ASPECTS OF GAUGE/GRAVITY DUALITY

By

Shankhadeep Chakrabortty

PHYS07200604032

INSTITUTE OF PHYSICS

BHUBANESWAR

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Synopsis

Carrying out reliable computations in strongly coupled systems are generally very difficult. This is largely due to the fact that, till to date, there does not exist a systematic formulation of theories with large coupling constants. Consequently, a need to search for indirect rutes becomes essential. Consider, for example, a strongly correlated condensed matter system. It can have descriptions in terms of quasiparticles. These are the collective exitations which are weakly interacting in free space. Similarly, some field theories may have a large number of internal degrees of freedom (N). It is then possible to use a perturbative $1/N$ expansion to explore certain strong coupling phenomena in these theories. One may also look for duality symmetries which generally relate a strongly coupled theory to another with a weak coupling. Indeed in late last century, such duality symmetries were conjectured to exist in string theory. The very fact that these symmetries relate strong and weak coupling, a direct proof of its existence becomes difficult. However, various indirect checks suggest that string theory is in fact endowed with such a symmetry. Inspired by these developments, in 1997, Maldacena proposed a dual description between a string theory defined in anti-de Sitter (AdS) space and a non-gravitational field theory defined on a conformal boundary of this space. Depending on the context, this is known as gauge/string duality, gauge/gravity duality or the AdS/CFT correspondence. This immediately opens up a door to study strongly coupled string theory (however in AdS) via computations in the dual weakly coupled gauge theory and vice versa. The main aim of this thesis is to use this correspondence to carry out a set of computations to explore certain features of strongly coupled gravity *as well as* that of strongly coupled gauge theory. We first briefly review some aspects of the AdS/CFT correspondence and then provide a summary of our work.

The simplest and maximally explored example of AdS/CFT correspondence involves the type IIB string theory. It states that IIB string theory formulated on a five dimensional AdS space times a five-sphere is dually related to a four dimensional Yang-Mills $SU(N)$ (YM) theory with four supersymmetries. Supersymmetries can be broken by turning on a temperature (T). On the gravity side, this is achieved by introducing a black hole in the AdS space. When the super-YM is strongly coupled, the dual has a description in terms of type IIB supergravity where many computations can be explicitly carried out. For example, through this duality, calculations of correlation functions in the gauge theory get mapped to the computations of amplitudes in the supergravity theory. There are several ways to generalize this system. For example, turning on a chemical potential (μ) in the gauge theory is equivalent to adding some gauge charges to the black holes and so on. AdS/CFT correspondence can also be generalized for non-conformal gauge theories arising from the gravitational description of Dp branes for $p \neq 3$. However, in spite of all these advances, till to date, the gravity dual of QCD remains elusive and though there are some similarities, gauge theories with gravity duals are very different from QCD in several aspects. Nonetheless, there is still considerable interests among researchers to further explore the consequences of this correspondence. One of the main reasons perhaps is to get an answer to the following question. Can we identify some universal features of these strongly coupled gauge theories with gravity duals? One may then hope that these results will be useful if a dual of QCD is discovered. Indeed there is a remarkable progress in this direction – a discussion of which follows in the next paragraph.

In [1], [2], gauge/gravity duality was used to show that the shear viscosity to the entropy ratio of four-dimensional $SU(N)$ YM theory with $\mathcal{N} = 4$ supersymmetries is $\hbar/(4\pi k_B)$ where k_B is the Boltzmann constant. Besides the fact that this low viscosity is expected from the estimation of RHIC data for quark-gluon-plasma (QGP), to which we will shortly come back to, this ratio was found to be universal for all the strongly coupled gauge theories, in the $N \to \infty$ limit, having a gravity dual [3]. Subsequently, it was found that there are various other quantities which show universal behavior too. Ratio of R-charge conductivity to the charge susceptibility is another quantity of this nature [3]. Further, at finite μ, certain combination among the thermal conductivity, temperature and the chemical potential is expected to be universal [4].

With the heavy ion experiments running at RHIC and LHC, it becomes important to have progress in our theoretical understanding of the QGP phase. It is believed that such a thermalized state of matter, exhibiting the deconfined phase of quarks and gluons, have been created in these experiments. Notwithstanding the earlier expectation that this is a weak coupling phase of QCD, large amount of evidences have, by now, accumulated which point towards the dominance of large coupling non-perturbative effects. One of the main evidence perhaps is that the medium can be described by ideal hydrodynamics leading to a low shear viscosity of the fluid. Possibility of having a large jet quenching factor with considerably high energy loss per unit length of moving partons in this medium is also believed to be an artifact of the strong coupling effects. While none of these arguments are definitive, we will proceed assuming that strong coupling effects are important and it is necessary to improve our understanding of heavy ion collision. Though studying QCD at large coupling is beyond the scope of our present knowledge, in the light of our previous discussion, we may wish to start searching for the strong coupling effects in the theories with gravity duals. One may hope that some of the predictions arising from such analysis may be universal for strongly coupled theories including QCD. The work in the first part of this thesis follows this spirit. Here we consider two models. First is a holographic QCD (hQCD) model discussed in [5]. Gravity dual of this model is the asymptotically AdS (aAdS) black hole in Einstein-Maxwell-Dilaton (EMD) system. We then calculate drag force, jet quenching parameter, screening length and binding energy of external quark-antiquark ($q\bar{q}$) pair for this model. In the second model, we consider $\mathcal{N} = 4$ SYM plasma and the motion of an external heavy quark in this medium. This has been well studied in the literature [6], [7]. Gravity dual of $\mathcal{N} = 4$ $SU(N)$ SYM is the standard AdS black hole. However, if one introduces large number of heavy quarks in the gauge theory, the bulk gets deformed. We explicitly construct this back reacted geometry and use this geometry to calculate the drag force on one of those heavy quarks while in motion [8]. In the next few paragraphs, we elaborate upon different aspects of these two models.

We start with a discussion of the hQCD model specially focusing on our computations of different dynamical observables. They are important in the context of real time dynamics of high energy partons in QGP medium. The hQCD model, that we consider, can reproduce various aspects of thermal QCD, such as: equation of states, confinement/de-confinement phase transition, etc. Spectrum of this model contains both quarks and antiquraks in fundamental representation. As mentioned earlier, the dual gravity theory is realized as an aAdS black hole solution in EMD system. In this work [9], we use this aAdS black hole background to accomplish the holographic computation of drag force on a heavy probe quark moving in the thermal medium of hQCD model. Along the way, we also calculate jet quenching parameter $\hat{q}$, screening length (L_s) and binding energy of a quark-antiquark pair ($q\bar{q}$).

To elaborate further, first we discuss about the drag force acting on the probe quark. It originates from the strong energy loss of a high-energy parton probing through the medium. In our calculation of drag force, probe quark represents the high energy parton. The gravity dual of this probe quark is well known [6]. In our aAdS black hole geometry, it is described by an infinitely long fundamental string. One of its ends is attached to the boundary of the bulk spacetime. The body of the string extends along the radial direction and the free end of the string goes parallel to black hole horizon. The AdS/CFT duality suggests an identification between the end point of the string and the probe quark. Furthermore, the body of the string represents the gluonic field in the thermal plasma. In this dual gravity picture, the string trails back and imparts a drag force on it's endpoint that is attached to the boundary. This drag force is obtained by calculating rate of change in string momentum. The qualitative study of the drag force shows that it increases with the velocity of the quark for fixed chemical potential and temperature. It also increases with temperature while chemical potential and velocity are kept constant. All these results are consistent with the fact that the study of drag force is meaningful only in the de-confined phase of hQCD model. Another important quantity that we analyse is the jet quenching parameter. It is conceived as a measure of suppression of heavy quark spectrum with high transverse momentum due to medium induced energy loss. The holographic method to compute this quantity needs a consideration of Wilson loop($\mathcal{C}$) traced out by a $q\bar{q}$ pair [10]. The Wilson loop is taken to lie along the light cone in the gauge theory. The gravity dual of this $q\bar{q}$ pair is represented as the two end points of a fundamental string, attached to the boundary of the bulk spacetime. Body of the string hangs along the radial direction and up to the horizon of aAdS black hole. The Wilson loop is mapped, in the dual theory, as the string world sheet. The jet quenching parameter ($\hat{q}$) is related to the thermal expectation value of the light-like Wilson loop operator, $\langle \mathcal{W}(\mathcal{C}_{light-like}) \rangle$ [11]. The holographic correspondence between thermal expectation value of the light-like Wilson loop operator in fundamental representation $\langle \mathcal{W}^{fund}(\mathcal{C}_{light-like}) \rangle$ and the exponential of the worldsheet action, e^{-S} leads us to obtain a working formula of $\hat{q}$ in dual gravity theory. Here, S stands for the worldsheet action of the fundamental string. The study of the qualitative behavior of $\hat{q}$ with respect to T and μ shows that, if μ is higher than a certain critical value, $\hat{q}$ decreases monotonically with all T whereas if μ is lesser than the same critical value, $\hat{q}$ is a multi-valued function of T in the lower T region and it decreases monotonically for with respect to T in higher T region. The multi-valued behavior of the jet-quenching parameter in low T region is related to the first order phase transition between confined phase and deconfined phase of hQCD model. Finally, we study the screening length (L_s) of $q\bar{q}$ pair in the frame work of hQCD model. It is defined as the maximum separation between a $q\bar{q}$ pair moving with a constant velocity in the plasma. If the separation between them exceeds L_s, they break off with no binding energy between them and thus they become screened in the QGP medium. As was pointed out in [12], the study of L_s requires the consideration of a time-like Wilson loop($\mathcal{C}_{time-like}$) as a world line of a $q\bar{q}$ pair. The computation of this quantity becomes simpler in the rest frame of $q\bar{q}$ pair. In this frame, the plasma flows with a constant velocity. Correspondingly, in the dual theory, the aAdS black hole background is boosted by a rapidity parameter. The holographic duals of the $q\bar{q}$ pair and the time like Wilson loop are constructed in a similar way as considered in the case of $\hat{q}$ parameter. However, in this case the body of the string hangs along the radial direction up to a certain position between the boundary and the blak hole horizon. When the q and $\bar{q}$ are separated beyond L_s, isolated strings are favorable energetically in the dual theory. Binding energy of q and $\bar{q}$ is related to the thermal expectation value of the time like Wilson loop operator, $\langle\mathcal{W}(\mathcal{C}_{time-like})\rangle$. Thereby, using the holographic mapping between $\langle\mathcal{W}^{fund}(\mathcal{C}_{time-like})\rangle$ and e^{-iS}, we calculate the binding energy in dual gravity. We obtain L_s from the boundary condition on radial coordinate of the background geometry. Subsequently, we study the behaviors of L_s and the binding energy. For a certain value of the chemical potential of the system, L_s varies with temperature in a definite way, consistent with result arising from the lattice QCD computations. In particular, for $\mu = 0.1$ our computation shows L_s behaves as $\frac{0.45}{T}$ whereas from lattice calculation it is $\frac{0.5}{T}$ [13]. The variation of L_s with respect to rapidity parameter indicates that the quark-antiquark pair dissociates at a lower temperature as it is moving with higher velocity [14]. If this qualitative behavior holds for QCD, it will have the consequence for quarkonium suppression in heavy ion collision. Additionally, our results show that for a smaller chemical potential the system allows for a larger L_s. The observation of binding energy reveals that for all values of quark-antiquark separation length less than L_s, there exists two separate branches of the binding energy. It is very important to note that the aAdS black hole geometry dual to hQCD gauge theory is too complicated to allow for a pure analytic approach. For all our results we had to use both numerical and analytical means.

The *other* model we consider is thermal $\mathcal{N} = 4\ SU(N)$ SYM plasma. A large number of heavy, static quarks are uniformly distributed over this hot plasma. In this set up, we calculate drag force on a heavy probe quark moving through the medium. Like other drag force calculations [6], we compute it holographically by constructing the appropriate gravity dual [8]. First, we construct the gravity dual of the quark distribution. In the dual theory, this represents large number of fundamental strings uniformly distributed over the bulk geometry. These strings are assumed to be non-interacting, static and infinitely long. One of the end points of each string is attached to the boundary and the body of the string is aligned along the radial direction. The bulk space time gets deformed due to the backreaction of the string distribution. We explicitly compute this *backreacted* geometry without doing any approximation. It turns out to be a deformed black hole in AdS space parametrized by it's mass and the density of strings and is stable under tensor and vector perturbation. As a result of the holographic computation, the drag force is expressible in terms of two parameters, e.g, density of heavy quarks and the temperature of the gauge theory. We find that the magnitude of the drag force increases with these parameters. We note that, in the study of the drag force on a heavy quark (say charm) passing through the QGP medium, the back-reaction of the plasma is usually neglected. In the context of $\mathcal{N} = 4$ SYM, our work [8] can perhaps serve as an attempt to compute such back-reaction effects.

The second part of this thesis includes our investigation which explores the AdS/CFT correspondence. Namely, here, we probe the strongly coupled bulk via computations in their weakly coupled boundary dual. In particular, by exploiting this strong-weak coupling duality, the transition probability among the states of the string theory on non-trivial background is studied. These states are the spherical brane like objects wrapped inside the sphere of either AdS_5 or S_5 of the $AdS_5 \times S^5$ geometry. In order to study transitions among string states using the weakly coupled gauge theory, we need to identify the states of the string theory with the gauge invariant operators of the dual theory. Once this is done, the gauge theory correlator with proper normalization gives the transition probability of the corresponding states. Following this idea, in this thesis we are particularly interested in a class of gauge/string duality, known as AdS_4/CFT_3 duality. According to this duality, the M theory on $AdS_4 \times S_7/Z_k$ is equivalent to $\mathcal{N} = 6$, $U(N) \times U(N)$ Chern-Simons gauge theory on the 3-dimensional boundary of the AdS_4 space. The gauge theory is called ABJM following the work of Aharony, Bergman, Jafferis and Maldacena [15]. As a further generalization, another theory was proposed by Aharony, Bergman, Jafferis, which is known as ABJ theory [16], by modifying the ABJM gauge group $U(N)_k \times U(N)_{-k}$ into $U(N_1)_k \times U(N_2)_{-k}$ for the same Chern-Simons matter fields, with $N_2 \geq N_1$. The states of our interest in M-theory are either spherical M2-brane growing in the AdS_4 or M5-brane wrapping on $S_5 \subseteq S_7$ and we like to study transition probability among them by using the AdS/CFT correspondence. Therefore in our work [17], we first extend our study to construct the correct gauge invariant operator in the ABJ theory by generalizing the Schur polynomial constructed for ABJM theory. We then find out the realization of the duality between giant gravitons and the Schur polynomials as the gauge invariant operators for both ABJ(M) theory. Further, we study the transition probabilities among giant gravitons as well as between giant gravitons and ordinary gravitons by analyzing the corresponding gauge theory correlators. Finally, we consider a particular non-trivial background produced by an operator with an R-charge of $O(N_2)$ and find, in presence of this background, due to the contribution of the non-planar corrections, the large (N_1, N_2) expansion is replaced by $\frac{1}{(N_1+M)}$ and $\frac{1}{(N_2+M)}$ respectively, where M is the number of columns in the representative Young diagram of the operator and is of order N.

In the limit $N = 2$, one finds some extra symmetry due to the fact that the $\mathbf{2}$ and $\bar{\mathbf{2}}$ representation of $SU(2)$ are equivalent. This extra symmetry enhances the supersymmetry of ABJM theory from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. Finally, in the special limit $N = 2$ and $k = 2$, ABJM theory becomes equivalent to another independent world volume theory of two M2 branes known as BLG theory, named after Bagger, Lambert, Gustavson [18], [19]. The gauge field as well as the matter fields in BLG theory are valued in a completely anti-symmetric ternary product satisfying the so-called fundamental identity and a Euclidean metric. This ternary product is also known as 3-algebra. Although this 3-algebra plays an important role in the formulation of multiple M2 brane theory, its rich mathematical structure makes the algebra very important to its own right. The consistent generalization of Kac-Moody and (centerless) Virasoro 2-algebras into respective 3-algebras motivates us to construct a further extension. In this work [20] we explicitly construct a classical w_∞ 3-algebra and show that our relation satisfies the 3-algebra "Fundamental Identity condition". We start with commutation relation defining $W_{1+\infty}$ algebra which is basically a higher spin generalization of Virasoro algebra. We define a star-product among the generators and using both of them we construct a 3-bracket. There is an explicit dependence scaling parameter q in this bracket. Finally we multiply all the generators with an arbitrary scale factor β and take the double scaling limit $q \rightarrow 0$ and $\beta \rightarrow \infty$ such that $q\beta^2 = 1$. By this method we obtain w_∞ 3-algebra which is completely anti symmetric in all the mode indeces of w_∞ generators and satisfies Fundamental Identity condition. We also figure out the geometric realization of the w_∞ 3-algebra. It turns out that the 3-algebra generators can be identified with the modes of the deformations of a geometry which is a direct product of 2-torus with a point.

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List of Publications

***[1]** *"Some aspects of QGP phase in a hQCD model"*,
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***[2]** *"Correlators of Giant Gravitons from dual ABJ(M) Theory"*,
T. K. Dey and **Shankhadeep Chakrabortty**,
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***[3]** *"Dissipative force on an external quark in heavy quark cloud"*,
Shankhadeep Chakrabortty,
Phys. Lett. B **705**, 244 (2011), arXiv:1108.0165 [hep-th].

[4] *"Some BPS configurations of the BLG Theory"*,
S. P. Chowdhury, K. Ray and **Shankhadeep Chakrabortty**,
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[5] *"Moduli and BPS configurations of the BLG theory"*,
S. P. Chowdhury, T. Koppe, K. Ray and **Shankhadeep Chakrabortty**,
Phys. Lett. B **695**, 285 (2011), arXiv:1005.3707 [hep-th].

[6] *"Proof of universality of electrical conductivity at finite chemical potential"*,
Sayan K. Chakrabarti, Sachin Jain, **Shankhadeep Chakrabortty**,
JHEP **1102**, 073 (2011) arXiv:1011.3499 [hep-th].

***[7]** *"w_∞ 3-algebra "*,
Alok Kumar, Sachin Jain and **Shankhadeep Chakrabortty**,
JHEP **0809:091**, (2008), arXiv:0807.0284 [hep-th].

A (*) indicates papers on which this thesis is based.

1
Introduction

1.1 Overview

Our understanding of strongly coupled gauge and gravity theories has increased considerably following Maldacena's proposal of AdS/CFT correspondence. This conjectured correspondence aims to establish an equivalence between a field theory and the string theory formulated in a curved background by relating a strongly coupled theory with a weakly coupled one. It is this dual nature of the correspondence that has opened up an avenue to probe strongly coupled theories via computations in their weakly coupled duals. Though the dual of the QCD is yet unknown, the theories for which this correspondence has been developed do share some similarities with the QCD. Consequently, one hopes that some of the universal features of these strongly coupled theories may also shed some light when the dual of the QCD is discovered.

In this thesis, we apply the AdS/CFT correspondence to address some issues in the strongly coupled gauge theories as well as in the string theory. The first part of our investigation is inspired by the recent results of the Relativistic Heavy Ion Collider (RHIC) at Brookhaven. It is believed that a strongly coupled quark-gluon plasma phase of matter has been discovered in this experiment. One of the main evidences favoring this is that the medium can be described by an ideal hydrodynamics leading to a low shear viscosity of the fluid. Possibility of having large jet quenching factor with considerably high energy loss per unit length of moving partons in this medium is also believed to be a consequence of the strong coupling effects. While none of these arguments are definitive, in this thesis, we will take a view that the strong coupling effects in the plasma are important and we need to improve our theoretical understanding. Following this spirit, in the first part of the thesis, we introduce a holographic QCD model discussed in [178]. The dual of this model is described by an asymptotically anti deSitter (AdS) black hole in Einstein-Maxwell-Dilaton system. This model has some similarities with the QCD. We will have occasions to discuss this in detail in the later chapters. By employing the holographic techniques, we calculate the drag force, the jet quenching parameter, the screening length and the binding energy of an external quark-antiquark pair within this model. We hope that some of the predictions arising from our analysis may be universal in strongly coupled theories including the QCD. We subsequently investigate a model within the well studied $\mathcal{N} = 4$ super Yang-Mills

(SYM) plasma. This admits a gravity dual which is an AdS-Schwarzschild black hole. We further introduce, instead of a single probe quark, a quark cloud of constant density. Our motivation here is to first calculate the dual back-reacted geometry. Using this new geometry, we holographically calculate the drag force on a single quark (from the cloud) when it is in motion. We note here that, generally in the study of the drag force on a heavy quark (say charm) passing through the QGP medium, the back-reaction of the plasma is usually neglected. Our work serves as an attempt to capture such effects at least for the SYM plasma.

In the second part of this thesis, our aim is to exploit the other side of the duality conjecture. Here we investigate some non-perturbative features of the gravity by looking at their weakly coupled boundary duals. It is well known that the type IIA string theory (or M-theory) contains non-perturbative semi-classical objects called the giant gravitons. Within the AdS_4/CFT_3 duality, we consider the transitions among the giant gravitons and also between the giants and the normal gravitons. This is done by mapping these computations into a set of equivalent computations on the boundary. The boundary gauge theory is now given by the $U(N_1) \times U(N_2)$, $\mathcal{N} = 6$ Chern-Simon-matter theory proposed by Aharony, Bergman and Jafferis. We first identify the giant gravitons with the Schur polynomials in the gauge theory and further study the transition probabilities by considering correlators involving the Schur polynomials. Finally, we consider a particular non-trivial background produced by an operator with large R-charge and find that, in the presence of this background, due to the contributions of the non-planar corrections, the large N_1, N_2 expansions get appropriately modified. For a particular choice of the gauge group, $N_1 = N_2 = 2$ the supersymmetry of the Chern-Simon-matter theory enhances from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. Further in this theory, the matter and the gauge field take values in a 3-algebra [130]. Motivated by the consistent generalization of Kac-Moody and (center-less) Virasoro 2-algebras into respective 3-algebras, we end this thesis with a construction of the w_∞ 3-algebra. Interestingly, this w_∞ 3-algebra satisfies the "Fundamental Identity condition" which can be thought of a generalization of the Jacobi identity of ordinary 2-algebra.

Before we go on to present our results in the later chapters, in the next section we give a brief introduction to the string theory and the Dp branes. Subsequent sections contain an elaboration of Maldacena's conjecture mainly focusing on the AdS_5/CFT_4, various tests and generalizations of the correspondence. We also provide some selective applications of this conjecture to prepare ourselves for the later chapters. This chapter ends with a discussion on giant gravitons of M-theory/type IIA theory and the AdS_4/CFT_3 version of the conjecture. A plan of the rest of the thesis has been provided at the end of this chapter.

1.2 D brane

1.2.1 A brief description of strings and D-branes

The notion of string theory is based on a radical idea that the elementary particles are actually different vibrational modes of an one-dimensional extended objects called *strings*.

They can be either *open* or *closed* depending on the boundary conditions [1–6]. Strings are characterized by a parameter, energy per unit length of the string, known as string tension (T_s)

$$T_{\rm s} \equiv \frac{1}{2\pi\alpha'} \quad \text{with} \quad \alpha' \equiv l_{\rm s}^2. \tag{1.1}$$

Here l_s is the string length and hence α' has a dimension of (length)^2. Excitement in this subject is primarily due to the fact that along with the other mediators of the interactions, string theory contains a spin two massless excitation in its spectrum. This massless excitation, known as the graviton, mediates the gravitational interaction. To reconcile the correct strength of this interaction, $\sqrt{\alpha'}$ has to be of the order of $10^{-33}cm$ which sets the length scale of string theory in a quantitative way. The interactions among the strings are determined by another parameter g_s which is dimensionless and related to the expectation value of a field which appears with the massless spectrum of the strings, known as dilaton. Physical consistency requires the five different types of superstring theories in 10 dimensions, namely the $typeI$, $typeIIA$, $typeIIB$, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. It has been shown that these five different string theories are, in fact, connected with each other via the S, T, U dualities [7, 8]. The main emphasis of this thesis lies on the $typeII$ theories. The worldsheet of the $typeII$ theories contain eight scalars (eight transverse coordinates of the string) and eight Majorana fermions. Eight Majorana fermions can be further considered as sixteen Majorana-Weyl fermions. Eight of them have left handed chirality (left-moving), and the rest of the fermions have right handed chirality (right- moving). Both of these right-moving and left-moving fermions enjoy periodic and anti periodic boundary conditions. We refer to the periodic boundary condition as the Ramond (R) sector and the anti-periodic boundary condition as the Neveu-Schwarz (NS) sector [9–11]. Based on the boundary conditions imposed on two different types of worldsheet fermions, we have four independent sectors of the spectrum, namely (R-R), (NS-NS), (R-NS), (NS-R). The (R-NS) and (NS-R) sectors give spacetime fermions. The massless spectrum of the (NS-NS) sector consists of spacetime boson, such as the graviton $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, the dilaton ϕ. The massless spectrum of the (R-R) sector also contains the spacetime boson such as the $p+1$ form fields A_{p+1}. To remove the unphysical tachyonic ground state from the spectrum, GS projection [12] is implemented in two different ways in the $R-R$ sector, and this procedure gives two different $typeII$ theories, called as the $typeIIA$ (even p) and the $typeIIB$ (odd p).

Besides the one dimensional strings, string theory contains p dimensional solitonic objects which are charged under A_{p+1}. These solitons can be thought of as a p dimensional hypersurface embedded in the space time on which the open strings can end. They are known as Dirichlet p-branes (Dp-branes) [13, 14]. The string end points attached to the Dp brane satisfy the Neumann boundary condition along the $p+1$ space-time directions and the Dirichlet boundary conditions along the $(9-p)$ spatial directions. Dp brane interacts with the open and the closed string. The minimal coupling of the Dp-branes with the form fields can be written as

$$\mu_p \int A_{p+1}, \tag{1.2}$$

where μ_p is the charge of the brane and is related to the tension T_p of the brane as

$$\mu_p = T_p(2\pi)^{\frac{7}{2}} {l_s}^4 g_s, \tag{1.3}$$

where,

$$T_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{1.4}$$

T_p is determined from the string amplitude of a process describing the exchange of closed strings between a pair of Dp-branes [13, 14]. There are two complementary pictures of the Dp brane, namely, a closed string picture (supergravity description) and an open string picture (supersymmetric gauge theory description). In the supergravity description, we consider a stack of N number of Dp branes charged under the $R-R$ form fields. This configuration is stable because the Dp branes are BPS object and consequently they do not exert force among each other. In this scenario, for large N, Dp-brane has solitonic, static description in terms of a metric and the $R-R$ form potential. On the other hand, the open string modes live on the Dp brane. The dynamics of these modes is encoded in a quantum field theory on the Dp brane. In particular, the massless vector modes along with the fermions and the transverse scalar fields furnish a supersymmetric gauge theory [17]. The detail exploration of these two complementary pictures give rise to the idea of gauge/gravity duality which is also known as AdS/CFT correspondence [18–21]. This duality, long expected to appear from the works of 't Hooft and Susskind [15, 16], got a strong support from a proposal due to Maldacena [18]. In the next couple of sub-sections we elaborate upon the supergravity as well as the gauge theory descriptions of the Dp brane leading to the correspondence.

