$
$$\int_{-\infty}^{\infty} d\lambda \, e^{-N\beta J\lambda^2/2} \prod_i \left(\sum_{s_i=\pm 1} e^{(\beta h+\beta J\lambda)s_i}\right) =$$
$$\int_{-\infty}^{\infty} d\lambda \, e^{-N\beta J\lambda^2/2}[2\cosh(\beta h + \beta J\lambda)]^N. \tag{4.10}$$

It is convenient to define a function (a VERY important one) $A(\lambda)$ as

$$Z = \int_{-\infty}^{\infty} d\lambda \, \exp[-N\beta A(\lambda)], \tag{4.11}$$

where $A(\lambda) = \frac{1}{2}J\lambda^2 - kT\ln[2\cosh(\beta h + \beta J\lambda)]$. For $N \to \infty$, the integral is just the right one for a saddle point evaluation - in fact the saddle point approximation will be exact in that limit. (The reason is the same as the maximum term theorem).

The partition function is given by $Z = \exp[-\beta N A(\lambda_0)]$, where λ_0 satisfies the saddle point equation:

$$\frac{\partial A(\lambda)}{\partial \lambda} = 0 \Longrightarrow \lambda = \tanh(\beta J\lambda + \beta h). \tag{4.12}$$

The free energy per particle is, of course, given by $f(T,h) = A(\lambda_0)$. The magnetization follows from the usual derivative rule

$$m = -\frac{\partial f}{\partial h} = -\frac{\partial A}{\partial \lambda_0}\frac{\partial \lambda_0}{\partial h} - \frac{\partial A}{\partial h}$$
$$= -\frac{\partial A}{\partial h} = \tanh(\beta J\lambda_0 + \beta h) = \lambda_0. \tag{4.13}$$

where the last one follows from the saddle point condition. Once we identify λ_0 as m, we are done.

1. Comments

(1) What we have done is really a Hubbard Stratanovich transformation in a trivial way!

(2) It is difficult to go beyond the saddle point approximation.

V. SOLUTION OF MEAN FIELD EQUATION

The mean field equation, Eq. 2.3, can be solved graphically. Since it is easy to redraw a straight line, we recast the MF equation as

$$\frac{kT}{qJ}x = \tanh x, \text{or, ingeneral,} \quad \frac{kT}{qJ}x = f(x), \qquad (5.1)$$

in zero external field ($x = \beta mqJ$). As mentioned already, what is important is the nonlinearity of the function on

Why? RHS, and we take $f(x)$ to be concave. The straight line $y = (kT/qJ)x$ obviously intersects $f(x)$ at the origin, so that $x = 0$ is a solution. There can be another solution provided kT/qJ is smaller than the slope of $f(x)$ at the origin. For the Ising case, $\tanh x$ has a slope 1, so that a nonzero solution is possible for

$$\frac{kT}{qJ} \leq 1. \qquad (5.2)$$

This identifies the critical temperature as $kT_c = qJ$.

For any other function $f(x)$, the corresponding condition is

$$kT \leq kT_c = qJf'(0) \qquad (5.3)$$

where prime denotes derivative. Remember that $f(x)$ gives the dependence of a single spin on the external field in absence of any interaction. As a result, the derivative is the zero field susceptibility of the free spin. This can also be written as the fluctuation of the spin $\langle s^2 \rangle$. For the Ising case, this average is identically 1. In general, $kT_c = qJ\langle s^2\rangle_0$.

`Problem` **V.1** *Derive the connection between susceptibility and $\langle s^2 \rangle$.*

A. Comments

(1) T_c depends only on the number of nearest neighbours and no other details of the lattice really matter. This is definitely wrong. Most serious is that the 1-d Ising model does not have any nonzero T_c. T_c is zero.

Why But MF gives a nonzero value.

(2) It is possible to get better estimate of T_c by concentrating on a cluster of neighbouring spins, rather than one. However, other properties discussed later on don't change much.

(3) It is possible to prove, quite generally that the actual critical point is lower than the mean field critical temperature. In other words, for the Ising model, $kT_c(\text{actual}) \leq qJ$. Try to prove this.

VI. MAGNETIZATION

We find that there is a critical point below which ($T < T_c$) there can be nonzero magnetization, even in the absence of any external field. This is called spontaneous symmetry breaking. The Hamiltonian has an up-down symmetry but the ground state doesn't. However, there is no unique ground state - for the Ising case it is two fold degenerate - that saves the problem! At any given temperature, for any finite system, one also expects, as one sums over all the configurations, to see no magnetization. It is actually the thermodynamic limit that causes the ground state behaviour to continue for some finite T, energy dominating over entropy. This spontaneous magnetization can be taken as a measure of order in the system and is called the order parameter.

What happens to the order parameter as one approaches T_c?

Take $T - T_c$ to be small, expand $\tanh x = x - x^3/3$, and solve the quadratic equation. You get $m \sim (T_c - T)^{1/2}$. Defining an exponent β by

$$m \sim \mid t \mid^{\beta}, \quad t \equiv (T - T_c)/T_c, \qquad (6.1)$$

we find the MF result $\beta = \frac{1}{2}$.

A. Comments

(1) This result is NOT dependent on tanh. By symmetry, we expect $f(x)$ to be an odd function. Thus, for small x, $f(x) \sim a_1 x + a_3 x^3 + ...$, so it's again a quadratic equation, yielding $\beta = \frac{1}{2}$.

(2) $m = 0$ is always a solution. For low temperatures, the lower free energy solution corresponds to $m \neq 0$. Show this.

(2) It is true most often that the mean field value of $m(T, h)$ is an upper bound for the actual magnetization at $h \geq 0$.

`Problem` **VI.1** *Discuss what happens to m as $T \to 0$.*

`Problem` **VI.2** *Can something exotic happen in Eq. 5.1, like say $f(x) = a_1 x + a_3 x^3 - a_5 x^5 \quad (a_i > 0)$? If it does, what can you say about magnetization? Take a simpler case $f(x) = a_1 x - a_2 x^2 + a_3 x^3$. Where is the transition? See what you can say about the nature of the transition.*

`Problem` **VI.3** *Keep T fixed and study the variation of magnetization with h, may be for small h. Discuss the behaviour for $T > T_c$ and $T < T_c$.*

VII. SUSCEPTIBILITY

A. Exponent γ

To study susceptibility, we introduce a small external field h and consider T close to T_c. Keeping leading order terms, we have from Eq. 2.3

$$m = \frac{T_c}{T} m + \frac{h}{kT} \quad \Longrightarrow \chi \equiv \frac{\partial m}{\partial h} \sim \frac{1}{T - T_c}. \tag{7.1}$$

This is the Curie-Weiss law. Defining the exponent γ as $\chi \sim (T - T_c)^{-\gamma}$, we have $\gamma = 1$. Also note that χ^{-1} would be negative below T_c. This catastrophy is avoided by the phase transition to a ferromagnetic state.

Note, again, that $\gamma = 1$ is really a consequence of $f(x) \sim x$ for small x. Such a linear response for a free spin is expected unless a spin couples strangely with the field.

Frankly speaking, the Curie Weiss law follows as an immediate consequence of the linear response. Take χ^0 as the susceptibility of a free spin. This means $m = \chi^0\ \mathsf{h}$ in a field h. This field h, in our case, is the actual field seen by a spin; it is obviously $h + qJm$. Ergo, $m = \chi^0(h + qJm)$, which means $\chi = \chi^0/(1 - qJ\chi^0)$. For an Ising type system, $\chi^0 \sim 1/T$. Hence the Curie Weiss law. Note this is how the amplification factor of a feedback amplifier is calculated!

B. Exponent δ

Let us now fix the temperature at T_c. We have to go to the cubic term in the expansion of tanh. To leading order, we have

$$h \sim m^3 \tag{7.2}$$

for $m, h \to 0$. Define the exponent δ via $h \sim m^\delta$. We get $\delta = 3$.

C. Comments

(1) The susceptibility diverges at T_c. A divergent susceptibility means that the system has a large response to a small change in the external parameter. The magnetization is nonzero as one crosses T_c from the high temperature side. This tendency near T_c for ordering is reflected in the huge response to a small ordering field. Such divergences are hallmark of most critical points.

(2) m and h are not related linearly at T_c. Linear relation would never give a divergent χ! That's why δ is important.

(3) The critical temperature has already been defined as the temperature at which spontaneous magnetization (symmetry breaking - if you want to be sophisticated) occurs. Let's call it $T_c^{\rm SB}$. Another way, easier experimentally, is the divergence of the susceptibility. If we reduce the temperature from a high value, at what point does χ diverge ? Let's call it $T_c^{\rm HT}$ (HT for high temperature). What guarantees that they are same? Sorry, there is none!

(4) It is possible to prove that actual $\gamma \geq 1$. Try to do it.

VIII. SPECIFIC HEAT

You may wonder why the greek symbol α is missing. That's what we do now.

In zero field, the free energy per particle is $f = -kT \ln[2\cosh(\beta qJm)]$. For $T > T_c$, $m = 0$, and so $f = -kT \ln 2$. The specific heat is zero! For $T < T_c$, $m \sim (T_c - T)^{1/2}$. For small x, $\cosh x \sim 1 + x^2/2$.... A straight forward expansion then gives a power series in $T_c - T$, showing that for $T \to T_c-$, specific heat remains finite but nonzero. In short, sp heat has a discontinuity at T_c. A divergent specific heat is described by an exponent $c \sim \mid T - T_c \mid^{-\alpha}$. In the mean field theory $\alpha = 0$.

A. Comments

(1) Free energy for $T > T_c$ is $-NkT \ln 2$, coming solely from the entropy of the completely disordered state. So, in MF, the system is in the completely disordered state ("entropic death") as soon as T_c is reached. why?

(2) There is a simple meaning to the transition temperature. Ground state energy is $-qJ/2$ per particle. The maximum energy possible (say on a hypercubic type lattice) is $+qJ/2$ when each neighbouring pair is antiparallel. kT_c is just the total width of the energy spectrum. In the MF approximation at T_c all the states become equally probable leading to the entropic death.

(3) So far we have obtained four exponents $\alpha = 0, \beta = \frac{1}{2}, \gamma = 1, \delta = 3$. These are all mere consequences of the linear and the cubic order terms of the expansion of tanh whose origin is in the behaviour of an isolated spin in a field. So where do the model, interaction etc come in?

IX. FLUCTUATION-DISSIPATION (RESPONSE)

So far we have focused on thermodynamic quantities. Let us now consider other statistical quantities, e.g., correlations etc. For this let us first see the connection between a response and microscopic correlations. To be explicit, we consider susceptibility.