1.2.2 D-branes as supergravity solution

In the closed string picture, the Dp brane is a solitonic solution of the $typeII$ theory. Among many important properties satisfied by the Dp brane solution, we note that it is a half BPS object preserving 16 of 32 spacetime supersymmetry, carries the charge of the ($R-R$) form field and it is isotropic in the transverse directions. The solution of our interest can be constructed by studying the effective low energy $typeII$ action that involves the massless bosonic modes, such as the metric $g_{\mu\nu}$, the dilaton ϕ, the $p+2$ form field strengths F_{p+2} from the ($R-R$) sector, the ($NS-NS$) 3-form fields and their supersymmetric partners [22–25]. In Einstein frame, the action can be written as,

$$I_E = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{|g|}(R - \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\sum_p \frac{1}{(p+2)!} e^{a_p \phi} F_{p+2}^2 + ...). \tag{1.5}$$

The dots in the action stand for the ($NS-NS$) 3 form fields and the fermionic fields. The dilatonic coupling a_p is defined as,

$$a_p = -\frac{1}{2}(p-3). \tag{1.6}$$

The ten dimensional gravitation constants, G_{10} is related to previously mentioned parameters g_s and l_s as follows,

$$G_{10} = 8\pi^6 g_s^2 l_s^8. \tag{1.7}$$

The Euler-Lagrange's equation of motion for the $g_{\mu\nu}$, dilaton ϕ and the $p+2$ form field strengths F_{p+2} can be written respectively,

$$\begin{aligned}
R^{\mu}_{\nu} &= \frac{1}{2}\partial^{\mu}\phi\partial_{\nu}\phi + \frac{1}{2(p+2)!}e^{a_p\phi}((p+2)F^{\mu\xi_2\ldots\xi_{p+2}}F_{\nu\xi_2\ldots\xi_{p+2}} - \frac{p+1}{8}\delta^{\mu}_{\nu}F^2_{p+2}),\\
\nabla^2\phi &= \frac{1}{\sqrt{g}}\partial_{\mu}(\sqrt{g}\partial_{\nu}\phi g^{\mu\nu}) = \frac{a_p}{2(p+2)!}F^2_{p+2},\\
&\partial_{\mu}(\sqrt{g}e^{a_p\phi}F^{\mu\nu_2\ldots\nu_{p+2}}) = 0.
\end{aligned} \tag{1.8}$$

The form field tensor F_{p+2} in the action is considered as electric. Using electric/magnetic duality in this set up, it is easy to define it's magnetic Hodge dual

$$\tilde{F}_{10-p-2} = e^{a_p\phi} * F_{p+2}. \tag{1.9}$$

It can be shown that under the following set of duality transformations,

$$a_p\phi \rightarrow -a_p\phi, (p+2) \rightarrow (10-p-2), F_{p+2} \rightarrow \tilde{F}_{10-p-2}, \tag{1.10}$$

the equations of motion (1.8) remain invariant [25]. Now we consider a particular class of supergravity solutions depicting the $D3$-brane. The $D3$-brane solution plays a major role to understand the most well studied example of AdS/CFT. It is important to note that the $D3$-brane carries the charge of the self dual five form field strength. Generalizations to $p \neq 3$ branes are rather straightforward. Imposing the self-duality condition $F_5 = *F_5$, a solution of (1.8) in the Einstein frame can be found as,

$$\begin{aligned}
ds^2 &= H^{-1/2}(-fdt^2 + \sum_{i=1}^{3}(dx^i)^2) + H^{1/2}(f^{-1}dr^2 + r^2(d\Omega_5{}^2)),\\
H &= 1 + (\frac{h}{r})^4, f = 1 - (\frac{r_0}{r})^4,\\
h^8 + r_0^4 h^4 &= \frac{Q^2}{16}, \ \ \phi = \text{Constant}, \ \ g_s = e^{\phi}.
\end{aligned} \tag{1.11}$$

h can be determined by the integration constants, r_0 and Q. Here we have considered (t, x_1, x_2, x_3) as the world-volume coordinates of the $D3$-brane and $r = \sum_{i=1}^{6} {y_i}^2$, where $\{y_i\}$ are the transverse coordinates which are orthogonal to the world-volume. For electric ansatz, the solution for five form field strength is obtained as,

$$F_{ti_1i_2i_3r} = \epsilon_{i_1i_2i_3}H^{-2}\frac{Q}{r^5}. \tag{1.12}$$

To make the form field self dual we replace $F_5 \longrightarrow F_5 + *F_5$. The parameter Q is related to the charge μ_3 of the $D3$ brane,

$$\mu_3 = \frac{\Omega_5 Q}{(2\pi)^{\frac{7}{2}} l_s^4 g_s}, \tag{1.13}$$

where Ω_5 is the volume of the 5-sphere, r_0 turns out to be the horizon of the metric where $f(r)$ vanishes. The singularity is localized at $r = 0$. The solution given in (1.11) is supersymmetric only when r_0 vanishes, which is then called the extremal $D3$ brane. The explicit form of a single extremal $D3$ brane solution is

$$\begin{aligned} ds^2 &= H^{-1/2}(-dt^2 + \sum_{i=1}^{3} (dx^i)^2) + H^{1/2}(dr^2 + r^2(d\Omega_5{}^2)), \\ h^4 &= \frac{Q}{4}, \;\; f(r) = 1, \;\; H = 1 + \frac{Q}{r^4}. \end{aligned} \tag{1.14}$$

Following eqn (1.14), one can construct an effective low energy supergravity solution for the N coincident extremal $D3$-branes. In this case, equation (1.3) should be generalized for N number of branes and hence

$$\mu_3 = NT_3(2\pi)^{\frac{7}{2}} l_s{}^4 g_s. \tag{1.15}$$

Now using (1.15), (1.13) and (1.4) we can write Q in terms of N, l_s and g_s.

$$Q = 16N\pi g_s l_s{}^4. \tag{1.16}$$

With these, the solution takes the final form:

$$\begin{aligned} ds^2 = H^{-1/2}(-dt^2 + \sum_{i=1}^{3} (dx^i)^2) + H^{1/2}(dr^2 + r^2(d\Omega_5{}^2)) \\ H = 1 + \frac{l^4}{r^4}, l^4 = 4N\pi g_s l_s{}^4. \end{aligned} \tag{1.17}$$

This N number of extremal $D3$-branes span along the three spatial directions x_1, x_2, x_3. From the gravitational point of view, this stack of branes is similar to that of a point particle with mass $M \sim NT_3$ in the six transverse directions. Therefore the metric depends only on the radial coordinate r. The solution (1.17) approaches the Minkowski geometry in the asymptotic limit ($r \gg l$)

$$ds^2{}_{Minkowski} = (-dt^2 + \sum_{i=1}^{3} (dx^i)^2 + dr^2 + r^2(d\Omega_5{}^2)), \tag{1.18}$$

with a small correction term which is of the order of $\frac{l^4}{r^4}$. Using the relations, $G_{10} \propto g_s{}^2 l_s{}^8$ and $M \propto NT_3 \propto \frac{N}{g_s l_s{}^4}$, we get $\frac{l^4}{r^4} \propto \frac{G_{10}M}{r^4}$ which can be thought of as the gravitational potential due to a massive object of mass M in the six transverse directions $\{y_i\}$. Thus l

is considered as the characteristic length scale of gravitational effect of N number of 3-branes. This effect is very strong in the region $r \ll l$ where the metric (1.17) becomes $AdS_5 \times S^5$ geometry.

$$ds^2{}_{AdS_5 \times S^5} = \underbrace{\frac{r^2}{l^2}(-dt^2 + \sum_{i=1}^{3}(dx^i)^2) + \frac{l^2}{r^2}dr^2}_{AdS_5} + \underbrace{l^2(d\Omega_5{}^2)}_{S^5}. \tag{1.19}$$

Hence we conclude that the far away geometry, sourced by N number of coincident $D3$ branes is a flat ten-dimensional Minkowski space, whereas close to the branes it represents a 'throat' geometry of the form $AdS_5 \times S^5$. An important property of this metric (1.17) is it's non-constant red shift factor, $H(r)^{-\frac{1}{4}} = (1 + \frac{l^4}{r^4})^{-\frac{1}{4}}$. The energy E_p of the configuration measured by an observer at a fixed position r differs from the energy E_∞ measured by an observer at infinity by this redshift factor and are related by

$$E_\infty = H(r)^{-\frac{1}{4}} E_p. \tag{1.20}$$

Therefore any excitation near the throat geometry appears energetically very small with respect to an observer at infinity. In the low energy limit, these exited modes decouple from the massless supergravity modes in the flat space time. So we get two decoupled theories. One is the close string theory with all higher excitations ($typeIIB$) near $AdS_5 \times S^5$ geometry and the other is the free supergravity in asymptotically flat spacetime.

We record here that AdS_5 is a space-time with a constant negative curvature. It can be geometrically realized as an embedding of a hypersurface in the six dimensional flat space.

$$X_0^2 + X_5^2 - \sum_{i=1}^{4} X_i^2 = l^2, \tag{1.21}$$

The metric of this space-time is obtained as,

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2. \tag{1.22}$$

The isometry group of AdS_5 is $SO(2,4)$. In addition to that, the S^5 has an isometry group $SO(6) \sim SU(4)$. We know that the superstring theories living on $AdS_5 \times S^5$ have 32 supercharges since the spacetime is maximally symmetric. These symmetries combine into the super-Lie group PSU(2,2|4). More details can be found in [21, 25].

So far we have discussed the N $D3$-branes in terms of the low energy supergravity description. This is the picture seen by the closed strings propagating in the bulk. In the next subsection, we study the same stack of N number coincident $D3$ branes from the open strings perspective.

1.2.3 D-branes and gauge theory

We pointed out earlier, the Dp brane has an equivalent description in terms of the open strings. To further elaborate on this, let us consider N number of coincident Dp branes in 10 dimensional flat spacetime. Like in previous subsection, we restrict our analysis for $p = 3$ in $typeIIB$ theory. According to the perturbative string theory, two kinds of excitation modes arise. One is the open string modes living on the world volume of the $D3$- brane and the other is the closed string modes living in the bulk of the theory. Indeed two different modes interact with each other [21]. The total action can be written as,

$$S_{total} = S_{brane} + S_{bulk} + S_{int}. \tag{1.23}$$

While the S_{brane} contains the $U(N)$ gauge fields, the Majorana fermions in the adjoint representation of $U(N)$ and the massive open string modes,the S_{bulk} includes massless and massive modes of the closed string sector. It is important to note that, for a single $D3$-brane, the maximally supersymmetric gauge theory living in the $3+1$ dimensional world volume corresponds to a single species of $U(1)$ gauge field. For N number of coincident $D3$ brane, one can associate a non-dynamical degree of freedom, namely the Chan-Paton level, with both the ends of the open strings. For each end of the open strings, the Chan-Paton level runs from 1 to N, giving rise to N^2 number of $U(1)$ gauge fields of different species. But N^2 is the dimension of the adjoint representation of $U(N)$ gauge group, so indeed we find the maximally supersymmetric $U(N)$ gauge theory living in the $3+1$ dimensional world volume of the N parallel $D3$-branes. Again there is a $U(1)$ part associated to the trace, which constitutes an Abelian sub-sector of $U(N)$. Excitation related to this $U(1)$ provides the dynamics of the brane's center of mass. Due to the overall transnational invariance, this $U(1)$ part decouples from the remaining $SU(N)$ part describing relative dynamics among the branes in the stack. Henceforth, we consider that the stack of N number of $D3$ branes corresponds to a $SU(N)$ gauge theory [26]. The effect of the low energy limit in 10 dimensions, i.e, taking $\alpha^{'} \to 0$ limit at finite g_s and N results into a decoupling between the massless and the massive modes of open string sector. However, interactions between the massless modes in this sector, controlled by the g_s, remain non-trivial. Finally, the massless effective gauge theory living on the world volume of the $D3$ branes is equivalent to the dimensional reduction of the $SU(N), \mathcal{N} = 1$ super Yang-Mills(SYM) gauge theory in 10 dimensions to 3+1 dimensions. Up to the two derivative level, this is exactly the $3+1$ dimensional $SU(N), \mathcal{N} = 4$ super Yang-Mills gauge theory. In particular, via dimensional reduction the ten-dimensional gauge field A_μ reduces to a four-dimensional gauge field A_a and six scalar fields ϕ_i. The ten-dimensional Majorana spinor reduces to the four 4-dimensional Weyl spinors λ_α^A, where A = 1, 2, 3, 4 is the R symmetry (will be explained shortly) index and $\alpha = 1, 2$ is the Weyl index. The action of the $3+1$ dimensional $SU(N)$,

$\mathcal{N}=4$ super Yang-Mills gauge theory is given by [27],

$$S_{\mathcal{N}=4} = \int d^4xTr\{-\frac{1}{2g_{YM}^2}\sum_{a,b} F^{ab}F_{ab} + \frac{\theta_I}{8\pi^2}\sum_{a,b} F^{ab}\tilde{F}_{ab} - i\sum_{a}\bar{\lambda}\bar{\sigma}^a D_a\lambda - \sum_{a}(D_a\phi)^2 + \frac{g_{YM}^2}{2}\sum_{i,j}[\phi_i,\phi_j]^2 + \text{fermion-scalar term}\}, \tag{1.24}$$

where g_{YM} is the SYM gauge coupling and θ_I is the instanton angle. The trace is taken over the $SU(N)$ indices. The gauge coupling is determined by the string coupling,

$$g_{YM}^2 = g_s. \tag{1.25}$$

$\mathcal{N} = 4$ SYM gauge theory further exhibits a $\mathcal{R}$-symmetry. This is a global rotational symmetry among the 6 scalars ϕ_i, and thus has a symmetry group $SO(6)_R = SU(4)_R$. The scalars have mass dimension 1 and the fermions have mass dimension $\frac{3}{2}$, thus all the terms in the action together with the measure, are dimensionless. This renders the theory to be scale invariant. The combination of this scaling symmetry and the Poincarè symmetry in the four dimensions gives the conformal symmetry $SO(2,4)$. $\mathcal{N} = 4$ supersymmetry in four dimensional space has 16 supercharges, but the conformal superalgebra doubles the number of the supersymmetry generators. So the total number of the supercharges becomes 32. Finally, together with the supersymmetry, the $\mathcal{R}$-symmetry and the conformal symmetry, the superconformal group of the four dimensional $SU(N)$, $\mathcal{N}=4$ SYM gauge theory becomes $PSU(2,2|4)$. This is also the supergroup of $AdS_5\times S^5$ geometry discussed in the previous subsection.

The rest of the action also simplifies in the low energy limit because of the following reasons. All the interaction terms in the S_{int} as well as the higher order corrections in the S_{bulk} come with positive powers of the ten-dimensional Newton's constant G_{10}. G_{10} is proportional to κ_{10}^2 where $\kappa_{10} \sim g_s\alpha^{'2}$. So in the low energy limit, the S_{int} becomes zero and S_{bulk} describes a free supergravity in the ten-dimensional bulk spacetime.

We note that in the low energy limit, the closed string picture of the extremal $D3$ brane gives a decoupling between the $typeIIB$ superstring theory (all excitations are allowed) near the throat geometry of $AdS_5 \times S^5$ and the super gravity in the asymptotically flat spacetime. In the open string picture of the extremal $D3$ brane, the low energy limit results into a decoupling between the $\mathcal{N} = 4$ SYM and the free supergravity in the bulk. As the both pictures describe the same extremal $D3$ brane, hence identification of the super gravity modes in both sides naturally leads to a conclusion that the $\mathcal{N} = 4$ SYM is dual to the $typeIIB$ superstring theory on $AdS_5 \times S^5$. The systematic exploration of this relationship builds the AdS/CFT correspondence which we discuss in the following section.

1.3 AdS/CFT correspondence

From our previous discussion it is clear that the action of the system is given by the contribution from the branes, the bulk and the interactions between them. In the low energy limit ($\alpha^{'} \to 0$), we have two decoupled theories, namely, the $\mathcal{N} = 4$, $SU(N)$ SYM living on the stack of branes and the free supergravity in the bulk. In the closed string picture, the low energy limit leads to a scenario where the same stack of N number of $D3$ branes can be described by two decoupled theories, namely, the $typeIIB$ superstring theory near the throat geometry of $AdS_5 \times S^5$ and the supergravity in the asymptotically flat spacetime. If we compare the low energy limit of both the open and the closed string pictures and take into account that both of them give the description of the same N number of $D3$ branes, we can match the free supergravities in the bulk for these two descriptions and also identify the gauge theory on the branes with the string theory on the throat, leading to the Maldacena conjecture [18]:

$\mathcal{N} = 4$, $SU(N)$ *super Yang-Mills theory in four dimensional Minkowski space is dual to type IIB superstrings theory on* $AdS_5 \times S^5$ *with* N *unit of R-R 5-form flux on* S^5.

The parameters of the theory are related as,

$$
\begin{aligned}
&g_s = g^2_{YM}, \\
&g^2_{YM} N = \lambda = \frac{l^4}{4\pi {l_s}^4}
\end{aligned}
\tag{1.26}
$$

The above statement is a strong version of Maldacena's AdS/CFT correspondence as it is a correspondence for any values of the parameters N and λ. However, it is very difficult to implement this version at the computational level. The very lack of a consistent formulation of full quantum string theory on a curved space such as $AdS_5 \times S^5$ plays the role of hindrance. Nevertheless, there are precise non-trivial limits restricting the range of the parameter space where explicit computations can be performed. For example, in the 't Hooft limit, ($N \to \infty, \lambda$ is fixed and finite) the SYM reduces to only planar sector and the string theory has to account for only tree level diagram ($g_s \sim 1/N \to 0$) whereas all the higher order loops get suppressed as sub-leading corrections. This particular limit is also known as the planar limit. There is another limit of this correspondence, known as the strong coupling limit ($\lambda \gg 1$), where the string theory side is more tractable. In this limit, the $\mathcal{N} = 4$ gauge theory becomes strongly coupled and thus enters into the non-perturbative regime. According to the relation (1.26), $\lambda \gg 1$ limit of the gauge theory translates into $l \gg l_s$ limit of the string theory. As a result, the string theory sector of the correspondence dramatically simplifies to the corresponding supergravity. $l \gg l_s$ limit can be equivalently understood as ${m^2}_s \gg R$, where the Ricci scalar $R \sim \frac{1}{l^2}$ provides the curvature scale of the spacetime and m_s is typically the mass scale of the string. In a low energy process, ${m^2}_s \gg R$ suggests us to keep only the massless string modes, i.e. the supergravity modes of the theory. Thus to summarize, 't Hooft limit suppresses the quantum nature of dual string theory and the strong-coupling limit reduces its stringy effect.

Moreover with the simultaneous consideration of the both limits, the full string theory reduces to a ten dimensional classical theory of supergravity.

In many cases, it is convenient in many cases to express the fields in the ten dimensional gravity theory into the tower of fields in AdS_5. This can be done by compactifying the ten dimensional action on S^5. This approach renders the original duality into an equivalence between the four dimensional, $\mathcal{N} = 4$ SYM at strong coupling and the gravity in AdS_5. This is commonly refereed to as AdS_5/CFT_4 correspondence. The advantage in this approach is twofold. Firstly, it makes the realization of the holographic principle manifest with the bulk spacetime being AdS_5 and the boundary being four-dimensional Minkowski spacetime. Secondly, this provides a unified footing to many different examples of the gauge/gravity duality. With the dimensional reduction on S^5 the ten dimensional action takes the following form,

$$S = \frac{1}{16\pi G_5}\int d^5x[\mathcal{L}_{gravity} + \mathcal{L}_{matter}]. \tag{1.27}$$

$\mathcal{L}_{gravity}$ is the well-known Einstein-Hilbert Lagrangian in five dimensions with a negative cosmological constant $\Lambda = -\frac{6}{l^2}$

$$\mathcal{L}_{gravity} = \sqrt{-g_{AdS_5}}[\mathcal{R} + \frac{12}{l^2}]. \tag{1.28}$$

$\mathcal{L}_{matter}$ is the Lagrangian sourced by the matter fields and G_5 is the five dimensional Newton's constant satisfying the following relation,

$$G_5 = \frac{G_{10}}{\pi^3 l^5}. \tag{1.29}$$

It is important to note that the AdS/CFT correspondence is a strong / weak type duality. In the non-perturbative gauge theory regime ($\lambda \gg 1$), the string theory lives on a weakly curved background ($l \gg l_s$) and is tractable at the level of computation. On the other hand, the perturbative regime of gauge theory ($\lambda \ll 1$) is computationally tractable but the string theory lives on a highly curved back ground ($l \ll l_s$) and goes beyond perturbative analysis. *In this thesis we will explore various aspects of both sides of this duality.* We will study certain non-perturbative aspects of the finite temperature gauge theory of the strong interaction via the the computation in the dual weakly coupled gravity theory. We will also analyze the transitions between the typical nonperturbative semiclassical objects called the giant gravitons (explained later) via the computations in the weakly coupled dual gauge theory. In the latter case of study, we exploits the AdS_4/CFT_3 correspondence. Here we briefly elucidate the correspondence and the limit we are interested in. The detailed discussion of this duality will be provided later.

The M-theoretic version of the AdS_4/CFT_3 correspondence: $U(N)_k \times U(N)_{-k}$, $\mathcal{N} = 6$ *three dimensional Chern-Simon gauge theory is dual to the M-theory on* $AdS_4 \times S_7/\mathbb{Z}_k$ *(*$k \ll N^{\frac{1}{5}}$*)*, where k is the Chern Simon level that takes only integral values.

We can compactify the M-theory on $AdS_4 \times S_7/Z_k$ to the $typeIIA$ string theory on $AdS_4 \times CP_3$. The duality takes the following form.

The string-theoretic version of the AdS_4/CFT_3 correspondence: $U(N)_k \times U(N)_{-k}$, *$\mathcal{N}=6$ three dimensional Chern-Simon gauge theory is dual to the $typeIIA$ string theory on $AdS_4 \times \mathbb{CP}_3$ ($N^{\frac{1}{5}} \ll k \ll N$),*

The illuminating part of the AdS_4/CFT_3 duality is its nature of again being strong/weak type. The Chern-Simon level k comes out as an overall factor in the action of the gauge theory. Eventually $g_{CS}^2 \equiv \frac{1}{k}$ becomes the coupling constant of the theory. For $k \gg 1$, the theory is weakly coupled and fits with the perturbative analysis. However, for both versions of the duality, we can define a 't Hooft coupling constant $\lambda = \frac{N}{k}$. Thus the 't Hooft limit, which is actually the planar limit of the theory, becomes $N \to \infty$, $k \to \infty$, where λ is kept fixed. By observing the strong/weak nature of the duality we can further take a limit in which the gauge theory has a free field description.

$$k \gg N, \lambda \ll 1, \tag{1.30}$$

Though a direct proof of this conjecture is still lacking, in the following we only provide some evidences in support of the correspondence focusing only on the AdS_5/CFT_4 version. Our discussion is mainly based on the matching of the symmetry, the spectrum and the correlators from the both sides.

1.3.1 Matching of the symmetry

We have already seen, in the low energy closed string picture, the isometry of $AdS_5 \times S^5$ spacetime and the spacetime supersymmetry combine in a super-Lie group PSU(2, 2|4). In the low energy open string scenario, the combination of the conformal symmetry, the $\mathcal{R}$ symmetry and the supersymmetry forms the same superconformal group. Moreover there is also a matching of symmetries in the non-perturbative sector of both sides of the duality. $\mathcal{N}=4$ SYM has a $SL(2,Z)$ S-duality symmetry. This symmetry transformation acts on the complex gauge coupling $\tau_{gauge} = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$, where θ is the instanton angle. The $typeIIB$ string theory is invariant under the same S-duality which acts on the axion-dilaton complex coupling $\tau_{string} = \frac{\xi}{2\pi} + ie^{-\phi}$, where ξ is the axion field.

1.3.2 Matching of the spectrum

The validity of the AdS/CFT correspondence is supported by the fact that the supersymmetric group of both sides of the duality is $PSU(2,2|4)$. Accordingly, the representation of the group should have a one to one correspondence in both sides of the duality. This realization leads to an idea that the gauge invariant local operators in the gauge theory maps into the local fields in the dual gravity. Here we discuss a particular sector of the gauge theory multiplet and its dual gravity multiplet and establish the operator-field correspondence [21].