Suppose a system is described by the hamiltonian $H = H_{\rm int} - h \sum s_i$ where $H_{\rm int}$ is the interacting piece - no matter what it is. The magnetization is

$$M = kT\frac{\partial \ln Z}{\partial h}, \tag{9.1}$$

and χ requires one more derivative of $\ln Z$ with respect to h. Take these brute force derivatives:

$$M = \frac{\sum_{\text{config}} (\sum s_i) e^{-\beta H_{\text{int}} + \beta h \sum s_i}}{Z},$$
$$kT\chi = Z^{-1} \sum_{\text{config}} (\sum s_i)^2 e^{-\beta H_{\text{int}} + \beta h \sum s_i} - \left(Z^{-1} \sum_{\text{config}} (\sum s_i) e^{-\beta H_{\text{int}} + \beta h \sum s_i} \right)^2, \tag{9.2a}$$

with $h = 0$ in the last equation. A simple rearrangement then produces

$$\boxed{\chi = (kT)^{-1} \sum_i \sum_j \langle (s_i - \langle s \rangle)(s_j - \langle s \rangle) \rangle}. \tag{9.3}$$

This is the famous fluctuation dissipation theorem. In fact there is no dissipation in our case and a better name would be fluctuation response theorem. We have already seen this for a free spin. This formula connects χ to the correlation in spin fluctuations, and not to spin correlations explicitly. In the high temperature phase, $\langle s \rangle = 0$, and both are equal. This is not the case in the low temperature phase.

For a translationally invariant system, one can get rid of one of the sums. Choose any arbitrary site as the origin and define a pair correlation function

$$g(\vec{r}) = \langle (s_o - \langle s \rangle)(s_r - \langle s \rangle) \rangle, \tag{9.4}$$

in terms of which χ can be written as

$$\chi = (kT)^{-1} N \sum_r g(\vec{r}). \tag{9.5}$$

N takes care of extensivity - we forget about it.

`Problem` **IX.1** *Any second derivative of the free energy can be connected to a correlation function. Prove this general statement.*

Show, in particular, that the specific heat is related to the energy energy correlation function. For the Ising case, this requires four spin correlation function.

A. χ and Long range order

For simplicity, let's replace the sum by an integral so that $\chi = \int d\vec{r} g(\vec{r})$. For the Ising case, s is bounded and so is g. We also know that χ diverges at least at T_c. Only way this can happen is from the divergence of the integral or the sum - it is the large distance property of $g(r)$ that controls the behaviour. We can conclude that, at least at T_c, $g(r)$ cannot be a short ranged function, but it has to decay because, for infinitely large distances the correlations should go to zero. In d dimensions, $d\vec{r} \sim r^{d-1}dr$ so that $g(r)$, at $T = T_c$, has to decay as

$$g(\vec{r}) \sim r^{-(d-2+\eta)} \tag{9.6}$$

with $\eta \geq 0$. For $T \neq T_c$, convergence requires that $g(r)$ should decay sufficiently faster than this - in almost all cases it decays exponentially, with a characteristic length scale ξ. The decay for any temperature can be written as

$$g(\vec{r}) \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}. \tag{9.7}$$

To be consistent with the power law decay at $T = T_c$, this length scale ξ has to diverge as one approaches the critical point

$$\xi \sim \mid T - T_c \mid^{-\nu}. \tag{9.8}$$

Two new exponents are required to describe the decay of the two point correlations, ν for $T \neq T_c$ and η for $T = T_c$.

Let's first evaluate them in MF, and then we discuss the physical picture.

1. *Comments*

(1) We have now defined a third way of characterizing the critical point. This is the temperature at which long range order sets in. Let's call it T_c^{LRO}. At the special point, there is no characteristic length scale in the problem - the system looks similar at all length scales (in the large distance limit of course, not at the lattice spacing!). That's why we get power law behaviour for the correlation. Power laws do not have any length scale.

From a rigorous point of view, "long range order" is defined as the case when the spin spin correlation does not go to zero in the large distance limit, i.e., $\lim_{r\to\infty} \langle s_0 s_r \rangle = const \neq 0$. There is a subtle difference between this and the spontaneous magnetization. However, the susceptibility is related to the fluctuation, which acquires a long range character only at T_c. The fluctuation decays sufficiently fast (like exponential) even in the ordered state. So far, one can prove rigorously, that $T_c^{\text{LRO}} \leq T_c^{\text{SB}} \leq T_c^{\text{HT}}$. The last inequality is known to be true for the Kosterlitz Thouless transition. For simple Ising type problems, they are known to be all equal.

(2) It is not guaranteed that all high temperature phases or the low temperature phases will have exponential decays. Power law decay of correlation throughout a range of temperature is not unheard of. These cases require special attention.

(3) A simple consequence of immense experimental importance is that if you freeze the structure of a critical system like a binary mixture at its critical point, it will have composition fluctuation at all length scales without any characteristic length!

B. ν, and η

We utilize the simple minded picture used to derive the Curie Weiss law at the end of Sec. VII A. Start with a periodic field $h_r = h_k \exp(i\ \vec{k}\cdot\vec{r})$. The response is also expected to be sinusoidal, $m_r = m_k \exp(i\ \vec{k}\cdot\vec{r})$. The response of magnetization need not be local. Think of an elastic body if you press at one point, it is going to affect other points also. Since the field is nonuniform, we define a nonlocal susceptibility $\chi(\vec{r}-\vec{r}')$, so that the magnetization can be written as $m(\vec{r}) = m^{h=0}(\vec{r}) + \int\ d\vec{r}'\chi(\vec{r}-\vec{r}')h(\vec{r}')$, where $m^{h=0}$ is the zero field magnetization, and χ is really the pair correlation function as in Eq. 9.5. Fourier transformation gives

$$m_k = m_k^{h=0} + \chi_k h_k, \tag{9.9}$$

where χ_k is the Fourier transform (FT) of $g(r)$. This χ_k gives the response of the system to a sinusoidally varying field of wave vector $\vec{k}$. In this Fourier space, $\chi_{k=0}$ is the zero field susceptibility that diverges. It really pays to study the correlation functions in Fourier space!

Now use the simple feedback argument, taking care that in a varying field, magnetization is not uniform. The internal field is accordingly $\sum_r Jm(\vec{r})$, with $m(\vec{r}) = \sum_k m_k \exp(i\vec{k}\cdot\vec{r})$. For site $\vec{r}$, we then have

$$m(\vec{r}) = \chi^0[h(\vec{r}) + \sum_{\rm NN} Jm(\vec{r})], \tag{9.10}$$

get k^2 so that after Fourier transformation,

$$m_k = \chi^0[h_k + aJk^2 m_k], \quad \Longrightarrow \quad \chi_k = \frac{\chi^0}{1 - aJk^2\chi^0}, \tag{9.11}$$

why? a being a constant. Using $\chi^0 \sim 1/T$, we rewrite this equation for χ_k as

$$\chi_k \sim [\xi^{-2} + k^2]^{-1}, \tag{9.12}$$

with $\xi \sim \mid T - T_c \mid^{-1/2}$. ξ is the length scale that determines the decay of χ_k, the FT of $g(r)$. Hence,

$$\nu = \tfrac{1}{2}. \tag{9.13}$$

Right at T_c, ξ^{-2} is zero. Thus, $\chi_k \sim k^{-2}$. The power law decay of Eq. 9.6 has a FT $k^{-2+\eta}$. So,

$$\eta = 0. \tag{9.14}$$

C. Comments

(1) χ_k or the Fourier transform of $g(r)$ is important experimentally. This is related to the structure factor or scattering. It is measured directly by light scattering, x-ray, neutron scattering etc.

(2) The Lorentzian shape of χ_k is also known as the Ornstein-Zernike formula.

(3) the width of the scattering function gives the correlation length.

X. RELATIONS AMONG EXPONENTS

So far we have calculated five different quantities and obtained five different exponents. How many do we need? Are they all independent? In fact they are not - only two are needed.

Let's think of the pair correlation function. It decays rapidly once we are on a scale greater than the correlation length ξ. Close to T_c, we can think of the system as blobs of highly correlated regions - the blobs are of size ξ^d in d dimensions. Inside a blob ($r << \xi$), the spins are blob highly correlated and at a simple level can be thought of as at T_c. On a bigger length scale $r >> \xi$, the blobs are independent. Basically, we are arguing that it is the correlation length that matters - all other length scales are unimportant.

We use this simple picture for the susceptibility, which is an integral of $g(r)$. We can cutoff the integral at $r \sim \xi$, and inside this region $g(r) \sim r^{-(d-2+\eta)}$. The integral $\int^\xi dr\ r^{1-\eta} \sim \xi^{2-\eta}$. Using the temperature dependence of ξ, we get the temperature dependence of χ as $\mid T - T_c \mid^{-\nu(2-\eta)}$. The net result is

$$\gamma = \nu(2-\eta). \tag{10.1}$$

The MF exponents do obey this relation. There are many such relations.

`Problem` **X.1** *Take free energy $\sim hm$ to argue that $2\beta + \gamma = 2 - \alpha$. MF exponents satisfy this.*

Prove, from thermodynamics, that $c_h - c_m = T\left(\frac{\partial m}{\partial T}\right)_h^2 \chi_T^{-1}$, where c_x is the specific heat with x constant. Since sp.heat is positive definite, show that $\alpha + 2\beta + \gamma \geq 2$. Now, argue that equality is the general rule. A special condition is required for the inequality. Find out that condition.

We can push such arguments a bit further. Consider the free energy density. This goes inversely as volume. But in the thermodynamic limit, correlation length is the only length available. No harm in expecting $f \sim \xi^{-d}$, and using the temperature dependence of ξ, we get $f \sim \mid T - T_c \mid^{d\nu}$. Compare this with $f \sim \mid T - T_c \mid^{2-\alpha}$. As how? a consequence, we must have $2 - \alpha = d\nu$ (HYPERSCALING). Unfortunately, the mean field exponents have no d dependence. Therefore, this hyperscaling can be obeyed only in a special dimension, which turns out to be 4. This is extremely important.

`Problem` **X.2** *What's wrong with hyperscaling? Exact exponents below this special dimension obey hyperscaling but not above. How can this be violated?*

XI. LANDAU THEORY

We discussed MF theory in the context of the Ising model but also pointed out that the detailed features of the model are not really important. It would be natural, indeed, to have a theory that does not include unnecessary details, and still be simple and rich enough to describe criticality. Landau developed such a scheme.