The four dimensional, $SU(N), \mathcal{N}=4$ SYM theory has four supersymmetry generators Q^A_α (and their complex conjugates $\bar{Q}_{\dot{\alpha}A}$), where $\alpha = 1,2$ is the Weyl-spinor index and A transforms as the **4** of the $SU(4)_R$ $\mathcal{R}$ symmetry group. The gauge theory has a unique vector multiplet which includes a vector field A_μ (μ belongs to $SO(1,3)$), four complex Weyl fermions λ^A_α, and six real scalars ϕ^i which transform as **6** of $SU(4)_R$. Using the field contents of the vector multiplet, one can construct the local operators in this theory as a gauge invariant observable. Since the fields in this theory transform in the adjoint representation of $SU(N)$, such operators involve a product of traces of the product of the fields. The operators are divided into two sectors, namely, the single trace operators and the multi-trace operators. Here we only concentrate on the single trace operators as in the 't Hooft limit the correlators of the multi trace operators, made of some set of fields, are suppressed by the powers of N compared to the correlators of the single trace trace operators involving same set of fields. We have already discussed that the gauge theory is a superconformal theory. The superconformal nature of the symmetry suggests one to classify the single trace operators into the chiral primary and the non-chiral primary operators. The chiral primary operators are annihilated by half of the supersymmetry generators. The representation of the superconformal algebra lies on the consideration of states with the lowest conformal dimension which are annihilated by the fermionic generators in the superconformal algebra and by the generators of the special conformal transformation. The rest of the states of higher dimensions can be build by acting the supersymmetry generators and the momentum operator P_μ on the lowest dimension state. The supercharges have the helicity $\pm\frac{1}{2}$ and the chiral primary operators in such representation have a range of helicities varying from $\lambda_h - 2$ to $\lambda_h + 2$. Here λ_h is the helicity of the operator with lowest dimension. The systematic computation of the representation of the chiral primary operators of the $\mathcal{N}=4$ SYM theory is very difficult. Therefore we focus on the known representation available in the literature. We start with the fact that the lowest components of the representation consists of only scalar fields. Let's consider the following operator,

$$\mathcal{O}_n = \mathcal{O}^{I_1 I_2 \ldots I_n} = Tr(\phi^{I_1}\phi^{I_2}\ldots\phi^{I_n}). \tag{1.31}$$

It can be proved that the short chiral primary representations can be built by treating the above operators as the lowest components, if and only if the indices $(I_1 I_2 \ldots I_n)$ form a symmetric trace-less product of n **6**'s of $SU(4)_R$ with $n = 2,3,\cdots,N$. By trace-less representations, we mean that when any two indices are contracted the result gives zero. This is a representation of weight $(0,n,0)$ of $SU(4)_R$. The superconformal algebra restricts the dimension of these fields $\mathcal{O}_n$ to be $[\mathcal{O}_n] = n$. As mentioned before, the full chiral primary representations can be achieved by the action of the supersymmetry generators and the momentum operator P_μ on the fields $\mathcal{O}_n$. The total number of states in this chiral primary representation is $256 \times \frac{1}{12}n^2(n^2-1)$, half of them are bosonic and half are fermionic. Since these short chiral primary multiplets are built on a scalar field of helicity zero, the helicity of the higher dimensional primaries ranges between -2 and 2. For example, to construct the bosonic field of $n+1$ dimension we have to consider the action of the two supercharges on $\mathcal{O}_n$. If the supercharges are of the same chirality (QQ) then we get a Lorentz scalar field in the $(2, n-2, 0)$ representation of $SU(4)_R$, and an anti-symmetric 2-form of the Lorentz group, in the $(0, n-1, 0)$ representation of $SU(4)_R$. Both of them are complex and the conjugate fields are constructed by the action of the two supersymmetric charges of opposite chirality ($\bar{Q}\bar{Q}$). Moreover the action of $Q\bar{Q}$ gives a real Lorentz vector in $(1, n-2, 1)$ representation of $SU(4)_R$. Now we consider a particular values of n, i.e. $n = 2$. This representation consists of a vector of dimension 3 ($SU(4)_R$ $\mathcal{R}$ symmetry current), a symmetric tensor field of dimension 4 (energy-momentum tensor), a complex scalar field of dimension 2.

We can now identify fields on AdS_5 with the operators in the dual conformal field theory. The single-trace operators in the field theory may be identified with the single-particle states in AdS_5, while the multiple-trace operators correspond to the multi-particle states. The conformal dimension of the field theory operator is identified with the mass dimension associated with the particle states in the dual gravity. Since the full $typeIIB$ string spectrum on $AdS_5 \times S^5$ is not yet known, we only concentrate on the states that arises from the dimensional reduction of the ten dimensional $typeIIB$ supergravity multiplet [35]. Firstly, the helicity of this multiplet ranges from -2 to 2. Here also we can achieve a short multiplet built on the lowest dimensional scalar field in $(0, n, 0)$ representation of the $\mathcal{R}$ symmetry group $SO(6)$ for $n = 2, 3, ...\infty$. The lowest dimensional scalar field in each representation is generated by a linear combination of the metric along S^5 and four form field which is indexed along S^5. For $n = 2$, the spectrum contains a massless graviton field, the massless $SU(4)_R$ gauge field. The massless graviton field is identified with the energy momentum tensor in the gauge theory sector and and the massless $SU(4)_R$ gauge field is identified with the $SU(4)_R$ symmetry current of the gauge theory.

1.3.3 Matching of the correlators

Another physical quantity, eligible for comparison in each of the theories is the correlation function. Since the gauge theory is conformally invariant, it has no scale. Consequently, there is no asymptotically free states and S-matrix of the theory. An important class of invariant, that a CFT does possess is the scaling dimensions of operators. These must be related to some other invariants in the gravity (string) theory. The supergravity in $AdS_5 \times S_5$ has a mass scale and the asymptotic mass eigenstates. From the equivalence of the correlators in both sides of the duality we can introduce a precise relation between the conformal dimension and the mass scale.

As argued in the Gubser-Klebanov-Polyakov-Witten prescription [19, 20], AdS/CFT correspondence maps the problem of finding quantum correlation function in the field theory side to a classical problem in the dual gravity. The correlation function of a local gauge invariant operator $\mathcal{O}(x)$ in the gauge theory can be calculated by deforming theory in the following way.

$$S \rightarrow S + \int d^4x \phi_0(x) \mathcal{O}(x), \tag{1.32}$$

where $\phi_0(x)$ is the source term. Boundary values of a bulk field ϕ plays the role of a source for the gauge-invariant operator $\mathcal{O}$ in such a way that the following relation holds:

$$W_{gauge}[\phi_0] = -\log\langle \mathrm{e}^{\int \mathrm{d^4x}\phi_0(\mathrm{x})\mathcal{O}(\mathrm{x})}\rangle_{\mathrm{CFT}} \simeq \text{onshell}\ \mathrm{S}[\phi_0(\mathrm{x})]_{\mathrm{sugra}}. \tag{1.33}$$

Here W is the generating functional for the connected Green's function in the gauge theory. In the bulk supergravity we have to solve equations of motion supplemented by the Dirichlet boundary condition on the boundary with the specified boundary value. Finally we plug the solution into the supergravity action and get back the "on-shell" action.

It is straight forward to compute the n-point correlation function. We need to take the source ϕ_0 to zero at the end of this calculation.

$$\langle T[\mathcal{O}(t_1,x_1)...\mathcal{O}(t_n,x_n)]\rangle = \frac{\partial^n S_{sugra}}{\partial\phi_0(x_1,t_1)....\partial\phi_0(x_n,t_n)}\Big|_{\phi_0\to 0}. \tag{1.34}$$

For an illustrative purpose, we consider the simplest of all, namely the two point function. For two-point function, only the quadratic part of the bulk action is needed. To this end, consider a massive scalar field in AdS_5. The generic form of the action is given as,

$$S \sim \int d^5x\sqrt{g}\left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{1.35}$$

We consider a metric form of AdS_5 space,

$$ds^2 = \frac{l^2}{z^2}(-dt^2 + \sum_{i=1}^{3}(dx^i)^2 + dz^2), \tag{1.36}$$

where $z \longrightarrow 0$ represents the boundary and $z \longrightarrow \infty$ represents the region deep inside the bulk. The action in this metric reads,

$$S = \frac{1}{2}\int dz d^4x \frac{l^3}{z^3}[(\partial_z\phi)^2 + (\partial_\mu\phi)^2 + \frac{m^2l^2}{z^2}\phi^2], \tag{1.37}$$

Now we consider the momentum space decomposition of the massive scalar field,

$$\phi(x^\mu, z) = \int d^4k e^{ik.x} f_k(z). \tag{1.38}$$

The equation of motion in terms of the Fourier modes is given by

$$f_k^{''} - \frac{3}{z}f_k^{'} - (k^2 + \frac{m^2l^2}{z^2})f_k = 0, \tag{1.39}$$

with $k^2 = g^{\mu\nu}k_\mu k_\nu$.

The equation of motion (1.39) has a solution which is a linear superposition of the Bessel functions, $z^2 I_{\Delta-2}(kz)$ and $z^2 K_{\Delta-2}(kz)$. The behavior of the above Bessel functions in the interior of the bulk AdS space ($z \to \infty$) is as follows,

$$I_{\Delta-2}(kz) \sim e^{kz}, \quad K_{\Delta-2}(kz) \sim e^{-kz}. \tag{1.40}$$

By demanding the regularity at $z \to \infty$ we set the coefficient of $I_{\Delta-2}(kz)$ to zero. In the above expressions we have introduced a parameter Δ related to the mass and the radius of curvature of AdS space

$$\Delta = 2 + \sqrt{4 + m^2 l^2}. \tag{1.41}$$

Δ comes as an exponent in the expression of the scalar field. To make it a real exponent one needs to impose the condition, $m^2 l^2 \geq -4$. This condition is known as Breitenlohner-Freedman (BF) bound and turns out to be very important for stability of the solution [21, 36, 37]. The boundary condition of the scalar field $\phi(x)$ is set to be

$$\phi(x, z)|_{z=\epsilon} = \phi_0(x)\epsilon^{4-\Delta}, \tag{1.42}$$

where we put a cutoff ϵ which is arbitrarily closed to the actual boundary $z = 0$. Using the above boundary condition, we normalize the function $f_k(z)$

$$f_k(z) = \phi_0(k) z^2 \epsilon^{2-\Delta} \frac{K_{\Delta-2}(k, z)}{K_{\Delta-2}(k, \epsilon)}. \tag{1.43}$$

Now we take this solution, plug it back into the equation (1.38). The on shell action for $\epsilon \to 0$ takes the form

$$\begin{aligned} S_{\text{onshell}} &= \frac{1}{2}\int_0^\infty \int_0^\infty \frac{d^4k d^4k^{'}}{(2\pi)^8} \delta^4(k + k^{'})\phi_0(k_\mu)\phi_0(k_\mu^{'})\Big(\text{Divergent term} \\ &\quad -2^{1-2(\Delta-2)}(\Delta - 2)k^{2(\Delta-2)}\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}\Big). \end{aligned} \tag{1.44}$$

Equation (1.44) has some divergent pieces in the limit $\epsilon \to 0$. However these terms can be subtracted off by adding some suitable counter terms in this action. According to our previous definition of the correlators, we take two derivatives on the action (1.44) to get

$$< \mathcal{O}(k)\mathcal{O}(k^{'}) >= -2^{1-2(\Delta-2)}(\Delta - 2)k^{2(\Delta-2)}\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}. \tag{1.45}$$

In the position space, the above 2-point function takes the form,

$$< \mathcal{O}(x)\mathcal{O}(x^{'}) >= 2\pi^{-2}\frac{\Gamma(\Delta)}{\Gamma(\Delta-1)}\frac{1}{|x - x^{'}|^{2\Delta}}. \tag{1.46}$$

From the position space realization of the 2-point function, the interpretation of Δ as the scaling dimension of $\mathcal{O}(x)$ becomes evident. Correlation functions of the non-scalar operators are also very useful to study the relation between the mass and the conformal dimension. A table of the mass-dimension formula in AdS_{d+1} with unit radius is given below.

- scalars: $\Delta_\pm = \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2})$,
- spinors: $\Delta = \frac{1}{2}(d + 2|m|)$,

- vectors: $\Delta_\pm = \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2})$,
- p-forms: $\Delta = \frac{1}{2}(d \pm \sqrt{(d-2p)^2 + 4m^2})$,
- spin-3/2: $\Delta = \frac{1}{2}(d + 2|m|)$,
- massless spin-2: $\Delta = d$.

We conclude this subsection with a remark that the AdS/CFT correspondence has passed many tests beyond the two-point functions. In particular the three point functions are well understood and are protected by the conformal invariance in the field theory side. Four point functions do not follow this protection, but have been studied widely. A detailed discussion of these higher point correlators are given in [38, 40–44].

1.4 Generalization of AdS/CFT correspondence

Having given a brief account of the AdS/CFT conjecture, in this section, we generalize this duality by introducing temperature, chemical potential and also by adding flavors to this gauge theory.

1.4.1 Finite temperature and chemical potential

One way of generalizing the gauge theory is to introduce a temperature. There exists two common ways. First is to perform a Wick rotation on the AdS geometry to bring it into the Euclidean signature and then compactify the Euclidean time. The inverse of this periodicity defines the temperature in the gravity theory. The other is to deform the pure AdS by introducing a black hole, keeping the asymptotic geometry unchanged. The Hawking temperature now becomes the temperature of the gauge theory [45–47]. In this thesis, we focus on the second approach. To further illustrate, we consider a stack of N number of black $D3$ brane. The geometry is described by a non-extremal version of the solution (1.11) where the radius of horizon (r_0) is finite.

$$\begin{aligned} ds^2 &= H^{-1/2}(-f(r)dt^2 + \sum_{i=1}^{3}(dx^i)^2) + H^{1/2}(f^{-1}(r)dr^2 + r^2(d\Omega_5{}^2)), \\ f(r) &= 1 - (\frac{r_0}{r})^4, \quad H = 1 + \frac{(\sqrt{r_0{}^8 + \frac{Q^2}{4}} - r_0^4)}{2r^4}, \end{aligned} \tag{1.47}$$

where Q, as mentioned in eq (1.13), is related to the charge of the black $D3$ branes. The above non-extremal geometry is asymptotically flat but, in the near horizon limit, it describes a black hole in the AdS space times a five sphere

$$ds^2{}_{AdS_{bh}\times S^5} = \frac{r^2}{\tilde{l}^2}(-fdt^2 + \sum_{i=1}^{3}(dx^i)^2) + \frac{\tilde{l}^2}{fr^2}dr^2 + \tilde{l}^2(d\Omega_5{}^2), \tag{1.48}$$

where the radius of the AdS_{bh} geometry is same as that of the S^5 and is given as

$$\tilde{l}^4 = \frac{(\sqrt{{r_0}^8 + \frac{Q^2}{4}} - r_0^4)}{2}. \tag{1.49}$$

We can associate a temperature T_H, known as the Hawking temperature, entropy and other thermodynamical quantities with the non-extremal solution. Additionally the solution satisfies all the laws of thermodynamics. One then associates this temperature, entropy and other thermodynamical observables with those of the finite temperature SYM [24]. The Hawking temperature is defined as [47],

$$T_H = -\frac{1}{4\pi}\left(\frac{dg_{tt}}{dr}\right)_{r=r_0} = \frac{\sqrt{2}}{\pi} r_0 \left[r_0^4 + \sqrt{r_0^8 + \frac{Q^2}{4}}\right]^{-\frac{1}{2}}. \tag{1.50}$$

The above temperature is identified with the gauge theory temperature. Hawking-Bekenstein formula then associates an entropy with the black hole [36].

$$S_{BH} = \frac{A}{4G_5}, \tag{1.51}$$

where G_5 is the five dimensional Newton constant and A is the horizon area. Because the horizon has an infinite volume, the black hole entropy diverges. To make the entropy finite we divide it by the total volume V and get a finite entropy density s. Finally, from the relation $\frac{G_5}{\tilde{l}^3} = \frac{\pi}{2N^2}$ and equation (1.50), we write the entropy density in terms of gauge theory parameters,

$$s = \frac{1}{4G_5}(\pi\tilde{l})^3 T_H^3 = \frac{\pi^2}{2} N^2 T^3. \tag{1.52}$$

The s is then interpreted as the entropy density of the dual gauge theory. Once we know the entropy, the other thermodynamical variables associated with the gauge theory can be easily derived. For example, we compute the pressure $P = \int s dT$, the energy density $\mathcal{E} = -P + Ts$ and the free energy density $F = \mathcal{E} - Ts$.

$$\begin{aligned} P &= \frac{\pi^2}{8} N^2 T^4, \\ \mathcal{E} &= \frac{3\pi^2}{8} N^2 T^4, \\ F &= -\frac{\pi^2}{8} N^2 T^4. \end{aligned} \tag{1.53}$$

Here, the factor N^2 multiplying various expressions is important. Since the degrees of freedom of the deconfined phase of the $SU(N)$ gauge theory is of the order N^2, one immediately suspects that the dual of the high temperature black hole geometry is a deconfined $SU(N)$ SYM. Indeed this was explicitly shown in a paper by Witten [20].

We can further introduce a chemical potential into the system. The $\mathcal{N} = 4$, $SU(N)$ SYM has a global $SU(4)$ $\mathcal{R}$-symmetry and one can consider the three independent $\mathcal{R}$ charges corresponding to the three Cartans of $SU(4)$. For each $U(1)$ Cartans we can turn on a chemical potential (μ). Dual of these $\mathcal{R}$ currents are the gauge fields A_μ in the bulk. Therefore, the dual of the finite temperature gauge theory at non-zero chemical potential are the charged black hole in AdS space.

1.4.2 Adding flavor

It is further possible to add the flavor degrees of freedom in the boundary gauge theory. The matter spectrum of the $\mathcal{N} = 4$, $SU(N)$ SYM contains the scalars and the Weyl fermions (transforming under the adjoint representation of $SU(N)$). Therefore, this theory does not include the quarks. It was first Karch and Katz [48] who introduced a way to include the fundamental degrees of freedom in this gauge theory set up. This is done by adding an extra stack of N_f Dp branes in addition to the pre-existed N number of $D3$ branes. The fundamental degrees of freedom can be obtained from the modes of the open string excitations with one end of the string on the stack of N $D3$ branes and the other end on a different stack of N_f Dp branes. We cannot distinguish between the branes of a stack on which a string ends. Thus we associate a $SU(N)$ symmetry group with N $D3$ branes and a $SU(N_f)$ symmetry group with N_f Dp branes [4,49]. We interpret the $SU(N)$ as the color and the $SU(N_f)$ as the flavor groups. The flavor branes are separated from the color branes by a distance L ($L \in (0, \infty)$). Further the so-called probe limit ($N_f \ll N$) ensures that the back-reaction of the additional N_f Dp branes on the near horizon geometry of the $D3$ branes can be neglected. The types of possible open string modes in this new setup come out as follows. The first one is of $3-3$ type which ends only on the $D3$ branes. The second one is, $p-p$ type which ends on the Dp branes. The remaining one is of $3-p$ type which connects the $D3$ and the Dp branes. All types of open strings interact among each other. The degrees of freedom associated with the $3-p$ strings transform under the bi-fundamental representation of the gauge group $SU(N) \times SU(N_f)$. In the low energy limit and for $p > 3$, the open string picture simplifies significantly. The $p-p$ strings living on the N_f Dp brane get decoupled from the $3-3$ and the $3-p$ sectors. Only interaction that remains, acts between the 3-3 strings and the $3-p$ strings. The gauge group, $SU(N_f)$ associated with the N_f Dp branes becomes a global symmetry group. In the closed string picture, the geometry is back reacted by the stack of N $D3$ branes. The Dp branes remains as probe brane in this back-reacted geometry. In fact such stable probe brane solutions can be explicitly constructed from their actions [48,50]. To summarize we can say that the $\mathcal{N} = 4$, $SU(N)$ SYM theory coupled to the N_f degrees of freedom transforming under the bi-fundamental representation of gauge group $SU(N) \times SU(N_f)$ is dual to the $typeIIB$ string theory in $AdS_5 \times S^5$ coupled to the open string living on the N_f number of probe Dp branes. So far we have not mentioned the values of p. In the $typeIIB$ theory, the possible configurations are $D3-D5$ and $D3-D7$ with specific orientations. The existence of the tachyonic mode in the ground state of the $D3-D5$ system excludes the possibility of having this configuration. So we are left only with the $D3-D7$ system. If we quantize the $3-7$ string, we get the fundamental degrees of freedom as N_f number of scalars and N_f number of Dirac fermions. All of them have equal mass,

$$M = \frac{L}{2\pi\alpha'}, \tag{1.54}$$

where L is the separation between the $D3$ and the $D7$ stacks. In literature, these scalars and the fermions are collectively called as "quarks". The inclusion of the $D7$ branes on top of the pre-existed the $D3$ branes, breaks the supersymmetry further from $\mathcal{N}=4$ to $\mathcal{N}=2$.

The inclusion of the fundamental dynamical quark opens up a window to study an important non-local operator in this context, called the Wilson loop. The loop can be traced out by the world line of the dynamical quark. In the next section we introduce the Wilson loop in the light of AdS/CFT.

1.5 Wilson Loop

Wilson loop is a gauge invariant non-local operator which plays an important role in the Abelian and the non-Abelian gauge theories. They are useful for evaluating the propagator of a particle interacting with the gauge fields, for studying the theory of confinement. Various approaches can be employed to study the Wilson loops, such as, the perturbation theory, the AdS/CFT techniques, the lattice methods, the localization techniques in the supersymmetric theories etc. For our purpose we only concentrate on the AdS/CFT techniques. For Non-Abelian gauge theories, Wilson loop operator is defined as,

$$\mathcal{W}(\mathcal{C})^r = Tr\mathcal{P}(i\oint_c (A_\mu dx^\mu)), \tag{1.55}$$

where $\mathcal{C}$ is the closed path on which the loop is computed. Here r stands for the either the fundamental (F) or the adjoint representation (A). The trace is taken over the gauge group involved in the theory. $\mathcal{P}$ is the path ordering [51]. $A_\mu = A^a_\mu T_a$ is the gauge field and T^a's are the generators of the gauge group. The Wilson loop can be thought of as the world line of a quark or antiquark. We have the probe quarks in the theory emerging from the probe $D7$ branes. In the dual gravity, we consider a fundamental string hanging from the probe $D7$ brane living in the AdS geometry. It is oriented along the radial direction. The end point of the string attached to the boundary is identified with the quark and the worldsheet traced out by the string is identified with the world line of the quark. The mass of the quark is proportional to the length of the string. The expectation value of the Wilson loop operator gives the vacuum to vacuum propagator of the quark along the loop $\mathcal{C}$. The identification between quark and the string naturally suggests that there must be a correspondence between the expectation value of Wilson loop operator and the partition function of the dual string world sheet A [52,53],

$$\langle \mathcal{W}(\mathcal{C})^r \rangle = Z_{string}[\partial A = \mathcal{C}]. \tag{1.56}$$

We are particularity interested in the case of the infinitely heavy quark. In the gravity dual we can achieve this by placing the probe $D7$ brane at the boundary of the AdS geometry. Simultaneous application of 't Hooft limit ($N \to \infty$ at fixed λ) and the strong coupling limit ($\lambda \gg 1$) simplifies the partition function.

$$\langle \mathcal{W}(\mathcal{C})^r \rangle = Z_{string}[\partial A = \mathcal{C}] = e^{iS(\mathcal{C})}. \tag{1.57}$$

$S(\mathcal{C})$ is the Nambu-Goto action [54] of the probe string world sheet dual to the world line ($\mathcal{C}$) of the quark. The consideration of the 't Hooft limit as well as the strong coupling limit is very crucial. Here, $N \to \infty$ at fixed λ leads to the fact that the string coupling $g_s \to 0$. So there will be no string loops which splits off from the worldsheet. $\lambda \gg 1$ signifies that the string does not fluctuate around its classical configuration. Using the relation specified in eqn (1.57), we now perform a simple computation of the potential energy of a quark antiquark pair ($q\bar{q}$) sitting in the vacuum configuration.

Let us consider a rectangular Wilson loop traced out by a $q\bar{q}$ pair seating at the boundary of the AdS_5 spacetime and separated by a distance L. We consider L to be along a boundary coordinate x_1 and the pair moves along the time direction up to an interval $\mathcal{T}$. We assume that $\mathcal{T} \gg L$. From the field theory side, we expect the $\langle \mathcal{W}(\mathcal{C})^r \rangle$ to

$$\langle \mathcal{W}(\mathcal{C})^r \rangle = e^{-iE_{tot}\mathcal{T}} = e^{-i(2M+V(L))\mathcal{T}}, \tag{1.58}$$

where E_{tot} is the total energy of the configuration, $V(L)$ is the potential energy and M is the mass of the quark/antiqurak. As the field theory is strongly coupled, we do the computation in the dual gravity. The AdS_5 metric parameterized by $x^\mu = t, x^1, x^2, x^3, z$ is given as,

$$ds^2_{AdS_5} = \frac{l^2}{z^2}(-dt^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + dz^2). \tag{1.59}$$

First we fix the parametrization of the worldsheet as well as the boundary conditions. We consider the worldsheet surface whose action $S(\mathcal{C})$ is to be extremized by the AdS_5 coordinates $x^\mu = x^\mu(\tau, \sigma)$, where τ, σ are the worldsheet parameters. In the $\mathcal{T} \gg L$ limit, we assume that the worldsheet surface is translationally invariant along the τ direction, Therefore it allows,

$$x^\mu(\tau, \sigma) = x^\mu(\sigma). \tag{1.60}$$

Due to the reparametrization invariance of the string world sheet action, we can choose a suitable gauge.

$$\tau = t, \sigma = x_1. \tag{1.61}$$

The Wilson loop is traced on a hyperplane defined by the boundary coordinates x^2, x^3 as,

$$x_2(\sigma) = \text{constant}, \;\; x_3(\sigma) = \text{constant}. \tag{1.62}$$

To fulfill the requirement that the worldsheet has the boundary specified by $\mathcal{C}$, we impose a boundary condition on the bulk coordinate z.

$$z(\pm\frac{L}{2}) = 0. \tag{1.63}$$

The Nambu-Goto action in the AdS_5 background takes the following form,

$$S_{NG} = -\frac{l^2\mathcal{T}}{2\pi\alpha'}\int_{-\frac{L}{2}}^{\frac{L}{2}} d\sigma \frac{1}{z^2}\sqrt{1+z'}, \tag{1.64}$$

where $z^{'}$ is $\frac{dz}{d\sigma}$. It is suitable now to introduce some dimensionless parameters in this analysis. We define $\sigma^{'}$ and y so that,

$$\sigma = L\sigma^{'}, z(\sigma) = Ly(\sigma^{'}). \tag{1.65}$$

Exploiting the symmetry $z(\sigma) = z(-\sigma)$ we get,

$$S_{NG} = -\frac{l^2\mathcal{T}}{\pi\alpha' L}\int_{0}^{\frac{1}{2}} \frac{d\sigma^{'}}{y^2}\sqrt{y^{'2}+1}. \tag{1.66}$$

The equation of motion of y takes the following form,

$$y^{'2} = \frac{{y_0}^4 - y^4}{y^4}. \tag{1.67}$$

y_0 is the turning point where $y^{'} = 0$. By symmetry consideration this should be at $\sigma^{'} = 0$. We can explicitly calculate y_0 by using the integration below

$$\frac{1}{2} = \int_0^{\frac{1}{2}} d\sigma^{'} = \int_0^{y_0} \frac{dy}{y^{'}}. \tag{1.68}$$

The result turns out to be $y_0 = \frac{\Gamma(\frac{1}{4})}{2\sqrt{\pi}\Gamma(\frac{3}{4})}$. Now, using the value of y_0 and changing the variable from σ' to y in equation (1.66) we compute integration. It is clear that the integrand is singular at $y = 0$. This divergence, coming out of this singularity, is due to the presence of infinite mass of $q\bar{q}$ pair. We can set a IR cutoff at $z = \epsilon$ where ϵ is arbitrary small number. Now we have to subtract $2M\mathcal{T}$ term from the exponential in equation (1.58). M can be calculated by considering the Nambu-Goto action representing two disjoint world sheets hanging from the boundary along z direction. The calculation is similar to the one already presented here. The final answer for M is $\frac{\sqrt{\lambda}}{2\pi\epsilon}$. After the subtraction, we take $\epsilon \to 0$ limit to achieve the finite $q\bar{q}$ potential.

$$V(L) = -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}\frac{\sqrt{\lambda}}{L}. \tag{1.69}$$

The potential is written in terms of gauge theory parameter. From the above discussion it becomes clear that in this set up the computation of Wilson loop boils down of evaluating worldsheet action of a classical string. One can also repeat the same computation for $q\bar{q}$ pair seating in the finite temperature gauge theory. Accordingly, in the dual gravity we have to replace the pure AdS_5 geometry by an AdS_5 black hole background. See [26] for the details.

Having discussed some generalizations of the AdS/CFT conjecture, we are now ready to review some applications of this correspondence. These will include an exploration of properties of the strongly coupled plasma phase of the gauge theory. We will also review certain expected features of the string/M theory on the AdS geometry by looking at their weak coupling boundary duals.

1.6 Quark- Gluon-Plasma and AdS/CFT

After the discovery of the asymptotic freedom in the QCD, it was realized that the confined phase of the quark and the gluons could reached a new deconfined phase of the matter at high temperature and at high number density. In the phase diagram of the QCD, this appears as a crossover (explained shortly after) from the confined to the deconfined phase. In the later phase, the quarks and gluons can freely move within the system [59]. This deconfined phase resembles the one which existed for a very short time period ($\sim 10\mu$ sec) after the Big Bang [80], or the one which still exists in the core of the neutron star [61]. This new phase of the matter is called the quark-gluon plasma (QGP). To further elaborate upon this new phase, it is instructive to consider the phase diagram of the QCD (see figure 1.1) in some detail.

Let us first focus on the region of low temperature and low baryon chemical potential (see figure 1.1). The degrees of freedom which become dominant at the low temperature ($T \leq 100Mev$) and at the zero baryon chemical density are the pions and the light mesons. As the temperature increases above 100 Mev, the massive resonance states are produced. According to the the pioneering work by Hagedorn [55], at some critical temperature around 160 MeV, the density of the hadrons becomes very large. In fact, it becomes so large that the hadron looses its composition and system starts behaving like a plasma made out of the quarks and the gluons. The new phase of matter we mentioned before is known as quark-gluon plasma (QGP) [56]. In this phase, the quarks and the gluons move freely within the system.

The temperature at which the transition between the hadronic phase and the QGP phase occurs is studied using the lattice QCD method. The computation is done at zero baryon chemical potential. It is found that, around the critical temperature many thermodynamical quantities increase very steeply [57]. This behavior signifies the occurrence of a phase transition. However, as mentioned in [58], there is no sharp phase transition that takes place around the critical temperature. Instead one observes a continuous crossover between the two phases. Correspondingly, the critical temperature is estimated not to be labeled by a point but by a region between 150-190 MeV. It is however not clear from the phase diagram, if the QGP phase is strongly coupled. We shall come back to this issue later in this section.

One can also study the phases of the QCD at zero temperature but at a finite baryon chemical density. The phase starts with a vacuum, remains same with the increase of the

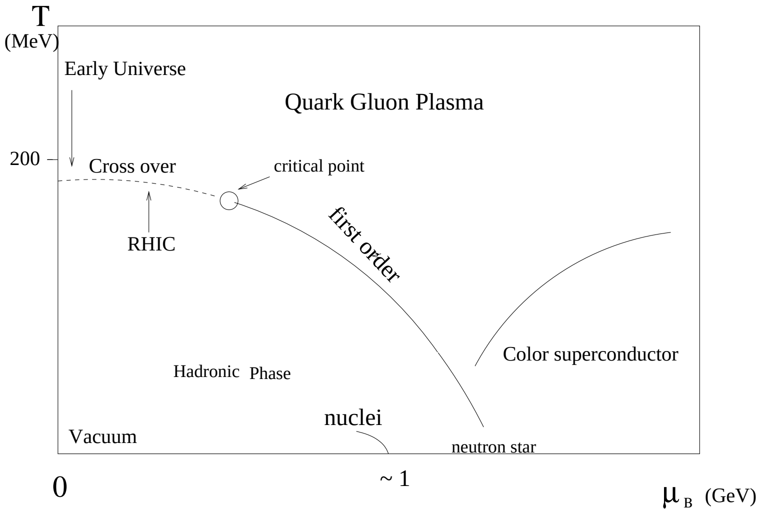


Figure 1.1: QGP phase diagram

potential up to a critical value (922 MeV). At and above this critical value of baryon density, a phase of nuclear matter is favored than the vacuum phase leading to a transition from the vacuum phase to the nuclear matter phase.

This is a first order phase transition which is extended from zero to finite but low temperature. Further increase of baryon density beyond 922 MeV by keeping the temperature at low value results into more and more compressed nuclear matter. If the baryon density continues to increase, at some critical value which is not properly known yet, again a transition occurs from the phase of the highly dense nuclear matter to a phase described in terms of quarks. The quark phase is known as the color superconductor phase.

When both the temperature and the baryon density are finite, it is very hard to study the QCD phases. However, the development of the lattice QCD shows that the crossover which is initially realized at zero chemical potential continues to occur even at finite but small values of the baryon density. Finally it smoothly extends up to the critical point in the phase diagram. Moreover, if the baryon chemical potential is increased further, the phase transition is expected to be a first order one. From theoretical speculations, it was thought that this QGP phase could be produced through the collision of two heavy nuclei with ultra relativistic energy [62]. In recent time, it is believed that at the Relativistic Heavy Ion Collider (RHIC) in BNL [63–69] or at the Large Hadron Collider (LHC) in CERN [71] this QGP phase has indeed been created.

The experimental observation at the RHIC is done at various stages. Two disk like heavy gold nuclei approach together with the maximum center of mass energy 200 GeV. They are Lorentz contracted by a factor $\gamma = \frac{E_{beam}}{M} \sim 110$, where E_{beam} is the energy per nucleon and M is the nucleon mass. After collision, a large momentum transfer ($Q \sim$ 1GeV) takes place among quarks, antiquarks and gluons (partons). As a result, secondary partons with large transverse momentum (p_T) are produced in very early time ($\sim \frac{1}{p_T}$) after collision. Consequently, soft collisions with small momentum exchanges take place. Again many more partons are produced and they get thermalized after $\sim$ 1fm/c. The thermalized phase expands hydrodynamically, cools down adiabatically and finally the hadronization occurs. The hadron matter interacts among themselves quasi-elastically and continues to expand till chemical freeze out (the inelastic interactions between the hadrons cease) and thermal freeze out (the elastic interactions between the hadrons cease) happens. During this process, the unstable hadrons decay down and stable ones move to the detector. The most intriguing quest that arises is whether the quark gluon plasma phase has at all been discovered in the RHIC. We do not yet have any direct evidence of the plasma as the particle reached at detector are those produced after the hadronization. However, the observation of the extreme level of thermalization, strong collective behavior, screening of the color fields in the intermediate process give indirect signatures of plasma formation. Very low values of the viscosity obtained from the experimental data suggest the plasma as a strongly coupled perfect fluid. Before we go on to study the properties of the strongly coupled plasma, in the next few subsections, we elaborate upon some features of the RHIC plasma.

1.6.1 Expectation of the pQCD

One of the most important motivations of the RHIC experiment is to study the phase structure of the QCD and the physics of the phase transition from a confined phase of colorless hadrons to a deconfined phase made of free quarks and gluons at very high energy and/or density. The other motivation is whether the QGP phase can be produced in a laboratory setup. Asymptotic freedom suggests that in the QCD, the enhancement of the energy scale in any process lowers the strength of the coupling. So the expectation from the perturbative QCD (pQCD) was, in the deconfined phase the quarks and the gluons interact very weakly and collectively behave like an ideal gas. The physics can be explained by the pQCD at and above the critical temperature T_c, where the T_c signifies the temperature at which the confinement/deconfinement phase transition occurs [72]. It was also argued from the perturbative study that the cascade of jets (stream of the free quark and gluons) produced in the RHIC should be similar to those formed in the individual proton-proton collision. If there is small deviation between them it should be captured by the perturbation technique [72, 73]. At low density, where the perturbative analysis breaks down, physics should be explained by the lattice QCD. In this approach the system can be thought of as a grand canonical ensemble and it resides over a space-time put in a lattice. This is a very useful technique to compute the kinematical observables, such as critical temperature, entropy etc. Using this method the numerical value of T_c turned out to be $\sim 150 - 180$MeV [68, 74].

1.6.2 Suppression of J/ψ and screening length .

So far we have discussed a few theoretical predictions of the quark gluon plasma. Let us now see if the analysis of the experimental results really agrees with the predictions. First consider the status of observing the phase transition between the confined phase and the QGP phase. Unfortunately, any direct observation of this phase transition turns out to be vary hard due to a number of experimental obstacles. The Au-Au collision in the RHIC takes place over a time scale of the order of $\sim$ 10 fm/c. It is not fully clear that what is taking place during this extremely small time interval, is a phase transition between two states in thermal equilibrium. Therefore, the signature of the phase transition remains elusive [63]. However, there are indirect but very elegant ways to observe the various stages of the matter after collision and confirm the signature of the QGP formation. For example, one of these approaches is to look at how the properties of the particles get modified by their interactions with the thermal medium. A very important phenomenon related to these modification of particle properties is suppression of the number of the J/ψ mesons observed by the detector [63, 70].

What actually happens is that the vector meson, consisted of a pair of quark and anti-quark, melts while passing through the plasma like medium. Consequently, fewer number of such mesons are observed by the detectors. One of these vector mesons is the J/ψ which decays into two muons. These leptons are very easy to detect. By comparing the cross-sections of the J/ψ meson production both for the Au-Au collision in the RHIC and the proton-proton collision, it becomes clear that the J/ψ production is suppressed by the medium induced effect. Thus the deconfined phase of the free quark and gluons can perhaps be identified with the QGP phase [75]. It is very crucial to understand the suppression of the J/ψ meson as a color screening effect. The high energy collisions produce the tightly bound qurkonium states made of heavy quark anti-quark pair ($q\bar{q}$). In particular, the resonant interaction of the charm quark(c) and the anti-charm quark($\bar{c}$) produces the J/ψ as a charmonium bound state. The $c\bar{c}$ bound states experience some modification of their vacuum properties (properties at the zero temperature) due to the presence of the hot QGP medium. Firstly, the confining potential acting between them dies out. Secondly, the color Coulomb interaction between the $c\bar{c}$ pair modifies due to the free color charges present in the medium. The free color charges actually screen the charge of $c(\bar{c})$ which is reduced in magnitude as seen by $\bar{c}(c)$. Eventually the color Coulomb interaction gets modified into the color Yukawa interaction $Exp[-r/L_s]$, where r is the separation distance between $c\bar{c}$ and L_s is the effective length scale known as screening length [76]. This length scale signifies the minimum separation length of $c\bar{c}$ beyond which the pair dissociates. We know that there is only one scale in the theory which is the medium temperature. So we associate the length scale L_s with the inverse of the temperature. This explains the fact that the increase of the temperature accelerates the dissociation of the charmonium bound states. Typically, the J/ψ dissociates in between $1.5T_c$ to $2.5T_c$. If the temperature is sufficiently high, the screening length is very small (less than the radius of bound) the J/ψ state melts into the free c and $\bar{c}$. They drift apart far away from each other. This screening length also depends on the relative velocity between the $c(\bar{c})$ pair and the rest frame of the QGP plasma. Experimental result shows that the L_s grows inversely with respect to the velocity.

1.6.3 Jet quenching.

Another strong signature of the QGP formation is Jet quenching. In the proton-proton collision at very high energy, there is always a production of parton (quark and gluons) pairs whose subsequent evolution leads to the fragmentation and the hadronization. Finally, the two jets of hadrons carrying high transverse momentum and propagating back-to-back in the center of mass frame are detected. Therefore, the detector can be triggered with the particles carrying high transverse momentum in one of these jets. The distribution of radiation in the azimuthal angle $\Delta\phi$ exhibits two prominent peaks at $\Delta\phi = 0$ (near side) and $\Delta\phi = \pi$ (away side). For Au-Au collision in the RHIC experiment, the away side peak at $\Delta\phi = \pi$ disappears. This observation compels us to imagine that the hard scattering happens near the edge of the collision region, so that the near side jet escapes to the detector, while the away side jet gets suppressed. But this kind of suppression of the particles with high transverse momentum is only conceivable if we consider the formation of plasma with the free color charges. While passing through a medium the away side jet interact with the free color charges and the color equivalent of Bremsstrahlung interaction takes place. This interaction results a quenching of hard scattered jets which are dragged and the high transverse momentum is dissipated in the medium. One can compute the ratio R_{AA} between the particle yield in the Au+Au collisions and the respective yield in the proton-proton collisions rescaled by the number of participating nucleons [77]

$$R_{AA} = \frac{d^2N^{Au-Au}/dp_T d\eta}{N_{coll} d^2 N^{p-p}/dp_T \eta}, \tag{1.70}$$

where N is the number of nucleon with transverse momentum p_T. N_{coll} is the total number of binary Au-Au at a given impact parameter, η is the pseudo-rapidity. According to the prediction of the pQCD, the nucleus-nucleus collision is a superposition of the collisions between the constituents nucleons and R_{AA} should be 1. But in the reality it is suppressed roughly by a factor of 5. This observation further shows that the process of jet quenching can not be explained by the pQCD methods. The theoretical way to describe the suppression or the quenching of partons in away side jet is achieved by computing the rate for energy loss due to the in medium effect. This energy loss is typically related to a transport coefficient called jet quenching parameter $\hat{q}$, which characterizes the parton interactions in the QGP medium.

1.6.4 Collective motion

Apart from the evidences which originate from the observation of modified properties of the matter, there is another signature of the QGP formation. Instead of looking for the individual parton, one can also observe the collective behavior of the system. Indeed, one of the main aims in the relativistic heavy ion collisions is to study the global features like the total and elastic cross sections. For this kind of experimental analysis, the object of interest is not the individual partons but mostly the macroscopically large fireball whose size exceeds the microscale of correlation inside the system. In the case of the QGP plasma this microscale is provided as the inverse of the system temperature [78]. In the RHIC experiments, the collisions between the Au nuclei with the mass number($A \sim 200$) and the size ($\sim 6fm$) produce an object which is at a very high energy and density. It is known as the fireball. The fireball expands hydrodynamically until the thermal and chemical freeze out take place. Theoretically, the collective behavior can be nicely explained by the relativistic fluid dynamics [79,80]. The stronger the strength of the interaction between the free quarks and the gluons is, the better is the explanation from the fluid dynamics. The velocity of the fluid has either the longitudinal (along the beam of jets) component or the transverse component. The transverse component can be divided again into two components. One is the radial flow which is present in the axially symmetric central collision. The other one is the elliptic flow which exists only for the non-central collision. By centrality we mean attaining the minimum value of the impact parameter of collision (length of a vector connecting the centers of the nuclei). We are mostly interested in those collective flows of the system which is contributed by the elliptic flow. During the process of QGP formation, fireball keeps an almond like shape for a non-zero value of the impact parameter. The shape is not spherically symmetric, but spatially anisotropic with respect to the reaction plane (plane which connects the centers of the two nuclei along the direction of the impact parameter of the collision). Because of this shape, when the fireball evolves hydrodynamically, this spatial anisotropy translates into a momentum anisotropy. The anisotropy in momentum can be realized as a Fourier distribution of the momentum in the azimuthal angle [81–86].

$$\frac{dN}{dyd^2p_T} = \frac{dN}{2\pi p_T dp_T dy}\Big[1 + \sum_{k=1}^{\infty} 2v_k \cos(k(\phi_p - \Psi_{RP}))\Big], \tag{1.71}$$

where N is the number of nucleons with transverse momentum p_T, ϕ_p is the azimuthal angle corresponding to p_T, y is the rapidity parameter and the Ψ_{RP} is the orientation angle of reaction plane. $v_k = v_k(y, p_T)$ is the k^{th} harmonic differential flow. The integrated flow representing the collective dynamics can be obtained by averaging the v_k over p_T and y. The second harmonics v_2 is related to the elliptic flow [82]. At lower energy, the negative values of v_2 implies that flow of the momentum out of the reaction plane. At higher energy, positivity v_2 signifies the in plane flow. It is important to note that if the QGP is weakly interacting, the whole system should expand like an ideal gas, isotropically in all directions. At higher energy the large, non zero v_2 indicates that the momentum of the in plane flow is always contributing more than that of the out of plane flow (momentum anisotropy). This implies that the system has a large pressure gradient and experiences a fast thermalization. Each species of the particles has a unique elliptic flow coefficients v_2 characterizing the unique azimuthal deformation of the momentum distribution. v_2 has been measured over a large class of hadron species, namely from the pion ($m_{pi} = 140$ MeV) to the Ω hyperon ($m_\Omega = 1672$ MeV). More than 99% hadron has a transverse momentum $p_T < 2$GeV. In this range the experimental data shows an excellent agreement with the hydrodynamics result. The value of the shear viscosity to the entropy density ratio (η/s) can be experimentally measured. It turns out that $\eta/s = .1 - .2 < 1$. This small value of the η/s ratio shows the QGP is the most ideal fluid ever discovered in a lab. By using the results ofthe kinetic theory one can deduce a relation between the shear viscosity and the coupling constant of the theory [87]. According to the kinetic theory, the mean free path (l_{mfp}) of a system is given as,

$$l_{mfp} \sim \frac{1}{n\sigma v}, \tag{1.72}$$

where n is the density of the particles, σ is the scattering cross-section, v is the typical particle velocity. Now using the typical temperature dependence of all three parameters, $n \sim T^3$, $\sigma \sim \lambda^2 T^{-2}$ and $v \sim 1$ we get,

$$l_{mfp} \sim \frac{1}{\lambda^2 T}. \tag{1.73}$$

λ is the coupling constant of the respective theory. The kinetic theory also prescribes the relation between the shear viscosity and the mean free path,

$$\eta \sim e\, l_{mfp}, \tag{1.74}$$

where e is the energy density ($e \sim T^4$). Using eqn (1.73) and eqn (1.74) we get the relation between the shear viscosity and the coupling constant.

$$\eta \sim \frac{T^3}{\lambda^2}. \tag{1.75}$$

From eqn (1.75) it is clear that when the viscosity is very small the coupling constant of the theory is very large. So the smallness of the η/s ratio again confirms the plasma is strongly coupled.

The above discussions of the quarkonium suppression, the jet quenching and the collective motion actually suggest that the phenomenological aspect of the QGP can not be explained theoretically only using the pQCD methods. One of the other suitable theoretical candidates is the lattice QCD. This is a theoretical tool, which is based on the numerical calculations of kinematical observables of thermal gauge theory, even at strong coupling. The quantitative behavior of all thermodynamical variables, the thermal equation of state, the critical temperature, the screening length etc are the successful triumph of this subject. However, there are severe limitations for which it does not play the role of the best suitable candidate. All the kinematical results have deceivingly small impacts when we shift from the weak coupling regime to the strong coupling regime. So the lattice QCD is somewhat insensitive to the typical strongly coupled phenomena. Moreover, in order to avoid some phase factors the lattice QCD uses the Euclidean signature of the metric. In this signature, the Euclidean time coordinate is made periodic. So any real time dynamics in Minkowski signature becomes difficult to analyze within the lattice methods. Considering these limitations of the pQCD and the lattice QCD, other avenues were proposed and explored. One of the most important options is perhaps the use of the gauge/gravity duality. However, we would like to highlight at this stage that the gauge theories admitting the gravity duals are the super Yang-Mills (SYM) theories and they are different from the real QCD in many aspects. In particular , SYM is conformal and non-confining. However, some of these differences get blur if we consider both these theories at temperature above the critical temperature. Then none of these strongly coupled theories are confining and both display the Debye screening

Many differences however still remain. Consider the $\mathcal{N} = 4$ SYM for example. Notice that the number of degrees of freedom (d.o.f) of the $SU(N)$, $\mathcal{N} = 4$ SYM with $N = 3$ is larger than those of the QCD. So if the observable (to calculate) is dependent on d.o.f in a non trivial way, the end result will differ for both theories. Furthermore, in the $\mathcal{N} = 4$ SYM theory, we take $N \to \infty$ and $\lambda \to \infty$ limit in order to perform the reliable computations in the bulk. In the QCD, the coupling constant remains large but finite and the number of color is three. Therefore while comparing with the QCD, the results of the $\mathcal{N} = 4$ SYM should be supplemented by some non-trivial correction terms. Lastly, the QCD has the d.o.fs in the fundamental representation of the gauge symmetry group. These fundamental d.o.f are three flavor, $N_f = 3$. They contribute to the thermodynamics obtained at temperature above T_c. But in the case of the $\mathcal{N} = 4$ SYM we always take either $N_f = 0$ or $N_f \ll N$. It is extremely difficult to keep $N \sim N_f$ in one hand and do computations in the dual gravity theory on the other.

From above discussion it is clear that, in spite of having few similarities, the plasma phase of the QCD at finite temperature is certainly different from the strongly coupled $\mathcal{N} = 4$ SYM at finite temperature. However, interestingly the studies on the transport coefficients and the dynamical observables of the various strongly coupled plasmas computed using the gauge/gravity duality show universal behavior [87–92, 174]. It is this universality which attracted attention of many researchers. Consequently, the gauge theories having gravity duals were explored even though they differ from the QCD.

To substantiate our arguments, let us review here some of the universal features of the strongly coupled plasma admitting the gravity dual. In [87–89, 91, 92], Kovtun, Son and Starinets discovered that the shear viscosity to entropy density ratio (η/s) is $\frac{1}{4\pi}$ for all such strongly coupled gauge theories. If the gauge theories are at finite temperature, then the universality of many physical coefficients ($\xi(T)$) like the speed of sound, the bulk viscosity, the diffusion coefficient, the DC conductivity of probe $U(1)$ charge emerge

$$\xi(T) = \xi_{CFT} + C_\xi(\Delta)(\frac{\pi\Lambda}{T})^{2(4-\Delta)}, \tag{1.76}$$

where the Λ and the Δ are the energy scale and the scaling dimension. $C_\xi(\Delta)$ is a constant [114, 115]. One can further show that the ratio of the R charge conductivity (σ) to the charge susceptibility (χ) at finite temperature and zero chemical potential is universal [92].

$$\frac{\sigma}{\chi} \geq \frac{\hbar c^2}{4\pi T}\frac{d}{d-2}, \tag{1.77}$$

where c is the velocity of light, $d \geq 3$ is the dimension of the gauge theory, T is the temperature. For a finite chemical potential, electrical conductivity shows a universal behavior for a large class of finite temperature gauge theory with gravity dual

The conductivity of gauge theories satisfying a constraint has universal form [174].

$$\sigma = \sigma_H \left(\frac{sT}{\epsilon + p}\right)^2, \tag{1.78}$$

where σ is the conductivity of gauge theory, σ_H is a geometric quantity evaluated at horizon, s, T, p, ϵ are the entropy density, the temperature, the pressure and the energy density. Such universal behaviors motivate us to calculate the transport coefficients and dynamical observables of the QGP like medium admitting a gravity dual. In the following subsections we review some of the holographic computations of such quantities, namely drag force on a heavy probe quark, jet quenching parameter and screening length of a quark-antiquark pair, in the context of the $\mathcal{N} = 4$ SYM plasma. Computations similar to these, will be performed in the next two chapters for different models.

1.6.5 Drag force in $SU(N)$ $\mathcal{N} = 4$ SYM gauge theory

When the high-energy partons move through the QGP medium, their energy loss is encoded by drag force they experience. Within the framework of the gauge/gravity duality, drag force on an external heavy quark, moving with a constant velocity in the $\mathcal{N} = 4$ super Yang-Mills plasma at finite temperature, is computed in [94–100]. The mass of the quark is generally assumed to be much larger than the typical energy scale associated with the medium (inverse of the temperature scale). The time scale of quark motion is assumed to be large compared to the relaxation time scale of the medium.

Here, we briefly describe the the drag force computation, following [95]. A computation similar to this will be employed in the next chapter. The high energy parton is represented by a heavy probe quark charged under the fundamental representation of $SU(N)$. It is holographically identified with the one of the endpoints of a fundamental string, attached to the boundary of the dual AdS background. The body of the string in the AdS bulk geometry is realized as the gluonic fields in the thermal plasma. The background, we consider, is a five dimensional AdS black hole, which is a part of the ten dimensional geometry defined in (1.48). This, in turn, introduces a non-zero temperature in the boundary. The dynamics of a fundamental string in the black hole background is described by the Nambu-Goto action. We consider the motion of the string along x^1 only. In the static gauge, $\tau = t$ and $\sigma = r$, the string dynamics can be specified by the function $x^1(t, r)$. Moreover, we choose the ansatz of the late time behavior associated with the string profile as: $x^1(t, r) = vt + \xi(r)$. Accordingly, the Nambu-Goto action takes the following form

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int dt dr \sqrt{1 + \frac{f}{H}(\partial_r \xi)^2 - \frac{(v)^2}{f}} = \int dt dr \mathcal{L}. \tag{1.79}$$

The equation of motion of the string can be calculated from (1.79)

$$\xi' = \pm(2\pi\alpha')\pi_\xi \frac{H}{h} \sqrt{\frac{h - v^2}{h - (2\pi\alpha')^2 \pi_\xi{}^2 H}}. \tag{1.80}$$

On the right hand side of (1.80), we consider the positive sign to take care of the trailing nature of the string profile [95]. To obtain a physical profile $\xi(r)$, we have to solve eqn (1.80) and further impose the following constraints

$$h = v^2,$$
$$\pi_\xi = -\frac{v}{\sqrt{1-v^2}}\frac{r_0^2}{l^2}. \quad (1.81)$$

The above two constraints make the quantity under the square root in the right hand side of eqn (1.80) always positive, therefore making ξ' always real. To evaluate the drag force we need to consider the conservation of world sheet current of space time energy-momentum of the test string, $P_\mu^\alpha = -\frac{1}{2\pi\alpha'}G_{\mu\nu}\partial^\alpha X^\nu$ around a closed loop $\mathcal{C}$ on the worldsheet,

$$\oint_{\mathcal{C}}(P_\mu^\tau d\sigma - P_\mu^\sigma d\tau) = 0. \quad (1.82)$$

The AdS/CFT dictionary suggests that the momentum loss of the heavy quark, during the interaction with the medium, can be holographically identified with the flow of total string momentum from boundary to the horizon. The rate of change of total momentum in a finite time interval (t_1 to t_2) gives the desired form of the drag force.

$$p_{t_1}^{x^1} - p_{t_2}^{x^1} = -\int_{t_2}^{t_1}\sqrt{-g}P_{x^1}^z dt, \quad (1.83)$$

where $p_t^{x^1}$ is the x^1 component of the total momentum at time t. The drag force is defined as

$$F_{drag} = \frac{dp_{x^1}}{dt} = -\frac{\pi\sqrt{g^2{}_{YM}N}}{2}T^2\frac{v}{\sqrt{1-v^2}}. \quad (1.84)$$

In the last expression we write the drag force in terms of gauge theory variables. The drag force has a quadratic dependence on the temperature. The form of the velocity dependence in the drag force expression is due to the relativistic motion of the quark.