There are various ways of introducing the Landau theory. We take the sequel of the MF theory developed in Sec. 4. In the infinite range model, we landed on an integral involving A which was easy to tackle. At the end we ultimately didn't do any computation and the free energy was given by A at its minimum. For a given T and h, this A depends on λ, that turns out to be the magnetization. The Landau approach is based on this. Anticipating the result, we define $A(m,h,T)$, to be called the Landau function or Landau free energy, or extremely loosely free energy, such that its minimum with respect to m should describe the thermodynamic property. We already assumed that m is homogeneous in space.

If not?

First a few technicality. The free energy we get from, say, the Ising Hamiltonian is a function of T and h - the intensive quantities or the externally imposed variables, $f = f(T,h)$. As you understand, these are the variables that couple to "operators" which depend on the internal degrees of freedom. These operators, after the stat mech averaging, are the thermodynamic quantities that should be proportional to the size of the system. Free energy itself is such an example. The intensive variables do not scale with size. The magnetization comes from a derivative of this free energy, $m = -\partial f/\partial h$. The extensivity of the derivative is a result of the same of f. It is, therefore, convenient to divide by the volume to get "densities".

Since the free energy contains most of the information about the critical singularity, it would be nice to have a simple expansion around say the critical point $(T_c, h = 0)$. Such an expansion has to be highly singular because, as we now know, m (first derivative) is well behaved on the high temperature side but is a multivalued function for $h = 0$ when $T < T_c$. This is hopeless! The situation is slightly better if one chooses some other thermodynamic potential.

Different thermodynamic potentials come through Legendre transformation. We consider $A(T,m) = f(T,h) - hm$ (We are sloppy with the variables.) An expansion in m may not be bad because $h \sim -\partial A/\partial m$ and $\partial h/\partial m \sim \chi^{-1} \to 0$, as $T \to T_c$. So crudely speaking the derivatives one would need for a Taylor expansion are not that bad. But, see, below T_c, two phases coexist at zero field. This means that any value of magnetization (within a range) is possible by appropriate choice of the volumes of the two phases, in zero field. This indicates the existence of a flat region in the $A(m,T)$ curve. That's again a source of difficulty.

The function we consider is not really the thermodynamic potential, because the conjugate variables (h,m), which are to be coupled by the equation of state, are treated as independent. The equilibrium value of m comes only after minimization. The gain is an analytic function. (Why not compare with the saddle point method discussed earlier?)

What about the structure of this Landau function? We want it to respect the symmetries of the problem. For the Ising model, there is an up down symmetry that's broken in the low temperature phase. We expand $A(T,m)$ in m, and use this symmetry to throw away terms not allowed. The T dependence will be handled separately. In short, for the problem in hand, $A(m)$ has to be an even function of m. If there is no external magnetic field, there should be no linear term. Hence,

$$A(m) = a_0(T) + a_2(T)m^2 + a_4(T)m^4 + ... \tag{11.1}$$

where the coefficients a_i depend only on T. We have already observed that the second derivative of A goes to zero at T_c. Nothing special can be said about a_4. Assume, and just assume, that these coefficients are analytic functions of t, amenable to Taylor series expansion. Obviously we will have

$$\begin{aligned} a_0(T) &= a_{00} + ... && (11.2a)\\ a_2(T) &= a_{21}(T - T_c) + O(t^2) && (11.2b)\\ a_4(T) &= a_{40} + ... && (11.2c)\\ & && (11.2d) \end{aligned}$$

Important observations for us are (i) a_2 changes sign at T_c, becoming negative for $T < T_c$, (ii) a_4 is positive. a_4 can be zero or negative. Those cases lead to multicriticality and require special attention.

`Problem` **XI.1** *Expand A of Eq. 4.11 and get the Landau expansion.*

Next, minimize A.

$$m[a_2(T) + 2a_4(T)m^2] = 0 \Longrightarrow m = 0, m = (a_2/2a_4)^{1/2} \tag{11.3}$$

For $T > T_c$, only real solution is $m = 0$. Since $a_2 < 0$ for $T < T_c$, we have $m \sim \mid t \mid^{1/2}$ - a result we already know. The exponent $1/2$ is really a consequence of linear t in Eq. 11.2b.

To get susceptibility, add $-hm$ to Eq. 11.1, and then minimize. The result is $\gamma = 1$. This we could have guessed. The coefficient of m^2, after all, is related to the inverse susceptibility. Also, recognize that taking such successive derivatives and then $h = 0$, is equivalent to look for the curvature of the Landau function - it's inverse gives the susceptibility.

Try to understand these from a plot of $A(m)$ with m for various T.

We can even go beyond exponents. Let's define amplitudes $C_\pm$ as $\chi_\pm \equiv C_\pm \mid t \mid^{-\gamma}$ for $t \gtrless 0$. For $T > T_c$, $\chi_+ \equiv 2a_2(T)$, while for $T < T_c$, using the value of m, $\chi_- \approx 4a_2(T)$. The exponents are the same on both sides of T_c, but the amplitudes are system specific because the coefficients of the Landau function are. The surprising feature is that, the amplitude ratio $C_+/C_- = 2$ is a universal number independent of the details. As a matter of fact, for each quantity that exists on both sides of T_c, one can define these amplitudes, and, believe it or not, the ratio is a universal number.

A. Comments

(1) Why is the Landau function not a thermodynamic potential? For the low temperature phase, there is an unstable region with a negative susceptibility. A thermodynamic potential cannot have this. This is a characteristic of any mean field solution. For example, van der Waal equation of state shows this in the isotherms. The solution is to draw the convex envelope that gets rid of the unstable branch. This is equivalent to drawing the common tangent through the two minima. This, in turn, corresponds to the famous equal area construction of Maxwell.

(2) Taking Legendre transforms is quite common. This is also done in all field theories purporting to discuss broken symmetries. The coefficients give the vertex functions.

If you think a little bit, you will realize that, in a lattice HOW? gas analogy, this transformation corresponds to a change in the ensemble.

XII. POTTS MODEL - A CONTRAST

Just to show the power of the meanfield theory and in the process learn about a very important model, we study the Potts model.

The model involves a generalization of the Ising variable. Suppose that at each site there is a spin s_i that can take q possible values. We don't care what the values are or what the objects are. There is an interaction that favours neighbors of equal spin values. The mean field hamiltonian is

$$H = -\frac{J}{N} \sum_{ij} \delta_{\rm Kr}(s_i, s_j), \qquad (12.1)$$

where $\delta_{\rm Kr}(s_i, s_j)$ is the Kronecker delta being equal to 1 if $s_i = s_j$, and zero otherwise. For $q = 2$, this model can be reduced to the Ising model.

Suppose x_p be the fraction of spins that are in spin state p, p running from 1 to q. Sure, $\sum x_p = 1$.

For large N, the energy and entropy are just

$$\frac{E}{N} = -\tfrac{1}{2} J \sum_p x_p^2, \quad \text{and} \quad \frac{S}{N} = -k \sum_p x_p \ln x_p, \quad (12.2)$$

[remember $\rho \ln \rho$?], so that the free energy per spin is

$$\beta A = \sum_p (x_p \ln x_p - \tfrac{1}{2} \frac{J}{kT} x_p^2). \qquad (12.3)$$

Compare this with the Ising case, Eq. 4.5. It looks like just a sum of independent Ising cases, but it isn't because of, ah! well, think why.

Motivated by the Ising case, we like to define an order parameter that would describe the ordered state. Suppose, $p = 1$ is the preferred state. Let us define m $(0 \leq m \leq 1)$ through

$$x_1 = \frac{1}{q}[1 + (q-1)m], \qquad (12.4a)$$

$$x_p = \frac{1}{q}(1-m), \quad \text{for } p = 2, ...q, \qquad (12.4b)$$

put them back in Eq. 12.3, and expand in m to get

$$\beta A(m) = \beta A(0) + \frac{1 + (q-1)m}{q} \ln[1 + (q-1)m] +$$
$$\frac{q-1}{q}(1-m)\ln(1-m) - \frac{q-1}{2q}\frac{J}{kT} m^2$$
$$= (q-1)[\frac{q - \beta J}{2q} m^2 - \frac{q-2}{6} m^3 + \frac{q^2 - 3q + 3}{12} m^4 + ... \quad (12.5)$$

If you have already solved the problem of exotic $f(x)$, you know that the negative cubic term implies a first order transition. The appearance of the cubic term is not surprising because no inversion symmetry is expected unless $q = 2$. At that q the coefficient of the cubic term vanishes as for the Ising case. For all real q, the m^4 term is positive.

The conclusion is that the Potts model, in the mean field approximation, shows critical behaviour only for $q \leq 2$ but a first order transition for $q > 2$.

`Problem` **XII.1** *Show that for $q > 2$, the transition point is $kT_c = J(q-2)\,[(q-1)\ln(q-1)]^{-1}$. The nonzero "magnetization" at T_c is $m_c = (q-2)/(q-1)$, and the latent heat is $L = J(q-2)^2/[2q(q-1)]$.*

A. Comments

(1) We see that the transition becomes first order for $q > 2$ for all d. It is known exactly that for $d = 2$, the transition is *not* first order for $q \leq 4$. This critical value of q is also d dependent. For $d > 4$, the critical value is 2 as in the mean field theory. Try to get the critical value as a function of d.

(2)

XIII. VALIDITY OF MFT

We have derived the mean field theory, tried to understand the approximations made, but as yet haven't answered the question of its validity for a given hamiltonian. This is done through Ginzburg criterion.

We repeatedly said that we are ignoring fluctuations. A quantitative statement would be this: Choose an appropriate volume Ω - which is large compared to the characteristic microscopic volume but less than the total volume - wait! I'll specify) - and the fluctuation of magnetization in this volume must be less than the magnetization $M_\Omega$in that region itself, $\delta M_\Omega^2 \quad << M_\Omega^2$. Let the number of spin in that region be N_Ω. By definition, the mean square fluctuation is

$$\delta M_\Omega^2 = \left\langle \left[\sum_i (s_i - \langle s \rangle) \right]^2 \right\rangle = N_\Omega \sum_\Omega [\langle s_0 s_i \rangle - \langle s \rangle^2]. \tag{13.1}$$

If Ω were the total volume, then it would have been the total susceptibility $\chi(\vec{k}=0)$ in the notation of Sec. 9. Now near the critical region the correlation length becomes very large, and this is the only length that controls the behaviour (that's the origin of universality). So it is natural to choose $\Omega \sim \xi^d$, as we did for hyperscaling. As $T \to T_c$, this volume becomes very large so that its per spin susceptibility can very well be taken as the susceptibility of the bulk. This enables us to write the fluctuation as

be-
ware

$$\delta M_\Omega^2 = const \; N_\Omega \chi(\vec{k}=0,t), \tag{13.2}$$

where the temperature dependence is shown explicitly.