1.6.6 Jet quenching parameter in $SU(N)$ $\mathcal{N}=4$ SYM gauge theory

As mentioned in [101, 105, 106, 109], the high energetic parton carrying a large transverse momentum interacts with the strongly coupled QGP medium while passing through it and and losses energy. In addition to that, the direction of parton's momentum with respect to the initial one, also gets changed. This phenomena is known as "transverse momentum broadening". While interacting with the medium, the pertons radiate gluons by a process which is a QCD analogue of Bremsstrahlung [102, 103]. The transverse momentum broadening is described by a probability function $P(k_\perp)$ defined as the probability of acquiring transverse momentum $k_\perp$ after propagating a distance $\mathcal{D}$ inside the medium. The normalization condition for $P(k_\perp)$ is,

$$\int\frac{d^2k_\perp}{(2\pi)^2}P(k_\perp) = 1. \quad (1.85)$$

We can define a parameter $\hat{q}$ related to the energy loss of the heavy parton in this scenario. It is defined as the mean transverse momentum acquired by a parton per unit distance it travels inside the medium

$$\hat{q} \equiv \frac{\langle k_\perp^2 \rangle}{\mathcal{D}} = \frac{1}{\mathcal{D}} \int \frac{d^2 k_\perp}{(2\pi)^2} k_\perp^2 P(k_\perp). \tag{1.86}$$

$\hat{q}$ measures the the radiative energy loss associated with the high energetic partons. This kind of energy loss sources the jet quenching phenomenon described earlier. $\hat{q}$ is known as the jet quenching parameter [105, 106, 109]. The probability distribution of the transverse momentum is expressed as a function of the Wilson lines traced by the heavy quark and antiquark [107–109]. This kind of calculation assumes that the gluonic medium does not get changed while interacting with the partons. So it can be treated as a background field. So the Wilson line describing the trajectory of the partons is computed in the presence of the background field. It is important to notice that a physical quantity in this scenario must contain at least two Wilson lines. This is so due to the fact that the medium average of any observable is accomplished at the level of the cross section. At this level only colorless states are allowed. Therefore the quantities of physical interests in the are [26],

$$P(k_\perp) = \int d^2 x_\perp e^{-ik_\perp \cdot x_\perp} \mathbb{W}_{Rep}(x_\perp), \tag{1.87}$$

with

$$\mathbb{W}_{Rep}(x_\perp) = \frac{1}{dim(Rep)} \langle Tr[\mathcal{W}^\dagger_{Rep}[0, x_\perp]\, \mathcal{W}_{Rep}[0,0]] \rangle, \tag{1.88}$$

where $\mathcal{W}_{Rep}\,[x^+, x_\perp]$ is defined as,

$$\mathcal{W}_{Rep}\left[x^+, x_\perp\right] \equiv P\left\{\exp\left[ig\int_0^{L^-} dx^- A^+_{Rep}(x^+, x^-, x_\perp)\right]\right\}. \tag{1.89}$$

Here we consider the boundary coordinates as t, x_1, x_2, x_3 and define the corresponding light cone coordinates as,

$$x_\pm = \frac{1}{\sqrt{2}}(x_0 \pm x_3). \tag{1.90}$$

Consequently, the Wilson lines are aligned along the light cone. $x_\perp$ is the transverse coordinates and Rep is the representation of the gauge theory. In pertutrbative regime of the gauge theory, all these quantities can be calculated [107, 109]. However, this prescription was not very useful for strongly coupled plasma until the development AdS/CFT correspondence because there was no known field theoretical computation of the Wilson line operator for the strongly coupled plasma. As a special case of (1.57), the Wilson line calculated over the light-like trajectories of the quark-antiquark pair moving in the conformal boundary of an AdS space corresponds to the minimal area of the dual string with both endpoints attach to the boundary.

$$\langle \mathcal{W}^F(\mathcal{C}) \rangle = \exp[-S(\mathcal{C})]. \tag{1.91}$$

The detailed holographic techniques for the computation of jet quenching parameter is described in $Chapter-2$. Here we briefly mention the final result [26] for $\mathcal{N}=4$ SYM plasma.

$$\hat{q}=\frac{\pi^{3/2}\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})}\sqrt{\lambda}T^3, \tag{1.92}$$

where T is the temperature of the $\mathcal{N}=4$ SYM gauge theory and λ is the 't Hooft constant.

1.6.7 Screening length in $SU(N)$ $\mathcal{N}=4$ SYM gauge theory

Screening length L_s between a pair of quark and antiquark, moving with a uniform velocity v, in a thermal plasma can be computed by using the AdS/CFT duality. For $\mathcal{N}=4$ SYM plasma, the computation has been done in [110]. In this computation the dual gravity background is described by an AdS_5 Schwarzschild black hole geometry which is exactly similar to the AdS part of (1.48). In the next chapter we shall holographically compute the screening length in detail. In those computations, we shall describe the dual gravity theory by a general background metric which can be reduced to an AdS_5 black hole geometry by suitable choice of the metric parameters. Therefore, following the same procedure described there one can reproduce the screening length for the $\mathcal{N}=4$ SYM plasma. According to the result given in [26], the screening length in this case depends on the velocity of quark-antiquark pair and the temperature of the plasma in the following way,

$$L_s\sim\frac{0.28}{T}\sqrt{1-v^2}. \tag{1.93}$$

In this thesis, we shall study the drag force, the jet quenching parameter and the screening length in the context of two gauge theory models admitting gravity duals. First one is a holographic QCD (hQCD) model discussed in [178]. Gravity dual of this model is an asymptotically AdS (aAdS) black hole in the Einstein-Maxwell-Dilaton (EMD) system. In the second model, we take $SU(N)$ $\mathcal{N}=4$ SYM plasma associated with the standard AdS black hole gravity dual. Then we add a uniform distribution of fundamental quark to this SYM plasma. We name this gauge theory setup as the quark cloud model. Due to the presence of these extra quarks in $\mathcal{N}=4$ SYM plasma, the dual AdS black hole geometry gets deformed. The description of the hQCD model and its gravity dual is given in $chapter2$ whereas the same for quark cloud model is given in $chapter3$.

In the following section we explore the application of the AdS/CFT to explore some features of the non-perturbative M theory/string theory.

1.7 Strongly coupled String/M theory and AdS/CFT

One of the hallmark of the strong/weak duality in the string theory is that it teaches us how to tame the non-perturbative sector of the string theory via the perturbative calculations in the dual field theory. The direct proof of this duality is not yet achieved as it presupposes the non-perturbative formulation of string theory. However, there are a class of non-perturbative half-BPS objects which are protected by the symmetry (especially super symmetry) of both sides of the duality and allow us to carry out various checks of this duality. Those states are annihilated by half of the supersymmetry generators and thus belong to a short representation of the supersymmetry. Therefore it is meaningful to extrapolate their properties in the weakly coupled regime to the strongly coupled regime (see [21] and reference therein). The $typeII$ string theory contains the Dp branes as a half BPS non-perturbative objects. We have already seen in the $typeIIB$ theory, the $D3$ brane plays a major role in the context of the strong/weak duality. Similarly one may ask if the Dp branes (p even) of the $typeIIA$ theory, being half BPS, can shed further light on the strong/weak coupling duality.

When the string coupling g_s is very large, an appropriate description in terms of a eleven dimensional theory is known as M theory in the literature. In the low energy limit, it has an effective description in terms of the eleven dimensional supergravity (11-D SUGRA). The solution of the 11-D SUGRA allows only two kinds of stable half BPS configurations. One is the three dimensional super-membrane called M-2 brane [117]. The other is the five dimensional super-membrane called M-5 brane [118]. Appropriate near horizon limit of the M2 brane facilitates a description of the AdS_4/CFT_3 correspondence. We first summarize this description.

1.7.1 M-2 and M-5 branes

We start with the 11-D SUGRA preserving $\mathcal{N} = 8$ supersymmetry with 32 supercharges. We look for the the stable $\frac{1}{2}$ BPS configuration preserving half of the supersymmetry. The mass less sector consists of a metric G_{MN}, three-form gauge field C_{MNP} and the fermion $\Psi_{M,\alpha}$. The action takes the following form,

$$S_{11d} = \frac{1}{2\kappa_{11}^2}\Big[\int d^{11}x\sqrt{-G}(R_{11} - \frac{1}{2}F^2) - \frac{1}{6}\int C \wedge F \wedge F\Big] + \text{fermionic terms}, \quad (1.94)$$

where, κ_{11} is the gravitational coupling constant in eleven dimensions and is related to the eleven dimensional Plank length l_p as,

$$2\kappa_{11}^2 = (2\pi)^8 {l_p}^9. \quad (1.95)$$

R_{11} is the eleven dimensional Ricci scalar, G is the determinant of the metric, and F is the four form field strength.

$$F_{LMNP} = \partial_{[L}C_{MNP]}. \quad (1.96)$$

Three form gauge potential C_{MNP} is related to the two-form gauge transformation parameter Λ_{PQ} in the following way,

$$C_{MNP} \rightarrow C^{'}_{MNP} = C_{MNP} + \partial_{[M}\Lambda_{NP]}. \quad (1.97)$$

The full theory is invariant under the local supersymmetry, preserving the 32 supercharges. However, we are interested in the $\frac{1}{2}$ BPS configuration of the theory preserving half of the supersymmetry. Since we are considering only the mass less bosonic sector of the theory, the $\frac{1}{2}$ BPS configuration can be achieved by demanding the vanishing of supersymmetric variation of the mass less fermion Ψ_M.

$$\delta\Psi_M = 0 = \nabla_M\epsilon + \frac{1}{12}\Big(\frac{1}{4!}\Gamma_M(dC)_{PQRS}\Gamma^{PQRS} - \frac{1}{2}(dC)_{MQRS}.\Gamma^{QRS}\Big)\epsilon \tag{1.98}$$

The stability of the solution suggests that the BPS configuration must be supported by flux sourced by the three form-field C_{MNP} and its dual six-form $\tilde{C}_{LMNOPQ} = \frac{1}{3!}\epsilon_{LMNOPQRST}C^{RST}$. The four-form field strength F_{MNPQ} arises from C_{NPQ} and satisfies eq.(1.96). The dual seven-form field strength is given as

$$\tilde{F}_{LMNOPQR} = \frac{1}{4!}\epsilon_{LMNOPQRSTUV}F^{STUV}. \tag{1.99}$$

The $\frac{1}{2}$ BPS configuration must be electrically and magnetically charged and acts as a source of the magnetic and the electric flux. A magnetically charged solution must act as a source for the four-form magnetic flux emanating through a 4-sphere enclosing the solution. In general, for D number of spatial dimensions a d-sphere encloses a $D-d-2$ dimensional object. Therefore the magnetic source of the four-form magnetic flux in the eleven space-time dimensions extends along $11-4-2=5$ spatial dimensions. So we conclude that the magnetic solution of the eleven dimensional supergravity is extended along the five spatial dimensions. This five dimensional supermembrane solution gives the effective description of M-5 brane in M-theory. From now on we refer this five dimensional object as M-5 brane. The magnetic charge of the M-5 brane is given by,

$$\int_{S^4} F = Q_{M5}, \tag{1.100}$$

where S^4 is a four-sphere enclosing the M-5 brane. Similar argument shows the solution of the 11D SUGRA, which acts as a source of seven form electric flux (dual to the four form magnetic flux), extends along the $11-7-2=2$ spatial dimensions. This is a two dimensional supermembrane solution giving an effective low energy description of M-2 brane in M-theory. From now on we refer this solution by M-2 brane. The electric charge of the M-2 brane is given as,

$$\int_{S^7} \tilde{F} = Q_{M2}, \tag{1.101}$$

where S^7 is a seven-sphere enclosing the M-2 brane. Now we describe the solution of eqn (1.94) representing a stack of N_{M2} number of M-2 branes. We choose the world-volume coordinates living on the branes as $y^a = y^0, y^1, y^2$ with y^0 being the timelike direction and the traverse coordinates as $x^I = x^1, x^2, ..., x^8$. By demanding the Lorentz symmetry $SO(1,2)$ in world volume coordinates, the rotational symmetry $SO(8)$ in transverse coordinates and the transnational symmetry in y^a coordinates, the solution takes the following form,

$$ds^2 = A_1(r)dy^a dy^a + A_2(r)dx^I dx^I. \tag{1.102}$$

The only non-zero component of electric field strength is given by,

$$F_{012r} = A_3(r), \tag{1.103}$$

where r is the radial distance $\sqrt{(x^1)^2 + ... + (x^8)^2}$ from the brane. The three functions $A_1(r)$, $A_2(r)$, $A_3(r)$ can be described by a single harmonic function $H(r)$

$$H(r) = 1 + \left(\frac{r_{M2}}{r}\right)^6, \qquad \partial_I \partial_I H(r) = 0, \tag{1.104}$$

where r_{M2} is given by

$$r_{M2}^6 = 32\pi^2 N_{M2} l_p^6. \tag{1.105}$$

It is related to the charge and the tension of the M-2 brane. finally we have $A_1(r)$, $A_2(r)$, $A_3(r)$ in terms of $H(r)$.

$$\begin{aligned} A_1(r) &= H(r)^{-\frac{2}{3}}, \\ A_2(r) &= H(r)^{\frac{1}{3}}, \\ A_3(r) &= -\frac{\partial}{\partial r}(H(r)^{-1}). \end{aligned} \tag{1.106}$$

We get the electric charge of the M-2 brane, which is quantized in N_{M2} units,

$$Q_{M2} = \int_{S^7} \tilde{F} = 6(r_{M2})^6 \Omega^7 = 2\pi^4 (r_{M2})^6, \tag{1.107}$$

where $\Omega^7 = \frac{\pi^4}{3}$ is the volume of the seven-sphere. The integration constant r_{M2} can be written in terms of the brane tension T_{M2}.

$$(r_{M2})^6 = \frac{(2\pi l_p)^9}{2\pi} \frac{N_{M2} T_{M2}}{2\pi^4}. \tag{1.108}$$

Thus we achieve a relation between the charge and the mass of a single M-2 brane

$$Q_{M2} = \frac{(2\pi l_p)^9}{2\pi} T_{M2}. \tag{1.109}$$

We can similiarly construct the solution for N_{M5} of M-5 branes. Here we briefly tabulate the results.

- World volume coordinates, $y^a = y^0, ..., y^5$.
- Transverse coordinates, $x^I = x^1, ..., x^5$.
- Symmetry $= SO(1,5) \times SO(5) +$ translational symmetry in y^a coordinates.

- $ds^2 = \tilde{A}_1(r)dy^a dy^a + \tilde{A}_2(r)dx^I dx^I$.
- Magnetic filed strength $= \tilde{F}_{012345r} = \tilde{A}_3(r)$.
- radial coordinate$r = \sqrt{(x^1)^2 + ... + (x^5)^2}$.
- $\tilde{H}(r) = 1 + \left(\frac{r_{M5}}{r}\right)^3, \quad \partial_I \partial_I \tilde{H}(r) = 0.$
- $r_{M5}^3 = \pi N_{M5} l_p^3$.
- $\tilde{A}_1(r) = \tilde{H}(r)^{-\frac{1}{3}},\ \tilde{A}_2(r) = \tilde{H}(r)^{\frac{2}{3}},\ \tilde{A}_3(r) = -\frac{\partial}{\partial r}(\tilde{H}(r)^{-1})$.
- Magnetic charge $Q_{M5} = 8\pi^2 r_{M5}^3,\ \ r_{M5}^3 = \frac{(2\pi l_p)^9}{2\pi}\frac{N_{M5}T_{M5}}{8\pi^2}$.
- Charge-mass relation for a single M-5 brane, $Q_{M5} = \frac{(2\pi l_p)^9}{2\pi} T_{M5}$.

Combining the expressions for the charges Q_{M2} and Q_{M5} we get,

$$Q_{M2}Q_{M5} = (2\pi l_p)^9 N_{M2}N_{M5} = (2\pi l_p)^9 N, \tag{1.110}$$

where $N = N_{M2}N_{M5}$ is an integer. The above relation is known as the Dirac quantization condition. Tensions of M-2 and M-5 branes are also related as follows,

$$T_{M2}T_{M5} = \frac{(2\pi)^2}{(2\pi l_p)^9} N. \tag{1.111}$$

The above two eleven dimensional solutions are the stable $\frac{1}{2}$ BPS configurations of the eleven dimensional supergravity preserving the $\mathcal{N} = 8$ supersymmetry with 16 supercharges. They stand for the low energy effective description of the M-2 brane and the M-5 brane put in the eleven dimensional flat background. However, in the context of the AdS_4/CFT_3 we need N_{M2} number of coincident M-2 branes probing the conical singularity of the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. The reason of considering the $\mathbb{C}^4/\mathbb{Z}_k$ geometry will be clear when we discuss the moduli space of the N_{M2} number of coincident M-2 branes probing this particular background. $\mathbb{Z}_k$ acts as a discrete symmetry group which rotates the phase of the four complex coordinates of $\mathbb{C}^4$. The over all geometry of the background is given by $\mathcal{M}_3 \times \mathbb{C}^4/\mathbb{Z}_k$, where $\mathcal{M}_3$ represents the three dimensional Minkowski space. Since the seven dimensional base geometry of the $\mathbb{C}^4/\mathbb{Z}_k$ is $S^7/\mathbb{Z}_k$, $\mathbb{C}^4/\mathbb{Z}_k$ can be viewed as a cone over $S^7/\mathbb{Z}_k$. In order to generate the solution describing N_{M2} number of coincident M-2 branes probing the conical singularity of the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ we follow two steps. First, we generate a solution of N_{M2} number of coincident M-2 branes on a flat space, $\mathbb{C}^4$. Then we take a orbifold of this geometry in a suitable way. In the $\mathbb{C}^4$ flat background the low energy effective description of the N_{M2} number of M-2 branes takes the following form,

$$ds^2 = H(r)^{-\frac{2}{3}} dy^a dy^a + H(r)^{\frac{1}{3}}(dr^2 + r^2 ds_{S^7}^2), \tag{1.112}$$

where $H(r)$ is same as in equation (1.104) We are interested in the geometry near to the stack of M-2 brane. It can be realized by taking the near horizon limit ($r \to 0$) of equation (1.112). In this limit, the metric simplifies significantly,

$$ds^2 = (\frac{r}{r_{M2}})^4 dy^a dy^a + (\frac{r_{M2}}{r})^2 (dr^2 + r^2 ds^2_{S^7}). \tag{1.113}$$

Redefining the radial coordinate as $\rho = \frac{r^3_{M2}}{2r^2}$, we get the final form the metric,

$$ds^2 = \tilde{R}^2 \{ \underbrace{\frac{dy^a dy^a + d\rho^2}{\rho^2}}_{AdS_4} + 4 \underbrace{ds^2_{S^7}}_{S^7} \}. \tag{1.114}$$

The geometry factorizes to a direct product of a AdS_4 space with radius $\tilde{R} = \frac{r_{M2}}{2}$ and a seven-sphere of radius $2\tilde{R} = r_{M2}$. Now we consider the action of orbifolding over the seven sphere part of the geometry given by (1.114). This orbifolding is more explicit when we write the metric $ds^2_{S^7}$ as a Hopf fibration of the S^1 circle over three dimensional complex projective plane $\mathbb{CP}_3$. S^7 can be coordinatized by four complex coordinates z^1, z^2, z^3, z^4 constrained to the condition,

$$|z^1|^2 + |z^2|^2 + |z^3|^2 + |z^4|^2 = 1. \tag{1.115}$$

The explicit parametrization of the complex coordinates is given by,

$$\begin{aligned}
z^1 &= \cos\zeta \sin\frac{\theta_1}{2} e^{i(\alpha + \frac{1}{4}\psi - \frac{1}{2}\phi_1)}, \\
z^2 &= \cos\zeta \cos\frac{\theta_1}{2} e^{i(\alpha + \frac{1}{4}\psi + \frac{1}{2}\phi_1)}, \\
z^3 &= \sin\zeta \sin\frac{\theta_2}{2} e^{i(\alpha - \frac{1}{4}\psi + \frac{1}{2}\phi_2)}, \\
z^4 &= \sin\zeta \cos\frac{\theta_2}{2} e^{i(\alpha - \frac{1}{4}\psi - \frac{1}{2}\phi_2)}.
\end{aligned} \tag{1.116}$$

The range of the parameters are , $\zeta \in [0, \frac{\pi}{2}]$, $\theta_1, \theta_2 \in [0, \pi]$, $\alpha, \phi_1, \phi_2 \in [0, 2\pi]$ and $\psi \in [0, 4\pi]$. α plays the role of an overall phase factor, The metric depicting the Hopf fibration of S^1 fiber over $\mathbb{CP}_3$ can be described as,

$$ds^2_{S^7} = (d\alpha + \omega)^2 + ds^2_{\mathbb{CP}_3}. \tag{1.117}$$

The Fubini-Study metric of $\mathbb{CP}_3$ space and the one-form potential of the Kähler form of the same space are given respectively,

$$\begin{aligned}
ds^2_{\mathbb{CP}_3} &= d\zeta^2 + \frac{1}{4}\cos^2\zeta \sin^2\zeta [d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2]^2 \\
&\quad \frac{1}{4}\cos^2\zeta [d\theta_1^2 + \sin^2\theta_1 d\phi_1^2] + \frac{1}{4}\sin^2\zeta [d\theta_2^2 + \sin^2\theta_2 d\phi_2^2], \\
\omega &= \frac{1}{4}\cos(2\zeta) d\psi + \frac{1}{2}\cos^2\zeta \cos\theta_1 d\phi_1 - \frac{1}{2}\sin^2\zeta \cos\theta_2 d\phi_2.
\end{aligned} \tag{1.118}$$

Now this $\mathbb{Z}_k$ quotienting over the seven-sphere is done by

$$z^I \to e^{\frac{2\pi i}{k}} z^I. \tag{1.119}$$

The action of the $\mathbb{Z}_k$ quotienting translates into the effect on the phase factor α, which identifies the total phase $\alpha \sim \alpha + \frac{2\pi}{k}$, where k is an integer. This identification actually shrinks the radius of the fiber S^1 by a factor $\frac{1}{k}$. By re-scaling the phase factor $\tilde{\alpha} = k\alpha$ we can rewrite the metric (1.117) in the desired form.

$$ds^2_{S^7/\mathbb{Z}_k} = \frac{1}{k^2}(d\tilde{\alpha} + k\omega)^2 + ds^2_{\mathbb{CP}_3}. \tag{1.120}$$

So the metric of the full-space time is now $AdS_4 \times S^7/\mathbb{Z}_k$.

$$ds^2 = \tilde{R}^2\{\underbrace{\frac{dy^a dy^a + d\rho^2}{\rho^2}}_{AdS_4} + 4\underbrace{\frac{1}{k^2}(d\tilde{\alpha} + k\omega)^2 + 4ds^2_{\mathbb{CP}_3}}_{S^7/\mathbb{Z}_k}\}. \tag{1.121}$$

By shrinking the radius of fiber S^1, the volume of the space $S^7/\mathbb{Z}_k$ can be made smaller by a factor k. In the orbifold space, to make the electric charge of the M-2 brane quantized in N_{M2} units, N_{M2} is needed to be kN_{M2}. Now correspondingly, the radius of the $\mathbb{CP}_3$ changes into $(32\pi^2 kN_{M2})^{\frac{1}{6}}$ (in Plank's length unit) and the same for the fiber S^1 reduces to $(\frac{32\pi^2 N_{M2}}{k^5})^{\frac{1}{6}}$. So to retain the successful perturbative description of M-theory, the limit $N \gg k^5$ should be maintained. When the radius of S^1 becomes vanishingly small, the eleven-dimensional supergravity reduces to an effective low energy description to the ten-dimensional $typeIIA$ superstring theory. The reduction of the metric (1.121) is similar to the one described in [119]

$$ds^2_{11} = e^{-\frac{2\phi}{3}} ds^2_{IIA} + e^{\frac{4\phi}{3}}(d\tilde{\alpha} + A_1)^2. \tag{1.122}$$

Comparing equation (1.121) and (1.122) one can identify the metric, dilaton and two-form field as

$$\begin{aligned} ds^3_{IIA} &== \frac{\tilde{R}^3}{k}(ds^2_{AdS_4} + 4ds^2_{\mathbb{CP}_3}), \\ e^{2\phi} &= \frac{\tilde{R}^3}{k^3}, \\ F_2 &= dA_1 = kd\omega. \end{aligned} \tag{1.123}$$

The four-form field remains the same as it is in the eleven dimensions. The action of the orbifolding further reduces the supersymmetry from $\mathcal{N} = 8$ to $\mathcal{N} = 6$.

1.7.2 ABJM theory

So far we have discussed the supergravity descriptions of M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ or the $typeIIA$ string theory on $AdS_4 \times \mathbb{CP}_3$. It is clear from section (1.3), that these descriptions represent the gravity side of the AdS_4/CFT_3 correspondence. To complete the circle, we need to introduce the three dimensional CFT description of the dual gauge theory side. It was first Aharony, Bergman, Jafferis and Maldacena (ABJM) who proposed the dual gauge theory providing the low energy physics of the world-volume of N number of M-2 branes probing the $\mathbb{C}^4/\mathbb{Z}_k$ geometry [120]. It is a $\mathcal{N} = 6$ Chern-Simon-matter theory with the matter fields transforming under the bi-fundamental representation of the $U(N)_k \times U(N)_{-k}$ semi-simple gauge group. k and $-k$ are the Chern-Simon levels with opposite signs and they take only integer values. Aharony, Bergman, Jafferis proposed a further generalization of the dual gauge theory where the gauge group is modified as $U(M)_k \times U(N)_{-k}$ [121]. This generalized theory is known as ABJ theory. The gauge sector of the ABJ(M) theory includes two gauge fields, $\mathbb{A}_\mu$ carrying the gauge index of $U(M)$ subgroup and $\hat{\mathbb{A}}_\mu$ carrying the same of $U(N)$ gauge group. The matter sector of the theory consists of four complex scalars (A_1, A_2, B_1, B_2) and four three dimensional Dirac fermions ψ_a and their complex conjugates $(A_1^\dagger, A_2^\dagger, B_1^\dagger, B_2^\dagger)$ and $\psi^{\dagger a}$, where the index a runs from 1 to 4. The scalar fields describe the complexified eight coordinates transverse to the world-volume of the M-2 brane. All the matter fields transform in the bi-fundamental representation $(M, \bar{N})$ and their conjugates in the anti-bi-fundamental representation $(\bar{M}, N)$ of the gauge group $U(M)_k \times U(N)_{-k}$. There is a global R symmetry under which ABJ(M) remains invariant. The gauge group of the R symmetry is given as $SO(6) \cong SU(4)$. The bi-fundamental scalars and the fermions transform as **4** and $\bar{\mathbf{4}}$ under R symmetry.