Problem **XIII.1** *Using the expression for pair correlation function, try to justify this formula.*

The magnetization is $M_\Omega = N_\Omega m$, m being the bulk magnetization per spin. The condition to be satisfied is

$$N_\Omega m^2 >> const \; \chi(\vec{k}=0,t). \tag{13.3}$$

The number of spins in the volume is expected to go like $N_\Omega \sim \Omega \sim \xi^d$. Use the exponents β, γ, and ν, to write the above inequality as

$$const \; t^{-\gamma} << t^{-d\nu + 2\beta}. \tag{13.4}$$

Consequently, the condition for the validity of MFT can be stated as $d\nu > \gamma + 2\beta$. If you recall the exponent relations, then $\gamma + 2\beta = 2 - \alpha$, so that the Ginzburg criterion is similar to hyperscaling $d\nu > 2 - \alpha$. In other words, we can define a critical dimension $d_c = (\gamma + 2\beta)/\nu = (2-\alpha)/\nu$ above which the mean field theory is self consistent in the sense that the fluctuation can really be ignored. If necessary, they can be treated perturbatively. Use the mean field exponents to get $d_c = 4$. This special dimension is called the UPPER CRITICAL DIMENSION.

A. Comments

(1) One should be careful about the connection between Ω and ξ. It is better to call Ω the correlation volume. This may have some other dependence on ξ, as for example in dipolar magnets. Such an extra factor of ξ changes the UCD.

(2) Loosely speaking, if we are at a very high temperature, then we do not expect fluctuations to dominate. One can always trust MFT in a very high temperature regime. The question one can then ask, if we start from such a region how close should we be to T_c to observe deviations? This can be estimated. A crude way to do this would be to compare the different length scales. The universal criticality is observed in a regime where the basic microscopic length scales are not important. We demand that in the critical region the correlation length $\xi = \xi_0 t^\nu$ should be greater than, say, the range of interaction, or the lattice spacing etc. Quite often, the microscopic length turns out to be pretty large requiring very small t, and one ends up seeing the conventional MF behaviour. This happens for old fashioned superconductors - but that's a different, and not so simple, story.

XIV. GOING BEYOND MFT

How do we go beyond MFT? We have seen that MFT gives universal quantities but they are too universal to be true! No doubt, its a failure of the Landau expansion but in what sense. We will see later on that any attempt to modify this function fails miserably. We have also seen, through Ginzburg criterion, that fluctuations are important. The right step evidently would be to incorporate fluctuation.

If fluctuations are important then uniform m is not a good approximation. Let's go back to Eq. 4.11. A saddle point evaluation was possible because of N (or equivalently volume) in the exponent. If $\lambda(\vec{r})$ has space dependence then one expects this to be changed to $\int d\vec{r} \; A[\lambda(\vec{r})]$, and the final integral (a functional integral) over all $\lambda(\vec{r})$. But this as such is not sufficient to handle fluctuation as we have seen in Sec 9 $-$ a k^2 term is needed.

Since A is a scalar, we take,

$$A[m(\vec{r})] = (\nabla m)^2 + a_2 m^2 + u m^4 - hm. \tag{14.1}$$

This is called the Landau-Ginzburg hamiltonian (or a ϕ^4 field theory. The partition function is given by

$$Z = \int \mathcal{D}m \exp[-\int A[m(\vec{r})]]. \tag{14.2}$$

If we now want to do a saddle point approximation with uniform m, we recover the Landau function of Eq. **??**. Any thing better is hard!

A. Gaussian Model

To have a feeling for fluctuation, we ignore the m^4 term, and set $h = 0$. The partition function then involves gaussian integrals, and is doable. The mm correlation function is easily seen to be given by Eq. 9.12, with $\xi^{-2} = a_2$. The exponents ν, η are the same as there. In fact there is no change in most of the exponents except for α. For this we need the free energy.

`Problem` **XIV.1** *Go to Fourier space, use equipartition theorem and get the correlation function.*

It is easy to show that the specific heat is given by the integral

$$c \sim \int^{\Lambda} d^d k [a_2 + k^2]^{-2}, \tag{14.3}$$

where Λ is a cutoff that may come from lattice spacing etc. Rescale k by $k a_2^{-1/2}$, so that for $T \to T_c$, and $d < 4$, $c \sim a_2^{-(4-d)/2}$. This gives a DIVERGENT specific heat with $\alpha = (4-d)/2$, for $d < 4$. For $d > 4$, the integral in the limit $T \to T_c$ diverges in a way that cancels out the scaling factor, leaving behind a finite answer. That means, for $d > 4$ the specific heat has a discontinuity at the critical point as in MFT. Fluctuations have no effect as predicted by the Ginzburg criterion.

The model we solved is called the Gaussian model and is the starting point to understand the full model. It is however ill defined for $T < T_c$.

Also note that, with this α, the exponents do satisfy the hyperscaling for $d < 4$. 4 again turns out to be the border line dimension.

XV. O(N) MODELS

The Landau theory can be generalized to any symmetry group. A case that occurs quite often is the $O(n)$ symmetry, where $\vec{m}$, the magnetization, is an n dimensional vector, and the system has full rotational invariance with respect to $\vec{m}$. For example, $n = 3$ corresponds to the Heisenberg model with three component spins, $H = -J \sum \vec{s}_i \cdot \vec{s}_j$.