There are various ways to write down the action of ABJ(M) theory. It can be written in terms of the $\mathcal{N} = 1$, $\mathcal{N} = 2$, $\mathcal{N} = 3$ and $\mathcal{N} = 6$ superspace formalism [122–125]. Here we discuss the action in terms of the $\mathcal{N} = 2$ superfields. In this formulation a $SU(2) \times SU(2)$ subgroup of $SU(4)$ R symmetry is manifest. The two vector fields $\mathbb{A}_\mu$, $\hat{\mathbb{A}}_\mu$ belong to vector superfields $\mathcal{V}$ and $\bar{\mathcal{V}}$ respectively. The matter sector in bi-fundamental representation can be arranged as the components of chiral superfields $\mathcal{A}^a$ and $\mathcal{B}_a$, where the index a runs from 1 to 2. $\mathcal{A}^a$ and $\mathcal{B}_a$ transform in $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \bar{\mathbf{2}})$ of the global group $SU(2) \times SU(2)$. They also transform as $(M, \bar{N})$ and $(\bar{M}, N)$ of the gauge symmetry group $U(M)_k \times U(N)_{-k}$. Following the convention of the $\mathcal{N} = 2$ super field expansion we can write down the expression for $\mathcal{A}$ and $\mathcal{B}$.

$$
\begin{aligned}
\mathcal{A} &= A(y) + \sqrt{2}\theta\zeta(y) + \theta\theta F(y), \\
\bar{\mathcal{A}} &= \bar{A}(\bar{y}) - \sqrt{2}\bar{\theta}\zeta^\dagger(\bar{y}) - \overline{\theta\theta F^\dagger}(\bar{y}), \\
\mathcal{B} &= B(y) + \sqrt{2}\theta\omega(y) + \theta\theta G(y), \\
\bar{\mathcal{B}} &= \bar{B}(\bar{y}) - \sqrt{2}\bar{\theta}\omega^\dagger(\bar{y}) - \overline{\theta\theta G^\dagger}(\bar{y}),
\end{aligned} \tag{1.124}
$$

where $\zeta, \zeta^\dagger, F, \bar{F}^\dagger, \omega, \omega^\dagger, \bar{G}^\dagger$ are the auxiliary fields. The superpotential manifestly showing

$SU(2) \times SU(2)$ global symmetry can be also constructed.

$$\begin{aligned} W(\mathcal{A},\mathcal{B}) &= \tfrac{1}{4!}\epsilon_{ac}\epsilon^{bd}Tr(\mathcal{A}^a\mathcal{B}_b\mathcal{A}^c\mathcal{B}_d), \\ \overline{W}(\bar{\mathcal{A}},\bar{\mathcal{B}}) &= \tfrac{1}{4!}\epsilon^{ac}\epsilon_{bd}Tr(\bar{\mathcal{A}}_a\bar{\mathcal{B}}^b\bar{\mathcal{A}}_c\bar{\mathcal{B}}^d). \end{aligned} \tag{1.125}$$

The total action including the contribution of the gauge fields and chiral superfields take the following form,

$$\begin{aligned} \mathcal{S} &= [\mathcal{S}_{CS} + \mathcal{S}_{mattar} + \mathcal{S}_{pot}], \\ \mathcal{S}_{CS} &= -\tfrac{ik}{8\pi}\int d^3x \int d^4\theta \int_0^1 ds Tr[\mathcal{V}\bar{D}^\alpha(e^{s\mathcal{V}}D_\alpha e^{-s\mathcal{V}}) - \bar{\mathcal{V}}\bar{D}^\alpha(e^{s\bar{\mathcal{V}}}D_\alpha e^{-s\bar{\mathcal{V}}}], \\ \mathcal{S}_{mattar} &= -\int d^3x \int d^4\theta Tr[\bar{\mathcal{A}}_a e^{-\mathcal{V}}\mathcal{A}^a e^{\bar{\mathcal{V}}} + \bar{\mathcal{B}}^a e^{-\bar{\mathcal{V}}}\mathcal{B}_a e^{\mathcal{V}}], \\ \mathcal{S}_{pot} &= \tfrac{8\pi}{k}\int d^3x \int d^2\theta\, W(\mathcal{A},\mathcal{B}) + \tfrac{8\pi}{k}\int d^3x \int d^2\bar{\theta}\, \overline{W}(\bar{\mathcal{A}},\bar{\mathcal{B}}). \end{aligned} \tag{1.126}$$

It is important to note that the super potential (1.125) is invariant under an extra global $U(1)$ symmetry.

$$\begin{aligned} \mathcal{A}^a &\to e^{i\alpha}\mathcal{A}^a, \\ \mathcal{B}_a &\to e^{-i\alpha}\mathcal{B}_a. \end{aligned} \tag{1.127}$$

So the R symmetry group is enhanced to $U(1) \times SU(2) \times SU(2)$. We can arrange the scalar and the fermionic fields into the multiplets of full $SU(4)_R$ symmetry.

$$\begin{aligned} Y^a &= \{A_1, A_2, B_1^\dagger, B_2^\dagger\}, \\ Y_a^\dagger &= \{A_1^\dagger, A_2^\dagger, B_1, B_2\}, \\ \psi_a &= \{\epsilon_{ab}\zeta^b e^{-\frac{i\pi}{4}}, -\epsilon_{ab}\omega^{\dagger b}e^{\frac{i\pi}{4}}\}, \\ \psi^{a\dagger} &= \{-\epsilon^{ab}\zeta_b^\dagger e^{\frac{i\pi}{4}}, \epsilon^{ab}\omega_b e^{-\frac{i\pi}{4}}\}, \quad a = 1,..4. \end{aligned} \tag{1.128}$$

By integrating out all the auxiliary fields from the action (1.126) we get

$$\begin{aligned} \mathcal{S}_{kinetic} &= \int d^3x[\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}Tr(\mathbb{A}_\mu\partial_\nu\mathbb{A}_\lambda + \frac{2i}{3}\mathbb{A}_\mu\mathbb{A}_\nu\mathbb{A}_\lambda - \hat{\mathbb{A}}_\mu\partial_\nu\hat{\mathbb{A}}_\lambda \\ &\quad -\frac{2i}{3}\hat{\mathbb{A}}_\mu\hat{\mathbb{A}}_\nu\hat{\mathbb{A}}_\lambda) - Tr(D_\mu Y_a^\dagger)(D^\mu Y^a) - Tr(\psi^{a\dagger}i\gamma^\mu D_\mu\psi_a)], \end{aligned} \tag{1.129}$$

$$\begin{aligned} \mathcal{S}_{V_{bos}} &= -\frac{4\pi}{3k^2}\int d^3x Tr(Y^aY_a^\dagger Y^bY_b^\dagger Y^cY_c^\dagger + Y_a^\dagger Y^aY_b^\dagger Y^bY_c^\dagger Y^c + \\ &\quad 4Y^aY_b^\dagger Y^cY_a^\dagger Y^bY_c^\dagger - 6Y^aY_b^\dagger Y^bY_a^\dagger Y^cY_c^\dagger), \end{aligned} \tag{1.130}$$

$$\mathcal{S}_{V_{fer}} = \frac{2\pi i}{k}\int d^3x Tr[Y_a^\dagger Y^a\psi^{b\dagger}\psi_b - Y^aY_a^\dagger\psi_b\psi^{b\dagger} + 2Y^aY_b^\dagger\psi_a\psi^{b\dagger} \tag{1.131}$$

$$-2Y_a^\dagger Y^b\psi^{a\dagger}\psi_b - \epsilon^{abcd}Y_a^\dagger\psi_bY_c^\dagger\psi_d + \epsilon_{abcd}Y^a\psi^{b\dagger}Y^c\psi^{d\dagger}], \tag{1.132}$$

where the covariant derivative acts as, $D_\mu Y^a = \partial_\mu Y^a + i\mathbb{A}_\mu Y^a - iY^a\hat{\mathbb{A}}_\mu$. The gauge transformation rule acting on the scalars and the gauge fields is induced by the operators

$(U, \hat{U}) \in U(M) \times U(N)$.

$$\begin{aligned} Y^a \to U Y^a \hat{U}^\dagger, \quad Y_a^\dagger \to \hat{U} Y_a^\dagger U^\dagger, \\ \mathbb{A}_\mu \to U \mathbb{A}_\mu U^\dagger - i U \partial_\mu U^\dagger, \quad \hat{\mathbb{A}}_\mu \to \hat{U} \hat{\mathbb{A}}_\mu \hat{U}^\dagger - i \hat{U} \partial_\mu \hat{U}^\dagger. \end{aligned} \tag{1.133}$$

The supersymmetry transformation of the ABJ(M) action is generated by 12 supercharges. In particular, the $U(N) \times U(N)$ ABJM theory provides the low energy physics of the world-volume of N number of M-2 branes probing the $\mathbb{C}^4/\mathbb{Z}_k$ geometry. Being a half-BPS object, the M-2 brane allows only 16 of the 32 supercharges. The orbifolding procedure further reduces the number of supercharges from 16 to 12. In three dimensions, they furnishes the $\mathcal{N} = 6$ supersymmetry algebra. The 12 supecharges combine into 6 real spinors which transform in the vector representation **6** of the R symmetry group $SU(4) \cong SO(6)$. In addition to the gauge symmetry and supersymmetry, ABJ(M) is invariant under Poincarè group, as well as the scale transformations. These symmetries together form a three-dimensional conformal group $SO(2, 3)$. The conformal symmetry, R symmetry and supersymmetry of ABJ(M) theory build up a superconformal group $OSp(6|4)$. Moreover ABJM theory preserves another $\mathbb{Z}_2$ parity symmetry under the exchange of the Chern-Simon indices k and $-k$. The ABJ theory does not respect parity symmetry due to the asymmetry between the orders of the gauge group $U(M) \times U(N)$. The realization of moduli space of ABJ(M) is one of the main cornerstone to understand the AdS_4/CFT_3 correspondence. We discuss mainly the moduli space of the $U(N) \times U(N)$ ABJM theory. The moduli space of the $U(L) \times U(M)$ ABJ theory is exactly same as the $U(N) \times U(N)$ theory, where $N = \min(L, M)$. For simplicity we consider the moduli space of the Abelian ABJM theory with gauge group $U(1) \times U(1)$. The generalization for $U(N) \times U(N)$ follows from the Abelian case. For the Abelian theory we consider only the bosonic sector. In this sector, the potential and the interaction terms vanish and the action reduces to a free field action. The four complex scalar fields are given by 1×1 matrices Y^1, Y^2, Y^3, Y^4. Apparently, the moduli space seems to be $\mathbb{C}^4$. But it can be shown that due to the presence of Chern-Simon term still there is a $\mathbb{Z}_2$ symmetry playing among the scalars, $Y^a \to e^{\frac{2\pi}{k}} Y^a \;\; a = 1, .., 4$ [120]. So the moduli space turns out to be $\mathbb{C}^4/\mathbb{Z}_k$. Now for non-Abelian $U(N) \times U(N)$ case the scalar potential vanishes for diagonal Y^a. This configuration actually gives the full moduli space of the theory. The $U(N) \times U(N)$ gauge symmetry reduces to the $U(1)^N \times U(1)^N \times S_N$, where S_N is the symmetry group permuting N number of diagonal elements. So generalizing the $U(1) \times U(1)$ case we get the moduli space as $\frac{(\mathbb{C}^4/\mathbb{Z}_k)^N}{S_N}$.

The ABJ(M) gauge theory enjoys a planar limit. We know that the $U(N) \times U(N)$ ABJM gauge theory has two parameters, the Chern-Simon level k and the order of the gauge group N. Both parameters take integer values. The Chern-Simon level k seats out side the action as an overall factor. Thus we can define the coupling strength of the ABJM theory as $g_{CS}^2 = \frac{1}{k}$. When $k \gg 1$ the gauge theory is weakly coupled and one can consistently use the perturbation theory. Like the $\mathcal{N} = 4$ SYM theory, here we can also define a 't Hooft coupling constant λ.

$$\lambda = g_{CS}^2 N = \frac{N}{k}. \tag{1.134}$$

Now it is straightforward to take the planar limit,

$$N \to \infty, k \to \infty, \quad \text{with } \lambda \text{ fixed.} \tag{1.135}$$

As N and k are integers λ must be a rational number but in the 't Hooft limit one can consider it as a continuous number. The $U(M) \times U(N)$ ABJ theory comes with two different orders of the gauge group, M and N. So theory contains an extra parameter $M - N$, where it is assumed that $M > N$. This extra parameter allows to introduce two distinct 't Hooft couplings,

$$\lambda = \frac{M}{k}, \quad \hat{\lambda} = \frac{N}{k}. \tag{1.136}$$

Correspondingly, the 't Hooft limit is defined as

$$M \to \infty, \;\; N \to \infty, \;\; k \to \infty, \quad \text{ with } \lambda, \; \hat{\lambda} \text{ fixed.} \tag{1.137}$$

Instead of treating λ and $\hat{\lambda}$ as the coupling constants, it is more convenient to define

$$\bar{\lambda} = \sqrt{\lambda\hat{\lambda}}, \quad \sigma = \frac{(\lambda - \hat{\lambda})}{\bar{\lambda}}, \tag{1.138}$$

where $\bar{\lambda}$ is now the equivalent coupling constant of the theory and σ is the deviation of coupling constant from the same in ABJM theory.

1.7.3 AdS_4/CFT_3 correspondence.

Let us further review the evidences in favor of the AdS_4/CFT_3 correspondence. The main reason behind this correspondence is the geometrical similarity between the *moduli space of vacua* in the gauge theory and the *manifold of vacua* in the dual M-theory. The moduli space of vacua of the ABJM theory is the N-th symmetric power of four dimensional complex space orbifolded by a discrete symmetry group $\mathbb{Z}_k$, i.e. $\frac{(\mathbb{C}^4/\mathbb{Z}_k)^N}{S_N}$. In the M-theory, parallel M-2 branes are $\frac{1}{2}$ BPS objects and there is no force acting between them. So one can consider N number of indistinguishable, non-interacting M-2 branes freely moving around the $\mathbb{C}^4/\mathbb{Z}_k$ singularity. Such configuration of M-2 brane stack corresponds to the vacuum of the theory. So the vacuum moduli space of the $ABJM$ theories has precisely the right form to be interpreted as the manifold of the M-theoretic vacua representing N number of coincident M-2 branes moving in the $\mathbb{C}^4/\mathbb{Z}_k$ transverse space [120]. Furthermore, in the gauge theory side of the correspondence, the coupling of the theory is given by $g^2_{CS} = \frac{1}{k}$. It is weakly coupled for $k \gg 1$. For large N, the theory admits a 't Hooft expansion in powers of $\frac{1}{N^2}$. One can associate a 't Hooft coupling $\lambda = \frac{N}{k}$ with this expansion. In the planar limit ($N \to \infty, k \to \infty,$ fixed and finite λ) only planar diagram survives. The theory has a perturbative description when coupling constant λ is small ($k \gg N$). The $\mathcal{N} = 6$ supersymmetry, the $SU(4) \times U(1)$ R symmetry and the conformal symmetry in three dimensions, $SO(2,3)$ combine into the $OSp(6|4)$ superconformal symmetry.

In the gravity side, we consider the near horizon limit of the geometry generated by the N coincident M-2 branes probing the $\mathbb{C}^4/\mathbb{Z}_k$ singularity. For simplicity, first we choose $k = 1$. The near horizon geometry of N M-2 branes becomes $AdS_4 \times S^7$ with the N units of four-form flux. It is always convenient to think of the S^7 as a Hopf fibration of a S^1 over a complex-projective plane $\mathbb{CP}^3$. The global isometry group of the full space is now $SO(2,3) \times SO(8)$. If we take the k^{th} order orbifolding over the S^7 part, the near horizon geometry becomes $AdS_4 \times S^7/\mathbb{Z}_k$. In the language of the Hopf fibration, this orbifolding reduces the radius of the S^1 fiber k times smaller than the original one. It also modifies the isometry group into $SO(2,3) \times SU(4) \times U(1)$. The components, $SO(2,3)$, $SU(4)$, $U(1)$ act on AdS_4, $\mathbb{CP}^3$, S^1 respectively. Reducing the M-theory on the S^1 fiber, one can recover the $typeIIA$ string theory on $AdS_4 \times \mathbb{CP}^3$ preserving same amount of symmetry. We know because of the orbifolding, the M-theory/string theory preserves only 12 out of 32 supercharges. The isometry and the supersymmetry of the gravity sector form a larger symmetry group $OSp(6|4)$ which is exactly the superconformal symmetry group of the ABJM theory.

These observations leads to the proposal that, the planar sector of the three dimensional, $\mathcal{N} = 6, U(N) \times U(N)$ ABJM theory is dual to the eleven dimensional supergravity on $AdS_4 \times S^7/\mathbb{Z}_k$ (the low energy effective description of N M-2 branes) or the ten dimensional $typeIIA$ supergravity in $AdS_4 \times \mathbb{CP}^3$ (the low energy effective description of N D-2 branes in $typeIIA$ string theory) [120].

It would be natural to expect some relation between the parameters of both sides of the correspondence. In the gravity sector, we have seen that after implementing $\mathbb{Z}_k$ orbifolding, the radius $\tilde{R}$ in eqn (1.121) changes as $\tilde{R}^6 = \frac{(32\pi^2 kN)}{2}$ (in Plank's unit). Keeping this in mind, we compare eqn (1.134) and (1.121) and get the relation between the parameters,

$$\lambda = \frac{N}{k} = \frac{2\tilde{R}^6}{\pi^2 k^2}. \tag{1.139}$$

The supergravity description of the duality to be good, we expect the large value of the 't Hooft coupling.

$$\lambda \gg 1 \rightarrow N \gg k. \tag{1.140}$$

We know that the consequence of the orbifolding on the fiber S^1 is the reduction of its radius into $(\frac{32\pi^2 N}{k^5})^{\frac{1}{6}}$. To have a meaningful supergravity description of the M-theory we need the radius of S^1 fiber to be large. This leads to

$$N \gg k^5. \tag{1.141}$$

When the gravity sector is the $typeIIA$ supergravity, by comparing (1.134) and (1.123) the relation between the parameters becomes

$$\lambda = \frac{N}{k} = \frac{2R^4}{\pi^2}, \tag{1.142}$$

where we have assumed that $R^2 = \frac{\tilde{R}^3}{k}$. The $typeIIA$ description is more appropriate than the M-theory when the radius of the fiber S^1 is small. This puts a restriction on the parameters.

$$k^5 \gg N. \tag{1.143}$$

We tabulate the three ranges of the parameter for which we can certainly exploit the duality.

$$\begin{aligned} \lambda \ll 1 : &\quad \text{Weakly coupled field theory.} \\ N \gg k^5 : &\quad \text{eleven dimensional supergravity} \\ k \ll N \ll k^5 : &\quad \text{type IIA supergravity.} \end{aligned} \tag{1.144}$$

The Chern-Simon level k is identified with the order of orbifold symmetry group in the M-theory side.

We now review the form of this duality in the ABJ theory. In this theory the gauge group is given by $U(M) \times U(N)$, where, we have assumed $M > N$ without the loss of any generality. We have already seen that for ABJ theory one may define two independent 't Hooft coupling constant. The dual supergravity theory builds up an effective description which allows the N number of M-2 branes moving in the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity and the $|M - N|$ number of fractional M-2 branes are localized at the singularity. By fractional M2-branes we can think of a configuration that represents a wrapping of a M5-branes on a vanishing three-cycle at the orbifold point. This effective geometry is given by $AdS_4 \times S^7/\mathbb{Z}_k$ with a background three-form field C_3 [121].

$$\frac{1}{2\pi} \int_{S^3/\mathbb{Z}_k \in S^7/\mathbb{Z}_k} C_3 = \frac{M - N}{k} + \frac{1}{2}. \tag{1.145}$$

By compactification over S^1 we get the $typeIIA$ supergravity on $AdS_4 \times \mathbb{CP}_3$ in the gravity side, with a background NS B-field B_2 having a non-trivial holonomy on $\mathbb{CP}^1 \in \mathbb{CP}^3$ [121]

$$\frac{1}{2\pi} \int_{\mathbb{CP}^1 \in \mathbb{CP}^3} B_2 = \frac{M - N}{k}. \tag{1.146}$$

There is an interesting property that is followed by the ABJ gauge theory due to its dual brane construction. An equivalence develops between the $U(M)_k \times U(N)_{-k}$ gauge theory with gauge coupling $\lambda, \hat{\lambda}$ and $U(N)_k \times U(2N - M + k)_{-k}$ with gauge coupling $2\hat{\lambda}, 2\hat{\lambda} - \lambda + 1$ [121].

There are further evidences for the AdS_4/CFT_3 duality which we have left out. For example, the spectrum of supergravity fields is in complete agreement with the spectrum of the chiral primary operators of the gauge theory [126]. For details we refer the readers to the reviews [127–129].

Before concluding this section, we would now like to digress for a moment and consider an important development in this area. It started with the works of Bagger, Lambert and Gustavson (BLG) where they proposed a model for multiple M-2 branes [130, 131].

1.7.4 BLG theory

BLG theory is a three-dimensional world-volume theory of the M2-branes. The eleven dimensional spacetime can be grouped into three dimensional world volume of M-2 brane ($\{x^\mu\},\ \mu = 0, 1, 2$) and the eight dimensional Euclidean space ($\{X^I\},\ I = 1, ..., 8$) transverse to the world volume. The eight transverse directions play the role of scalar fields in the world volume gauge theory. The theory has the explicit $\mathcal{N} = 8$ supersymmetry and $SO(8)$ R symmetry. As because the theory is supersymmetric there are also eight $Spin(1, 2)$ fermions which can be collectively thought of as a single eleven-dimensional Majorana spinor, Ψ with 32 independent real components. The spinor satisfies the following property,

$$\gamma_{012}\Psi = -\Psi, \tag{1.147}$$

where γ_μ is the world volume gamma matrix. In addition, there is a gauge field which contributes to a Chern-Simon term.

The action of the theory takes the following form,

$$\begin{aligned}\mathcal{L} = Tr\big(-\tfrac{1}{2}(D_\mu X^I)(D^\mu X^I) + \tfrac{i}{2}\bar{\Psi}\gamma^\mu D_\mu\Psi + \tfrac{i}{4}\bar{\Psi}\Gamma_{IJ}\{X^I, X^J, \Psi\} - \tfrac{1}{12}\{X^I, X^J, X^K\}^2\big)\\ + \tfrac{1}{2}\epsilon^{\mu\nu\lambda}\big(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \tfrac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\big).\end{aligned} \tag{1.148}$$

Here $\mu = 0, 1, 2$ designates the world volume directions, $I = 1, \ldots, 8$ denotes the flavors and $a = 1, 2, 3, 4$ are the indices of the gauge theory the containing gauge fields $A_{\mu_{ab}}$. γ and Γ are, respectively, the three- and eight-dimensional gamma matrices. f^{abcd} are the structure constants of the ternary algebra, while the ternary bracket is written as $\{\ ,\ ,\ \}$ (we shall explain the ternary algebra in the next section). $-\frac{1}{12}\{X^I, X^J, X^K\}^2$ is the sextic potential term. The gauge theory generators which furnish the ternary algebra are denoted as T_a. Correspondingly, we define the metric tensor for raising and lowering the gauge indices.

$$h_{ab} = tr(T_a T_b). \tag{1.149}$$

The scalars and the fermion take the value in the ternary algebra as

$$X^I = h^{ab}X^I_a T_b, \tag{1.150}$$

$$\Psi = h^{ab}\Psi_a T_b. \tag{1.151}$$

The covariant derivative in the above action (1.148) is given as,

$$D_\mu X^{Ia} = \partial_\mu X^{Ia} + f^a{}_{bcd}A^{cd}_\mu X^{Ib} \tag{1.152}$$

The gauge transformation of the fields are given by,

$$\begin{aligned}\delta X^I_a &= -f_{abcd}\Lambda^{bc}X^{Id}\\ \delta\ \Psi^a &= -f^{abcd}\Lambda_{bc}\Psi^d\\ f_{abcd}\delta A^{ab}_\mu &= f_{abcd}D_\mu\Lambda^{ab},\end{aligned} \tag{1.153}$$

where Λ_{ab} are the gauge transformation parameters.

The theory is invariant under the supersymmetry transformations [130]

$$\delta X^I = i\,\overline{\theta}\,\Gamma^I\Psi, \tag{1.154}$$

$$\delta\Psi = D_\mu X^I\gamma_\mu\Gamma^I\theta - \frac{1}{6}\Gamma^{IJK}\{X^I, X^J, X^K\}\,\theta, \tag{1.155}$$

$$\delta A_\mu(\phi) = i\,\overline{\theta}\,\gamma_\mu\Gamma^I\{\Psi, X^I, \phi\}, \tag{1.156}$$

where ϕ represents either a X^I or Ψ and θ denotes the parameter of the supersymmetry variation. The supersymmetry transformations close on-shell up to the translation and the gauge transformation. During the process it becomes evident that the structure constant satisfies the following identity which is known as the fundamental identity,

$$f^{abc}{}_g f^{efg}{}_d = f^{efa}{}_g f^{gbc}{}_d + f^{agc}{}_d f^{efb}{}_g + f^{abg}{}_d f^{efc}{}_g. \tag{1.157}$$

The only possible solution of the above identity is given by,

$$f^{abcd} = \frac{2\pi}{k}\epsilon^{abcd}, \tag{1.158}$$

where k is the Chern-Simon level. The realization of the theory is furnished by a $SO(4)$ gauge theory where X^I and the fermion Ψ transform as vectors under the gauge group [130]. The metric is taken to be Euclidean,

$$h_{ab} = \delta_{ab}. \tag{1.159}$$

The ternary bracket takes the following form,

$$\{X^I, X^J, X^K\} = \epsilon^{abcd}X^I_a X^J_b X^K_c T_d. \tag{1.160}$$

The classical vacuum moduli space of the BLG theory was systematically pointed out by Lambert and Tong [132]. Later the same realization of moduli space was achieved in a gauge invariant way [133]. The analysis shows, for $k = 1$ the moduli space is given by,

$$\mathcal{M}_{k=1} \cong \frac{\mathbf{R}^8 \times \mathbf{R}^8}{\mathbf{Z_2} \times \mathbf{Z_2}}. \tag{1.161}$$

While for $k = 2$ the moduli space takes the form,

$$\mathcal{M}_{k=2} \cong \frac{(\mathbf{R}^8/\mathbf{Z_2}) \times (\mathbf{R}^8/\mathbf{Z}_2)}{\mathbf{Z}_2}. \tag{1.162}$$

Both structures of moduli spaces agree with the fact that BLG theory is a theory of two M-2 branes moving in an orbifolded background. $\mathbf{R}^8/\mathbf{Z_2}$.

1.7.5 3-algebra

We have mentioned the ternary algebra or 3-algebra while describing the BLG theory. Though in mathematics, the ternary algebra structure [136–138] was developed quite a while back, its realization in physical problems remained elusive until recently. It only attracted attention after its appearance in the context of the multiple M-2 branes. A ternary algebra is a vector space V endowed with a triple product,

$$V \otimes V \otimes V \to V. \tag{1.163}$$

In particular we can define a linear adjoint map on the elements of the vector space V,

$$Ad_{A,B}(X) = \{X, A, B\}. \tag{1.164}$$

It is required that the linear adjoint map should act as a derivation on the algebra,

$$\begin{aligned} ad_{A,B}(\{X,Y,Z\}) = \{ad_{A,B}(X), Y, Z\} + \{X, ad_{A,B}(Y), Z\} \\ +\{X, Y, ad_{A,B}(Z)\}. \end{aligned} \tag{1.165}$$

The consistency of the derivation rule further requires that the elements of the vector space should satisfy some identity known as fundamental identity.

$$\{\{X,Y,Z\},A,B\} = \{\{X,A,B\},Y,Z\} + \{X,\{Y,A,B\},Z\} + \{X,Y,\{Z,A,B\}\}\ , \tag{1.166}$$

One may consistently develop a one to one mapping between the 3-algebraic structure and the 2-algebraic structure when both of them have properly defined metrics. This property makes the 3-algebra structure very useful for constructing the gauge symmetry of the M-2 brane. There are also some generalization of 3-algebras which have indefinite metrics for some physical reasons. For example, those theories allow the negative normed states to be gauged away. This kind of 3-algebra structure reduces the original theory into the Yang-Mills gauge theories of the Dp-branes. [140, 145–150]. There are also infinite dimensional 3-algebras with positive definite metric constructed from Nambu bracket [135]. Theories based on this algebra are related to M-5 branes [141–144]. Apart from the physical applications, there are also successful attempts to understand the rich mathematical structure of the 3-algebra. In particular, the consistent generalization of the Kac-Moody, the (centerless) Virasoro 2-algebras, the classical Heisenberg 2-algebra into respective 3-algebras are achieved [236–238]. In BLG theory we have already seen that the only possible realization of the ternary algebra is $SO(4)$ three algebra. The basic 3-algebraic structure of the theory is followed from the previous general discussion. $SO(4)$ 3-algebra is a vector space with basis T^a, $a = 1, \ldots, 4$, along with a trilinear antisymmetric product,

$$\{T^a, T^b, T^c\} \sim \epsilon^{abc}{}_d T^d, \tag{1.167}$$

where ϵ^{abcd} is the mostly anti-symmetric invariant tensor in four dimensions. The raising and the lowering of the indices is accomplished by the consideration of a inner product,

$$h^{ab} = tr(T^a T^b) = \delta_{ab}. \tag{1.168}$$

The structure constant ϵ^{abcd} is totally anti-symmetric in all of its indices.

$$\epsilon^{abcd} = \epsilon^{[abcd]}. \tag{1.169}$$

The trilinear product also satisfies a generalized version of Jacobi identity known as “fundamental identity".

Having introduced BLG theory and the role of 3-algebraic structure, we briefly mention the connection between ABJM and BLG. In ABJ(M) theory, when the orders of the gauge group $U(N_1)_k \times U(N_2)_{-k}$ satisfy the limit $N_1 = N_2 = 2$, one finds some extra symmetry due to the fact that the $\mathbf{2}$ and $\bar{\mathbf{2}}$ representation of $U(2)$ are equivalent. This extra symmetry enhances the supersymmetry of the ABJM theory from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. Finally, in the special limit $N = 2$ and $k = 2$, the ABJM theory becomes equivalent to the BLG theory. Within this limit, the moduli space of the both theories exactly matches.

In the last few paragraphs we have briefly discussed the BLG theory and the role of ternary algebra in this theory. We further have reviewed the basic mathematical structure of the ternary algebra. Finally we have briefly mentioned the connection between the BLG theory and the ABJM theory. One of the non-perturbative objects string theory has are the giant gravitons. Properties of this objects can be understood by exploiting the AdS/CFT correspondence.

1.7.6 Giant graviton

We previously learned that there exists an one to one mapping between the primary operators of the gauge theory and the fields in the gravity sector. Consider for example the case of the AdS_5/CFT_4. Then the gauge theory is described by the $\mathcal{N} = 4$ SYM in four dimensional Minkowski space and the dual gravity theory is described by the $typeIIB$ supergravity in $AdS_5 \times S^5$ spacetime. By dimensional reduction of massless supergravity fields on five-sphere we get a tower of massive fields in the AdS_5 spacetime. The massive fields correspond to the Kaluza-Klein excitations [151]. We assume that the Kaluza-Klein states maintain a BPS bound. They are characterized by set of quantum numbers. Among them, angular momentum l due to the motion along S^5 plays an important role and takes only integer values. The energy/mass of these sates is given by $m = \frac{l}{R}$, where R is the radius of both AdS_5 and S^5. The particles of mass m have the Schwarzschild radius [152],

$$r_S \sim \sqrt{\frac{g_s^2}{R^5} m} = \frac{1}{m}\sqrt{\frac{l^3}{N^2}}. \tag{1.170}$$

While introducing eqn (1.170) we have used the relation $g_s N \sim R^4$. It is pointed out in [152] that when $l \gg N^{\frac{2}{3}}$, the Schwarzschild radius associated with the Kaluza-Klein becomes larger than the Compton wavelength. As a consequence of that the linearized supergravity approximation breaks down. So the dimensional reduction mechanism prescribed in [151] does not fit into the picture. McGreevy, Susskind and Toumbas [153] first gave the correct supergravity picture of these massless states with large angular momentum l. They showed the solution corresponding to the large angular momentum, blows up into the spherical brane. Actually, those massless supergravity states rotate around the five sphere in the presence of a Ramond-Ramond magnetic flux and get polarized with a finite dipole moment. As they rotate along the S^5, the dipole moment streches due to the interaction with magnetic flux. As a result, the size of these states increases and thus they become spherical branes. This is a special case of the Myers effect [155, 156]. These branes are wrapped around the spherical part of the $AdS_5 \times S^5$ geometry. They are in a dynamical equilibrium because the contraction due to the tension of the brane is precisely canceled by the expansion due with the coupling of the angular momentum to the background flux field. The existence of this spherical brane becomes a realization of stringy exclusion principle observed in the context of the AdS/CFT correspondence [157]. In the gauge theory, this principle suggests that a family of chiral primary operators terminates at some maximum weight as the gauge symmetry group has a finite rank. In terms of the dual gravity description, these operators corresponds to the single particle massless supergravity states expanding into spherical branes. They grow into the spherical part of the $AdS_5 \times S^5$ geometry with a radius proportional to the angular momentum associated with their motion along the five sphere. The radius of these spherical branes must be smaller than the radius of the sphere. The AdS/CFT correspondence relates the radius of sphere in the gravity theory with the rank of the gauge group in the dual gauge theory. In other words, the angular momentum associated with the gravity states corresponds to the $\mathcal{R}$ charge of those primary operators in the dual gauge theory. Thus stringy exclusion principle puts an upper bound on the angular momentum. This spherical brane configuration is known as Giant Graviton (sphere giants). It is important to note that the stability of giant graviton requires the presence of Ramond-Ramond flux in the background. Moreover, the large value of angular momentum associated with those states renders the supergravity approximation. Combining both facts, we come to a conclusion that understanding the giant graviton in the gravity pictures requires the study of the $typeIIB$ string theory in $AdS_5 \times S^5$ geometry in the presence of Ramond-Ramond flux. But due to the presence of the significant nonperturbative effects, this kind of theory is very little understood so far. Therefore the giant graviton is considered as a non-perturbative object in the gravity theory. However, strong/weak duality of the AdS/CFT correspondence allow us to study the properties and representation of giant graviton in the dual perturbative field theory ('t Hooft constant $\lambda \ll 1$). There are also generalization of this spherical brane configuration of giant graviton which is wrapped inside the spherical part of the $AdS_5 \times S^5$ geometry. Hashimoto, Hirano, Itzhaki introduced the dual giant gravitons which expands into AdS_5 part of the full $AdS_5 \times S^5$ geometry (AdS giants) [158]. The radius of the dual giant graviton is again proportional to its angular momentum. But as it grows into the non-compact AdS_5 spacetime, unlike the spherical giant gravitons, there is no upper-bound for the angular momentum. So to summarize we can consider the giants and the dual giants as stable, $\frac{1}{2}$ BPS spherical $D3$ branes wrapping a three cycle inside either S^5 or AdS_5. Moreover, both of the stable, $\frac{1}{2}$ BPS giants and the dual giants are realized in the AdS_4/CFT_3 set up [160–166]. In $AdS_4 \times S^7/\mathbb{Z}_k$, these giant gravitons are either the spherical M2-brane growing in the AdS_4 or the M5-brane wrapping an $S^5 \subset S^7$. Whereas for $AdS_4 \times \mathbb{CP}^3$, the M-2 brane is replaced by a $D2$ brane growing into AdS_4 and instead of M-5 brane, $D4$ brane wrapping on some circle of $\mathbb{CP}^3$.

The polarizability and the stability of giant graviton can be studied in quantitative way. Here, we discuss this issue in the context of sphere giants in the AdS_5/CFT_4 set up. This discussion remains valid for AdS giants also. Moreover, same kind of analysis for both sphere and AdS giants is developed in case of the AdS_4/CFT_3 correspondence. We start our discussion with a simple configuration of a electric dipole which is composed of a pair of unit charges with opposite polarity. The dipole lives on a 2-sphere and performs an orbital motion on the equator. Consider them to be joined together by a perfect spring. When these particles are not moving, the spring shrinks to a zero length. The masses of the charges are negligible. We now place a magnetic monopole at the center of this sphere, which sources a uniform magnetic flux. Due to the orbital motion, the constituent charges of the dipole experience the Lorentz force in opposite direction. There is another force acting between the two unit charges via the perfect spring. The system rotates by maintaining a dynamical equilibrium. The analysis of [153] shows that as the angular momentum of the system associated with the rotation increases, the size of the spring stretches further. The stretching continues until the charges are at the opposite ends of the sphere. We assume that the radius of the sphere is R and the magnetic field has a flux density B. The total flux is quantized as $2\pi N = BR^2 Vol_{S^2}$. The momentum P associated with the dipole is $\sim 2BR$ when it stretches upto the two opposite ends of the sphere. So the angular momentum L is $PR \sim BR^2 \sim N$. Therefore, if the dipole never exceeds the size of the sphere, the angular momentum can not exceed the value N.

Now we extend the above analysis into the case spherical $D3$ brane (the spatial part of brane world volume is S^3) wrapped inside the three-cycle of S^5. $D3$ brane couples to a five form Ramond-Ramond field with a constant flux density B. Similar to previous case, the total flux is quantized inside S^5 of radius R.

$$Vol_{S^5} BR^5 = 2\pi N. \tag{1.171}$$

The radius R of the S^5 is given as,

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s. \tag{1.172}$$

Let us now consider the action of the spherical $D3$ moving in S^5. The action consists of two part,

$$\mathcal{S}_{total} = \mathcal{S}_{DBI} + \mathcal{S}_{CS}, \tag{1.173}$$

where $\mathcal{S}_{DBI}$ is the Dirac-Born-Infeld action for spherical $D3$ brane and $\mathcal{S}_{CS}$ is the Chern-Simon action which induces a coupling of the brane to the five form Ramond-Ramond flux. Before evaluating the DBI action for spherical $D3$ brane we fix the parametrization of background five-sphere.

$$\begin{aligned} X_1 &= \sqrt{R^2 - r^2}\cos\phi, \\ X_2 &= \sqrt{R^2 - r^2}\sin\phi, \\ X_3 &= r\cos\theta_1, \\ X_4 &= r\sin\theta_1\cos\theta_2, \\ X_5 &= r\sin\theta_1\sin\theta_2\cos\theta_3, \\ X_6 &= r\sin\theta_1\sin\theta_2\sin\theta_3, \end{aligned} \tag{1.174}$$

where, $X_I,\ I = 1, ..., 6$ satisfy the constraint, $\sum_{I=1}^{6} X_I^2 = R^2$. Using this background parametrization one can calculate the world volume metric of the spherical $D3$ brane.

$$ds^2_{wv} = -(1 - (R^2 - r^2)\dot{\phi}^2)dt^2 + r^2 d\Omega_3^2. \tag{1.175}$$

Here we assume that the spherical $D3$ brane (S^3) wraps on the S^5. So r is constant and $dr = 0$. The Ω_3 is parameterized by $\theta_1, \theta_2, \theta_3$. Now using eqn (1.175) we can write the explicit form of $\mathcal{S}_{DBI}$.

$$\mathcal{S}_{DBI} = -T_{D3}\int\sqrt{(1 - (R^2 - r^2)\dot{\phi}^2)}r^3 dt d\Omega_3. \tag{1.176}$$

So the kinetic term of the total Lagrangian takes the form,

$$\mathcal{L}_k = -T_{D3}r^3\Omega_3\sqrt{(1 - (R^2 - r^2)\dot{\phi}^2)}. \tag{1.177}$$

Using eqn (1.172) and the relation $T_{D3} = \frac{1}{2\pi^3 l_s^4 g_s}$ we get the final form of Lagrangian,

$$\mathcal{L}_k = -\frac{N}{R^4}r^3\sqrt{(1 - (R^2 - r^2)\dot{\phi}^2)}. \tag{1.178}$$

The $\mathcal{S}_{CS}$ can also be explicitly obtained.

$$\mathcal{S}_{CS} = \int_{\Sigma_5} F, \tag{1.179}$$

where F is the five form Ramond-Ramond field. Σ_5 is a five manifold in S^5. The boundary of the Σ_5 is identified as a 4-dimensional surface swept out by the brane during an orbital motion. Corresponding Lagrangian is given as,

$$\mathcal{L}_{CS} = \frac{\mathcal{S}_{CS}}{T} = BVol(\Sigma)T^{-1} = Br^4RVol_{S_5}\frac{\dot{\phi}}{2\pi} = \dot{\phi}N\frac{r^4}{R^4}. \tag{1.180}$$

To derive eqn (1.180) we used eqn (1.171) and the form of volume element of S^5, i.e. $Vol_{S_5} = \int Rr'^2 dr' d\phi d\Omega_3$. $\dot{\phi}$ can be interpreted as angular velocity of the brane. T is the time period of orbital motion executed by the brane. Now the total Lagrangian can be written as,

$$\mathcal{L}_{tot} = -\frac{N}{R^4} r^3 \sqrt{(1-(R^2-r^2)\dot{\phi}^2)} + \dot{\phi} N \frac{r^4}{R^4}. \tag{1.181}$$

It is straightforward to calculate the angular momentum from the total Lagrangian.

$$l = \frac{\partial \mathcal{L}_{tot}}{\partial \dot{\phi}} = \frac{N r^3 (R^2-r^2)\dot{\phi}^2}{R^4 \sqrt{(1-(R^2-r^2)\dot{\phi}^2}} + N\frac{r^4}{R^4}. \tag{1.182}$$

The energy of the system is given by

$$\mathcal{H} = l\dot{\phi} - \mathcal{L}_{tot} = \frac{N r^3}{R^4 \sqrt{(1-(R^2-r^2)}} = \sqrt{(\frac{N r^3}{R^4})^2 + \frac{(l - N r^4/R^4)^2}{R^2 - r^2}}. \tag{1.183}$$

Since R is the maximum value allowed for r, the angular momentum is bounded by N. The variation of total energy $\mathcal{H}$ with respect to r, keeping l fixed, we obtain a stable minimum at $r^2 = \frac{l}{N}R^2$. The total energy $\mathcal{H}$ at this minimum is given as $E = \frac{l}{R}$ provided l is large and $N > l$. So we conclude that that massless gravitons in the $typeIIB$ supergravity with the large angular momentum are described by the stable BPS spherical $D3$ branes.

We now consider the above giants from the gauge theoretic perspective. In the $SU(N)$, $\mathcal{N} = 4$ SYM theory, the $\frac{1}{2}$ BPS operators can be constructed out of the six real scalars ($\{\phi_i\}$ $i = 1,..,6$). These six real scalars transform under $(0, l, 0)$ representation of the $SU(4)$ $\mathcal{R}$ symmetry group. $(0, l, 0)$ representation is the symmetric trace-less representation of $SO(6)$. The $\frac{1}{2}$ BPS operators saturate a lower bound on their conformal dimensions which is related to their $\mathcal{R}$ symmetry charge. These operators transform under the adjoint representation of the gauge group $SU(N)$ and can be realized as the single trace or multiple trace operators. The trace is taken over the adjoint indices of the gauge group making them gauge invariant.

One can construct three complex scalars out of these six real scalars.

$$Z_1 = \phi_1 + i\phi_2, \;\; Z_2 = \phi_3 + i\phi_4, \;\; Z_3 = \phi_5 + i\phi_6. \tag{1.184}$$

These complex scalars transform as $Z_a \to G^\dagger Z_a G$, where $G \in SU(N)$ and a runs from 1 to 3. The simplest $\frac{1}{2}$ BPS operators are constructed as, $\prod_{n_i} \left[Tr(Z_a^l)\right]^{n_i}$, where l is the $\mathcal{R}$ charge of the operator. Using the AdS/CFT correspondence one can map a single particle state in the gravity theory with a chiral primary operator in the dual gauge theory. The chiral primary operator is realized as a single trace operator that we are interested in. The angular momentum of the gravity state corresponds to the $\mathcal{R}$ charge of the trace operator. Consequently, The multi-particle state corresponds to the multi trace operator. These multi-particle states of the gravity constitute a Fock space and satisfy an orthogonality condition. The orthogonality condition tells that the inner product of two elements of the Fock space describing m and n number of particles respectively, is zero for $m \neq n$. If the trace operators are the correct gauge invariant operators then the orthogonality condition satisfied by the states of gravity should have a proper realization in the dual gauge theory. Balasubramanian, Berkooz, Naqvi and Strassler first showed that this is not true when the $\mathcal{R}$ charge associated with the trace operators is large compared to the order of the gauge group $SU(N)$ [159]. In the free field limit, using the Wick contraction the exact two point correlator involving the complex scalars can be constructed as,

$$\langle (Z_a)_{ij}(x)(Z_a)^{\dagger}_{kl}(y)\rangle = \frac{\delta_{il}\delta_{jk}}{(y-x)^2}, \tag{1.185}$$

where i, j, k, l are the gauge indices. When we consider the correlators of gauge invariant trace operators we perform necessary contraction over the gauge indices. Let us consider,

$$\begin{aligned} \mathcal{O}_1 &= Tr((Z_a)^l) \\ \mathcal{O}_2 &= Tr((Z_a)^{l_1})Tr((Z_a)^{l_2}) \qquad l_1 + l_2 = l. \end{aligned} \tag{1.186}$$

Using proper modification of eqn (1.185) and ignoring the space time dependence for the moment we have,

$$\frac{\langle \mathcal{O}_1^{\dagger}\mathcal{O}_2\rangle}{\langle \mathcal{O}_1^{\dagger}\mathcal{O}_1\rangle\langle \mathcal{O}_2^{\dagger}\mathcal{O}_2\rangle} \sim \frac{\sqrt{l_1 l_2 l}}{N}. \tag{1.187}$$

By the operator-state correspondence in AdS/CFT, the above eqn suggests that the states created by the trace operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are orthogonal to each other in the free field limit ($N \to \infty$) only when the $\mathcal{R}$ charge l of the relevant operator is less than $N^{\frac{2}{3}}$ [159]. So we conclude that for large $\mathcal{R}$ charge ($l > N^{\frac{2}{3}}$), the $\frac{1}{2}$ BPS trace operators are no more the proper gauge invariant operators which obey the operator-state correspondence. We identify the gravity state of large angular momentum with the giant graviton state in the $typeIIB$ theory. The dual gauge theory operator must be characterized by a large $\mathcal{R}$ charge. This identification can easily be justified for the spherical $D3$ brane. We know that there is a bound on $\mathcal{R}$ charge of a chiral primary operator (trace operator) in the gauge theory. This naturally agrees with the fact that the angular momentum of the $D3$ brane should also have an upper limit. Therefore we conclude that the trace operator is not a correct description of the gauge invariant operator with large $\mathcal{R}$ charge, dual to the giant graviton state. Using the tensor decomposition of the $SO(6)$ $\mathcal{R}$ symmetry group in the $SU(N)$, $\mathcal{N} = 4$ theory, Ramgoolam, Corley, Jevicki pointed out that the correct gauge invariant operator, dual to the giant graviton state in the $typeIIB$ supergravity is described by Schur polynomial operator [167]. These Schur operator is labeled by the large $\mathcal{R}$ charge in $SU(N)$, $\mathcal{N} = 4$ theory. The realization of orthogonality can be verified by computing two point correlator involving the Schur polynomial. To construct the two point function of Schur polynomial we need to know some mathematical preliminaries about this polynomial. So before going into the details of the operator-state mapping between Schur polynomial and giant graviton state we briefly discuss some mathematical properties of that polynomial.

1.7.7 Schur polynomial

One can construct the relevant two point function by modifying eqn (1.185) in case of Schur polynomial. However, for gauge invariant combination, we need to take the sum over free gauge indices. In general the sum takes the form,

$$\sum_{i_1,i_2\cdots i_n} \delta^{i_1}_{i_{\sigma(1)}} \delta^{i_2}_{i_{\sigma(2)}} \cdots \delta^{i_n}_{i_{\sigma(n)}} = N^{C(\sigma)}, \tag{1.188}$$

where each index $i_1, \cdots i_n$ runs over integers from 1 to N and $C(\sigma)$ is the number of cycles in the permutation σ. To avoid the cumbersome expression involving the stings of delta functions, we consider the following short-hand notation,

$$\sum_I \delta\left(\begin{array}{c} I(n) \\ I\big(\sigma(n)\big) \end{array} \right) = N^{C(\sigma)}, \tag{1.189}$$

The all possible permutations form a group called permutation group. A special case of the permutation group is the symmetric group S_n. One may define a delta function on the S_n group algebra. The delta is 1 when the argument is the identity element of S_n and 0 otherwise. It is useful to write this delta function as an expansion of the group characters of S_n.

$$\delta(\rho) = \frac{1}{n!} \sum_R d_R \;\; \chi_R(\rho). \tag{1.190}$$

Here, the sum is taken over the representations R of S_n. The representations R are connected with the Young Diagrams with n boxes. d_R is the dimension of a representation R. $\chi_R(\rho)$ is the group character of the element $\rho \in S_n$ in the same representation. We can evaluate d_R by exploiting the properties of Young diagram.

$$d_R = \frac{n!}{\prod_{i,j} h_{i,j}}, \tag{1.191}$$

where i and j represent the the rows and columns of the Young diagram respectably and $h_{i,j}$ is the hook number of each box of the same diagram. The orthogonality condition of the characters is given as,

$$\sum_\sigma \chi_R(\sigma^{-1})\, \chi_S(\sigma\alpha) = \frac{\delta_{RS}\, n!}{d_S} \chi_S(\alpha), \tag{1.192}$$

where σ and α belong to the symmetry group S_n.