The Landau Ginzburg model can be written as

$$A[\vec{m}(\vec{r})] = (\nabla \vec{m})^2 + a_2 m^2 + u m^4 - h m_\alpha, \tag{15.1}$$

where $(\nabla \vec{m})^2 = \sum_{\alpha=1}^{n} (\nabla m_\alpha)^2$, and the magnetic field is in the α direction.

In MFT, we take uniform magnetization. No need for repetition to show that the magnetization shows identical behaviour as for the Ising case.

For susceptibility etc one has to worry about the components. Similarly, the pair correlation function depends on the spin component index. You must have recognized by this time that this correlation function comes form double derivatives with m. In the Fourier space, we have, in zero field,

$$[g_{\alpha\beta}(\vec{k})]^{-1} = \delta_{\alpha\beta}[k^2 + a_2 + 4a_4 m^2] + 8 a_4 m_\alpha m_\beta. \tag{15.2}$$

The magnetization direction is called the longitudinal uniform m is not a good approximation. Let's go back to Eq. 4.11. A saddle point evaluation was possible because of N (or equivalently volume) in the exponent. If $\lambda(\vec{r})$ has space dependence then one expects this to be changed to $\int d\vec{r}\ A[\lambda(\vec{r})]$, and the final integral (a functional integral) over all $\lambda(\vec{r})$. But this as such is not sufficient to handle fluctuation as we have seen in Sec 9 – a k^2 term is needed.

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The Landau Ginzburg model can be written as

$$A[\vec{m}(\vec{r})] = (\nabla \vec{m})^2 + a_2 m^2 + u m^4 - h m_\alpha, \qquad (16.1)$$

where $(\nabla \vec{m})^2 = \sum_{\alpha=1}^{n} (\nabla veck)\})]$. In the limit $N \to \infty$, the sum in the integral can be replaced by an integral, and the whole thing can be evaluated by a saddle point method (stationary phase method). I leave the details for you to work out and get the exponents. It is straight forward but needs some work.

The point of the exercise is to observe the introduction of the delta function to represent a constraint. With the Fourier representation, one can even think of a modified hamiltonian (with complex parameters - but rest assured every thing is real!). Note the similarity with the Lagrange undetermined multiplier method to tackle constraints. The sermon here is that, given a Hamiltonian, if I can identify the order parameter as a function of the microscopic variables, I can introduce it as a δ function and obtain $Z(m)$ as a function of the order parameter m. The effective Hamiltonian is expressed in terms of m - and that's the Landau-Ginzburg hamiltonian.

A. Comments

(1) The exponents of the spherical model are related to the exponents of the Gaussian model (GM). Any exponent x for the gaussian model and the corresponding one x_s for the spherical model are related by $x_s = x/(1-\alpha)$, where $\alpha = (4-d)/2$, the specific heat exponent for GM. The specific heat exponents are related by $\alpha_s = -\alpha/(1-\alpha)$. This is called Fisher renormalization. This occurs whenever there is a constraint in the system, and original $\alpha > 0$. The specific heat turns out to be non-divergent.

(2) The spherical model is also the $n \to \infty$ limit of the $O(n)$ model.

(3) Study the low temperature behaviour of the spherical model and you will find exact similarity with Bose-Einstein condensation. Any speculation on what happened to quantum features?

XVII. PROBLEMS

`Problem` **XVII.1** *Start with the Ising Hamiltonian and define the order parameter as $M = \sum s_i$. Using this as a constraint, with necessary approximations or expansions, obtain Eq. 15.4.*

`Problem` **XVII.2** *Test your expertise of MFT by deriving the van der Waal equation of state for n particles with pairwise interaction as given by the hamiltonian*

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2m} + \tfrac{1}{2} \sum_{ij} U(\vec{r}_i - \vec{r}_j). \qquad (17.1)$$

Take $U(\vec{r}) = \infty$ if $r \le r_0$.

`Problem` **XVII.3** *Cubic systems. Spins on a cubic crystal are expected to have terms of cubic symmetry. Consider n component spins. The L-G form is*

$$H = a_2(T) m^2 + u m^4 + v \sum_{\alpha=1}^{n} m_\alpha^4. \qquad (17.2)$$

Discuss the mean field behaviour for positive and negative u, v.

`Problem` **XVII.4** *Fluctuation driven first order transition: Consider superconductivity (or scalar electrodynamics). It requires a complex order parameter ψ and a vector potential $\vec{A}$. The L-G form is*

$$F\psi, \vec{A} = \int d\vec{r}\, [a_2 \mid \psi \mid^2 + u \mid \psi \mid^4 +$$
$$\gamma \mid (\vec{\nabla} - i q_0 \vec{A})\psi \mid^2 + \mu \sum_{i>j} (\nabla_j A_i - \nabla_i A_j)^2]. \qquad (17.3)$$

It is quadratic in A, DO the averaging over A to define an effective L-G form for ψ. Show that the free energy, in the process, acquires a cubic term that makes the transition first order. Are you surprised?

[1] H. E. Stanley, *Introduction to .....*
[2] M. E. Fisher, in *critical phenomena* Ed. by
[3] K. Huang, *Statistical Mechanics*
[4] *Phase transitins and critical phenomena* ed. by C. Domb and M. L. Green, Vol. 1 and 2