As mentioned in [168], same Young diagram can be associated with both unitary group and S_n representations. This is due to the fact that we can always construct a vector space on which both unitary group and S_n can act and the vector space itself simultaneously diagonalizes the operators belonging to both groups. The Schur polynomials can be defined as the characters of the unitary group in their irreducible representations

$$\chi_R(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\; tr(\sigma\, U). \tag{1.193}$$

If we trivially identify the unitary group with the identity group, eqn (1.193) gives the dimension of a representation of the unitary group

$$Dim_N(R) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) N^{C(\sigma)}. \tag{1.194}$$

Using the properties of Young diagram, we can explicitly evaluate $Dim_N(R)$.

$$Dim_N(R) = \prod_{i,j} \frac{(N-i+j)}{h_{i,j}}, \tag{1.195}$$

where i and j represent the the rows and columns of the Young diagram respectably and $h_{i,j}$ is the hook number of each box of the same diagram. Now using both eqns.(1.191) and (1.195) one may find out a relation of the product of the weights of the Young diagram (f_R) with dimensions of the representations ($d_R, Dim_N(R)$),

$$f_R = \prod_{i,j} (N-i+j) = \frac{n! Dim_N(R)}{d_R}. \tag{1.196}$$

For our purpose it is important to extend the definition of Schur polynomial involving a system of complex matrices transforming under adjoint representation of the $SU(N)$ gauge group

$$\chi_R(Z) \;\; = \;\; \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) Tr(\sigma Z),$$

where Z is the any one of the complex scalars of $\mathcal{N} = 4$ SYM gauge theory. The trace in the right hand side of the above equation is taken as,

$$Tr(\sigma Z) \equiv \sum_{i_1, i_2 \cdots i_n} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}}. \tag{1.197}$$

R is the representation associated with a specific Young diagram with n boxes, which labels the irreducible representation of both unitary group and S_n. It is very important to mention here the product rule for Schur polynomials. Let us consider three irreducible representation R_1, R_2 and S. R_1, R_2 and S have n_{R_1}, n_{R_2} and n_S boxes in their respective Young diagrams in such a way that the relation $n_S = n_{R_1} + n_{R_2}$ is satisfied. According to the rule, the product of two Schur polynomials having irreducible representation R_1 and R_2 respective becomes,

$$\chi_{R_1}(Z)\chi_{R_2}(Z) = \sum_S g(R_1, R_2; S)\chi_S(Z). \quad (1.198)$$

In the right hand side of the above equation there is a quantity $g(R_1, R_2; S)$ which is known as Littlewood-Richardson number. This quantity counts the number of times the S appears in the direct product of R_1 and R_2. We can generalize the relation for $\chi_{R_1}(Z)\chi_{R_2}(Z)\cdots\chi_{R_m}(Z)$ as follows,

$$\begin{aligned}\prod_{i=1}^{m}\chi_{R_i}(Z) &= \sum_{S_1,S_2\cdots S_{m-2},S} g(R_1,R_2;S_1)g(S_1,R_3;S_2)\cdots g(S_{m-2},R_m;S)\chi_S(Z) \\ &= \sum_S g(R_1,R_2\cdots R_m;S)\chi_S(Z). \quad (1.199)\end{aligned}$$

1.7.8 Two-point function

Now using the machinery discussed above we compute the two point correlators of the Schur polynomials defined in eqn (1.197). We note that the Schur polynomials involve the three complex scalars in $\mathcal{N} = 4$ SYM gauge theory. They might be the correct gauge invariant operators dual to the giant graviton states in the gravity theory provided the orthogonality condition of gravity Fock space would have a realization via the two point correlators in the gauge theory. Let us compute the two point correlators $\langle\chi_R(Z)\chi_S(Z^\dagger)\rangle$, where we have assumed that R and S are the irreducible representations of two different Young diagrams.

$$\begin{aligned}&\langle\chi_R(Z)\chi_S(Z^\dagger)\rangle \\ &= \langle\sum_\sigma \frac{\chi_R(\sigma)}{n!}\ Tr(\sigma Z)\sum_\rho \frac{\chi_S(\rho)}{n!}\ Tr(\rho Z^\dagger)\rangle \\ &= \sum_{i_1,i_2\cdots i_n}\ \sum_{j_1,j_2\cdots j_n}\sum_\sigma \frac{\chi_R(\sigma)}{n!}\sum_\rho\frac{\chi_S(\rho)}{n!} \quad (1.200) \\ &\quad \langle Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_n}_{i_{\sigma(n)}}(Z^\dagger)^{j_1}_{j_{\rho(1)}}(Z^\dagger)^{j_2}_{j_{\rho(2)}}\cdots(Z^\dagger)^{j_n}_{j_{\rho(n)}}\rangle \\ &= \sum_{i_1,i_2\cdots i_n}\ \sum_{j_1,j_2\cdots j_n}\sum_\alpha\sum_{\sigma,\rho}\frac{\chi_R(\sigma)}{n!}\frac{\chi_S(\rho)}{n!} \quad (1.201) \\ &\quad \delta^{i_1}_{j_{\alpha\rho(1)}}\delta^{i_2}_{j_{\alpha\rho(2)}}\cdots\delta^{i_n}_{j_{\alpha\rho(n)}}\ \ \delta^{j_1}_{i_{\alpha^{-1}\sigma(1)}}\delta^{j_2}_{i_{\alpha^{-1}\sigma(2)}}\cdots\delta^{j_n}_{i_{\alpha^{-1}\sigma(n)}} \\ &= \sum_\alpha\sum_\sigma\frac{\chi_R(\sigma)}{n!}\sum_\rho\frac{\chi_S(\rho)}{n!}\ N^{C(\sigma^{-1}\alpha\rho^{-1}\alpha^{-1})}, \quad (1.202)\end{aligned}$$

Now introducing the delta function mentioned earlier in the context of symmetry group S_n,

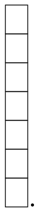
$$\begin{aligned}
&\sum_{\alpha,\sigma,\rho,\beta} \frac{\chi_R(\sigma)}{n!} \frac{\chi_S(\rho)}{n!} N^{C(\beta)} \delta(\beta^{-1}\sigma^{-1}\alpha\rho^{-1}\alpha^{-1}) \\
&= \frac{1}{(n!)^2} \sum_{\alpha,\sigma,\beta} \chi_R(\sigma)\, \chi_S(\alpha^{-1}\beta^{-1}\sigma^{-1}\alpha)\, N^{C(\beta)} \\
&= \frac{1}{n!} \sum_{\sigma,\beta} \chi_R(\sigma)\, \chi_S(\beta^{-1}\sigma^{-1})\, N^{C(\beta)} \\
&= \sum_{\beta} \frac{1}{d_R}\, \delta_{RS}\, \chi_S(\beta^{-1})\, N^{C(\beta)} \\
&= n!\, \frac{Dim_N(R)}{d_R}\, \delta_{RS} \\
&= f_R \delta_{RS}
\end{aligned} \tag{1.203}$$

In the final step of the above derivation we have used eqn(1.195) and eqn (1.196). It is clear from the last line that the presence of the delta function makes the two point function non zero only when both R and S are exactly same. Therefore the orthogonality of gravity Fock space is correctly realized in the dual gauge theory. Consequently, Schur polynomial turns out to be the correct $\frac{1}{2}$ BPS gauge invariant operator dual to the giant graviton state in the gravity theory.

1.7.9 Mapping

The detailed description of the rules of mapping between the Schur polynomial and the giant graviton states are prescribed in [167]. Here, we briefly review them. This mapping is based on a couple of important facts. Firstly, the Schur polynomials are labeled by the Young diagram associated with them. Secondly, the stringy exclusion principle can be realized in terms of the Young tableaux representation associated with the corresponding gauge invariant operator with $\mathcal{R}$ charge. The Young tableaux that we are considering is composed of a certain number of boxes. The total number boxes in each tableaux has a connection with the $\mathcal{R}$ charge of the Schur polynomial. In particular, each box is associated with a complex scalar field Z in the $\mathcal{N} = 4$ SYM theory and possesses 1 unit of $\mathcal{R}$ charge.

The mostly antisymmetric representation of Schur polynomial operator made of the complex scalar fields in the $SU(N)$, $\mathcal{N} = 4$ SYM theory is described by a Young tableaux of single column. We are interested in the gauge invariant operator of large $\mathcal{R}$ charge represented by the Schur polynomial. So the corresponding Young tableaux should have large number of rows. As this Young tableaux furnish the mostly antisymmetric representation of a unitary group of order N, their group theoretic properties tell that the maximum number of rows must be N. Therefore this restriction puts a cut-of on the value of $\mathcal{R}$ charge associated with the Schur polynomial. We know that the stringy exclusion principle suggests the sphere giant to have exactly same kind of cut-of on their angular momentum. Thus we identify the sphere giants with a certain class of the Schur polynomial represented by the mostly antisymmetric Young tableaux.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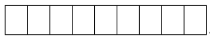
Similarly, the mostly symmetric representation of the Schur polynomial is given by a Young tableaux made of a single row. From the representation theory it is known that there is no restriction on the number of columns. So the Schur polynomial associated with the diagram actually carries arbitrarily large $\mathcal{R}$ charge. Therefore from previous discussion we identify this certain class of Schur polynomial represented by the mostly symmetric Young tableaux with the AdS giant carrying arbitrarily large angular momentum.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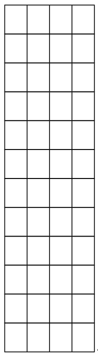
If the Young diagram has k number of columns and each of them have equal number of rows ($\sim N$),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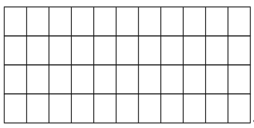
then it represents a sphere giant wrapping the 3-cycle of the spherical part of the $AdS_5 \times S^5$ geometry with a winding number k. The angular momentum and the energy of this sphere giant are scaled up by a factor k. If the lengths of the k number columns are not equal but differ from each other by a number which is very small compared to N then the diagram can be obtained by fusing a Young diagram of several columns of equal length and a very small Young diagram. So in the dual gravity picture is a interacting system of sphere giants and Kaluza-Klein graviton.

Similar the Young diagram having k number of rows, where each of them have equal but arbitrarily large number of column,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represents a AdS giants wrapping the AdS part of the $AdS_5 \times S^5$ geometry with a winding number k. The energy and the angular momentum of the graviton is scaled out by a factor k. If the number of columns are not equal then the dual gravity picture is an interacting system made of the AdS giants and the Kaluza-Klein graviton.

As explained in [169–171] when the number of vertical boxes in a Young diagram grows, the dual grows in the spherical part of the $AdS_5 \times S^5$ geometry. When the number of rows grows the dual grows in the AdS part of the $AdS_5 \times S^5$ geometry. However, if the number of boxes are so large that the $\mathcal{R}$ charge comes out as $\mathcal{O}(N^2)$, the dual would have a energy large enough to back-react the $AdS_5 \times S^5$ geometry. Therefore, operator with $\mathcal{O}(N^2)$ boxes corresponds to a new geometry in the gravity sector.

So far we have discussed the state operator mapping in the context of the AdS_5/CFT_4 duality. We shall again discuss the same issue for the AdS_5/CFT_4 duality. We shall again discuss the same issue for the AdS_4/CFT_3 duality in $Chapter 4$. We shall compute, in particular, the probability amplitude for various transition among these giants. Above identifications will play the key role in our computations.

1.8 Plan of the thesis

Exploiting AdS/CFT correspondence, in this thesis, we explore some properties of strongly coupled gauge theory ($chapter : 2-3$) and that of M/$typeIIA$ string theory ($chapter : 4-5$). A brief summary of various chapters is provided below.

In $chapter 2$, we qualitatively study various dynamical observables, important in the context of real time dynamics of high energy partons moving in QGP medium via the computations in a dual hQCD gravity model represented by a asymptotically AdS (aAdS) black hole in Einstein-Maxwell-Dilaton (EMD) system. We briefly describe the construction of this hQCD gravity background, namely, that follows from Einstein-Maxwell-Dilaton system by. Here, we provide an asymptotically AdS (aAdS) black hole solution important for the hQCD model. Finally we calculate the drag force, the jet quenching parameter, the screening length and the binding energy of external quark-antiquark ($q\bar{q}$) pair for this model. We study the qualitative features of this quantities in terms of the chemical potential and the temperature of the boundary gauge theory. Since the aAdS black hole solution turns out to be too complicated for a pure analytic approach, we extensively use numerical methods in the computation of various physical quantities.

In $chapter 3$, we estimate the dissipative force on an external quark in the presence of evenly distributed heavy quark cloud within the finite temperature $\mathcal{N}=4$ strongly coupled super YangMills. This is again computed holographically by constructing the corresponding gravity dual. We show that the gravity background gets deformed due to the backreaction of a string distribution which, in turn, is dual to the quark cloud density. We explicitly construct this $backreacted$ geometry. This deformed black hole is parametrized by it's mass and the density of strings. We further analyze the stability of the gravity dual for vector and tensor perturbations. Finally, we study the qualitative behavior of the drag force as a function of the cloud density and the temperature of the boundary theory.

In $chapter4$, we study a class of nonperturbative semi classical objects called giant gravitons living in M-theory/$typeIIA$ background via the computation in dual weakly coupled three dimensional $U(N_1) \times U(N_2)$, $\mathcal{N} = 6$ Chern-Simon-matter gauge theory known as ABJ(M) theory, with $N_2 > N_1$. We show that the correct gauge invariant operator dual to giant graviton state is described by the Schur polynomial. Following the work [166], we generalize the Schur polynomials of ABJM theory to ABJ theory and also identify them with the gravitons and giant gravitons in the dual gravity. With the aim of finding out the transition probability among giant gravitons and between giants and gravitons, we discuss CFT factorization and its probabilistic interpretation. Further, we compute correlators among giant gravitons as well as between giant gravitons and ordinary gravitons through the corresponding correlators of ABJ(M) theory. Finally, we consider a particular non-trivial background produced by an operator with an $\mathcal{R}$ charge of $\mathcal{O}(N_1^2)$ and find, in presence of this background, due to the contribution of the non-planar corrections, the large (N_1, N_2) expansion is replaced by $1/(N_1 + M)$ and $1/(N_2 + M)$ respectively, where M ($\mathcal{O}(N_1)$) is the number of extra boxes added in the extended Young diagram representing a gauge invariant operator carrying the $\mathcal{R}$ of the order of N_1^2.

We have already mentioned that the ABJ(M) theory is a description of the $\mathcal{N} = 6$ superconformal gauge theory of multiple M-2 branes. In a particular limit of this gauge theory, we can reinterpret it as another alternative superconformal gauge theory of two M-2 branes, known as BLG theory. This theory preserves $\mathcal{N} = 8$ supersymmetry. The gauge fields and the matter fields transform under a gauge group consisted of generators satisfying a structure of ternary algebra or 3-algebra. Motivated by the importance of 3-algebra in world volume gauge theory of multiple M-2 branes, in $chapter5$, we introduce a novel 3-algebra called w_∞ 3-algebra. We start with the generators of $W_{\infty+1}$ algebra. Then we define a lone-star product among these generators to define a ternary bracket. Moreover, we take a double scaling limit of the elements of this ternary bracket. We show the effect of double scaling transforms the generators of $W_{\infty+1}$ algebra into the w_∞ generators and the ternary bracket takes a completely antisymmetric structure with respect to the indices of the elements belonging to those w_∞ generators. We explicitly check the preservation of fundamental identity associated with this ternary bracket and thus confirm the existence of a novel 3-algebra, namely w_∞ 3-algebra. Finally, we give a geometric representation of this w_∞ 3-algebra.

In $chapter6$, we give a brief summary of our results.

2

Some aspects of QGP phase in a hQCD model

The dynamical features of the strongly coupled QGP plasma are encoded in the properties of the probe particles as well as in the collective behavior of the system. The phenomena of the jet quenching of high energy partons with large transverse momentum, the energy dissipation of a heavy moving probe quark, the quorkonium suppression due to the charge screening carry the the signatures of the strong coupling nature of the medium. Our purpose in this chapter is to compute some of the quantities within a holographic QCD model (hQCD). It is constructed in [178] by employing the so called potential reconstruction approach [175–178]. Some properties of the hQCD model were studied in [175]. The model has a deconfinement phase transition and an associated phase diagram [178]. The equation of states resembles quite closely to the one expected from the lattice QCD results. The gravity dual of this model is given by an AdS black hole of Einstein-Maxwell-Dilaton system. The model however has its own limitation and differs in many aspects from the real QCD. We will have occasions to discuss these limitations later in this chapter.

In section 2, we briefly review the potential reconstruction approach to the Einstein-Maxwell-Dilaton system by introducing a nontrivial coupling between the dilaton field and the Maxwell field. In section 3, we discuss the generic black hole solutions with asymptotical AdS boundary, and in particular present an analytic black hole solution. In section 4, we calculate the drag force in this hQCD model. Jet quenching parameter and screening length are discussed in section 5 and 6, respectively. We end this chapter by a discussion.

2.1 Einstein-Maxwell-Dilaton system

In this section, we use the potential reconstruction approach [175, 178] to study a 5D Einstein-Maxwell-Dilaton (EMD) system. Unlike in [178] here we allow a nontrivial coupling between the gauge and the dilaton field.

$$S_{5D} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g^S}e^{-2\phi}\left(R^S + 4\partial_\mu\phi\partial^\mu\phi - V_S(\phi) - \frac{Z(\phi)}{4g_g^2}e^{\frac{-4\phi}{3}}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.1}$$

where the action (2.1) is written in string frame, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field, $Z(\phi)$ is an arbitrary function of dilaton field ϕ and $V_s(\phi)$ is the dilaton potential. In Einstein frame we can rewrite the action as [175]

$$S_{5D} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g^E}\left(R - \frac{4}{3}\partial_\mu\phi\partial^\mu\phi - V_E(\phi) - \frac{Z(\phi)}{4g_g^2}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.2}$$

where $V_S = V_E e^{\frac{-4\phi}{3}}$. The metrics in these two frames are connected by the scaling transformation

$$g^S_{\mu\nu} = e^{\frac{4\phi}{3}} g^E_{\mu\nu}. \tag{2.3}$$

The Einstein equations from the action (2.2) read

$$E_{\mu\nu} + \frac{1}{2}g^E_{\mu\nu}\left(\frac{4}{3}\partial_\mu\phi\partial^\mu\phi + V_E(\phi)\right) - \frac{4}{3}\partial_\mu\phi\partial_\nu\phi - \frac{Z(\phi)}{2g_g^2}\left(F_{\mu k}F_\nu{}^k - \frac{1}{4}g^E_{\mu\nu}F_{kl}F^{kl}\right) = 0 \tag{2.4}$$

where $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. We consider an ansatz for the matter field as $A = A_0(z)dt, \;\; \phi = \phi(z)$ and for the metric

$$ds_S^2 = \frac{\ell^2 e^{2A_s}}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right), \tag{2.5}$$

in the string frame, where $i = 1, 2, 3$, ℓ is the radius of AdS_5 space, and A_s, the warped factor, is a function of the coordinate z. In the Einstein frame, the metric reads as

$$\begin{aligned} ds_E^2 &= \frac{\ell^2 e^{2A_e}}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right), \\ &= \frac{\ell^2 e^{2A_s - \frac{4\phi}{3}}}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right), \end{aligned} \tag{2.6}$$

with $A_e = A_s - 2\phi/3$. With this metric, the (t,t), (z,z) and (x_i, x_i) components of Einstein equations are respectively

$$\begin{aligned} b''(z) + \frac{b'(z)f'(z)}{2f(z)} - \frac{b'(z)^2}{2b(z)} + \frac{4}{9}b(z)\phi'(z)^2 + \frac{A_0{}'(z)^2 Z(\phi)}{6g_g^2 f(z)} + \frac{V_E(\phi)b(z)^2}{3f(z)} &= 0, \\ \phi'(z)^2 - \frac{9b'(z)f'(z)}{8b(z)f(z)} - \frac{9b'(z)^2}{4b(z)^2} - \frac{3A_0{}'(z)^2 Z(\phi)}{8g_g^2 b(z)f(z)} - \frac{3V_E(\phi)b(z)}{4f(z)} &= 0, \\ f''(z) + \frac{3b'(z)f'(z)}{b(z)} + \frac{4}{3}f(z)\phi'(z)^2 + \frac{3f(z)b''(z)}{b(z)} - \qquad (2.7) \\ \frac{3f(z)b'(z)^2}{2b(z)^2} - \frac{A_0{}'(z)^2 Z(\phi)}{2g_g^2 b(z)} + V_E(\phi)b(z) &= 0, \end{aligned} \tag{2.8}$$

where $b(z) = \ell^2 e^{2A_e}/z^2$, and $A_0(z)$ is the electric potential of the Maxwell field. Manipulating these equations one can obtain following two equations which do not contain the dilaton potential $V_E(\phi)$,

$$A_s''(z) + A_s'(z)\left(\frac{4\phi'(z)}{3} + \frac{2}{z}\right) - A_s'(z)^2 - \frac{2\phi''(z)}{3} - \frac{4\phi'(z)}{3z} = 0, \tag{2.9}$$

$$f''(z) + f'(z)\left(3A_s'(z) - 2\phi'(z) - \frac{3}{z}\right) - \frac{z^2 Z(\phi) e^{\frac{4\phi(z)}{3} - 2A_s(z)} A_0'(z)^2}{g_g^2 L^2} = 0. \tag{2.10}$$

Eq.(2.9) is our starting point to find exact solutions of the system. Note that Eq.(2.9) in the EMD system is the same as the one in the Einstein-dilaton system considered in [176] [177] and the last term in Eq.(2.10) is an additional contribution from electrical field. In addition, the equation of motion of the dilaton field is given by

$$\frac{8}{3}\partial_z\left(\frac{\ell^3 e^{3A_s(z) - 2\phi} f(z)}{z^3}\partial_z\phi\right) - \frac{\ell^5 e^{5A_s(z) - \frac{10}{3}\phi}}{z^5}\partial_\phi V_E(\phi) + \frac{Z'(\phi) b(z) A_0'(z)^2}{2g_g^2} = 0. \tag{2.11}$$

Maxwell field should satisfy

$$\frac{1}{\sqrt{-g^E}}\partial_\mu\left(\sqrt{-g^E} Z(\phi) F^{\mu\nu}\right) = 0. \tag{2.12}$$

From these equations of motion, once $A_s(z)$ is given, we can obtain a general solution to the system, which takes the following form

$$\begin{aligned}
\phi(z) &= \int_0^z \frac{e^{2A_s(x)}\left(\frac{3}{2}\int_0^x y^2 e^{-2A_s(y)} A_s'(y)^2\,dy + \phi_1\right)}{x^2}\,dx + \frac{3A_s(z)}{2} + \phi_0, && (2.13)\\
A_0(z) &= A_{00} + A_{01}\left(\int_0^z \frac{y e^{\frac{2\phi(y)}{3} - A_s(y)}}{Z(\phi(y))}\,dy\right), && (2.14)\\
f(z) &= \int_0^z x^3 e^{2\phi(x) - 3A_s(x)}\left(\frac{{A_{01}}^2\left(\int_0^x \frac{y e^{\frac{2\phi(y)}{3} - A_s(y)}}{Z(\phi(y))}\,dy\right)}{g_g^2\ell^2} + f_1\right)dx + f_0, && (2.15)\\
V_E(z) &= \frac{e^{-2A_s(z) + \frac{4\phi(z)}{3}} z^2 f(z)}{\ell^2} 2\Big(- \frac{e^{-2A_s(z) + \frac{4\phi(z)}{3}} Z(\phi(z)) z^2 A_0'(z)^2}{4g_g^2\ell^2 f(z)}\\
&- \frac{2\left(3 + 3z^2 A_s'(z)^2 + 4z\phi'(z) + z^2\phi'(z)^2 - 2zA_s'(z)\left(3 + 2z\phi'(z)\right)\right)}{z^2}\\
&- \frac{f'(z)\left(-3 + 3zA_s'(z) - 2z\phi'(z)\right)}{2zf(z)}\Big), && (2.16)
\end{aligned}$$

where $\phi_0, A_{00}, A_{01}, f_0, f_1$ are all integration constants and can be determined by suitable UV and IR boundary conditions. When $Z(\phi) = 1$, the general solution reduces to the one given in [178].

2.2 General asymptotical AdS black hole solutions

Since we are interested in the black hole solutions with AdS asymptotics, we impose the boundary condition $f(0) = 1$ at the AdS boundary $z = 0$, and require $\phi(z), f(z), A_0(z)$ to be regular at black hole horizon z_h as well as at AdS boundary $z = 0$. There is an additional condition $A_0(z_h) = 0$, which corresponds to the physical requirement that $A_\mu A^\mu = g^{tt} A_0 A_0$ must be finite at $z = z_h$.

We can parameterize the function $f(z)$ in Eq.(2.13) as

$$f(z) = 1 + \frac{A_{01}^2}{2g_g^2\ell^2} \frac{\int_0^z g(x)\left(\int_0^{z_h} g(r)dr \int_r^x \frac{g(y)^{\frac{1}{3}}dy}{Z(\phi(y))}\right)dx}{\int_0^{z_h} g(x)dx} - \frac{\int_0^z g(x)dx}{\int_0^{z_h} g(x)dx}, \quad (2.17)$$

where $f_0 = 1$, $f_1 = -\frac{A_{01}^2}{4g_g^2\ell^2} \frac{\int_0^{z_h} g(x)\int_0^x \frac{g(y)^{\frac{1}{3}}}{Z(\phi(y))}dy+1}{\int_0^{z_h} g(x)dx}$ and

$$g(x) = x^3 e^{2\phi(x)-3A_s(x)}. \quad (2.18)$$

We expand the gauge field near the AdS boundary to relate the two integration constants to the chemical potential (μ) and the charge density, respectively,

$$A_0(z) \sim A_{00} + A_{01}\frac{e^{\frac{2\phi(y)}{3}-A_s(y)}}{Z(\phi(y))}z^2 + \cdots, \quad (2.19)$$

with

$$A_{00} = \mu, \quad (2.20)$$

$$A_{01} = \frac{\mu}{\int_0^{z_h} y\frac{e^{\frac{2\phi}{3}-A_s(y)}}{Z(\phi(y))}dy} = \frac{\mu}{\int_0^{z_h} \frac{g(y)^{\frac{1}{3}}}{Z(\phi(y))}dy}. \quad (2.21)$$

The temperature of the black hole can be determined through the function $f(z)$ in (2.17) as

$$T = \frac{1}{4\pi}|f'(z)|_{z=z_h} = \left|\frac{A_{01}^2}{4\pi g_g^2\ell^2}\frac{g(z_h)\int_0^{z_h} g(r)dr\int_r^{z_h}\frac{g^{\frac{1}{3}}(y)}{Z(\phi(y))}dy - g(z_h)}{\int_0^{z_h} g(x)dx}\right|. \quad (2.22)$$

The entropy density can be found from the Bekenstein-Hawking formula and is given by

$$s = \frac{A_{area}}{4G_5V_3} = \frac{\ell^3}{4G_5}\left(\frac{e^{A_s-\frac{2}{3}\phi}}{z}\right)^3\Big|_{z_h}, \quad (2.23)$$

where V_3 is the volume of the black hole spatial directions spanned by coordinates x_i in (2.6).

2.2.0.1 An analytical black hole solution

In this subsection, we construct an analytical solution of the Einstein-Maxwell-Dilaton system by using Eq.(2.13-2.16) with $Z(\phi) = 1$. We impose the constrain $f(0) = 1$, and require $\phi(z), f(z)$ to be regular at $z = 0$, and z_h. We give the solution in Einstein frame

$$ds_E^2 = \frac{\ell^2 e^{2A_e}}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right), \tag{2.24}$$

with

$$\begin{aligned}
A_e(z) &= \log\left(\frac{z}{z_0\sinh(\frac{z}{z_0})}\right), \\
f(z) &= 1 - \frac{4V_{11}}{3}(3\sinh^4(\frac{z}{z_0}) + 2\sinh^6(\frac{z}{z_0})) + \frac{1}{8}V_{12}^2\sinh^4\left(\frac{z}{z_0}\right), \\
\phi(z) &= \frac{3z}{2z_0}, \\
A_0(z) &= \mu - \frac{2g_g\ell}{z_0}V_{12}\sinh^2\left(\frac{z}{2z_0}\right),
\end{aligned} \tag{2.25}$$

where z_0 is an integration constant and V_{11}, V_{12} are two constants from the dilaton potential

$$V_E(\phi) = -\frac{12 + 9\sinh^2\left(\frac{2\phi}{3}\right) + 16V_{11}\sinh^6\left(\frac{\phi}{3}\right)}{\ell^2} + \frac{V_{12}^2\sinh^6\left(\frac{2\phi}{3}\right)}{8\ell^2}. \tag{2.26}$$

The two integration constants V_{11} and V_{12} then can be expressed in terms of horizon z_h and chemical potential μ as

$$\begin{aligned}
V_{11} &= \frac{3\cosh^4\left(\frac{z_h}{2z_0}\right)\left(\frac{\mu^2 z_0^2\sinh^4\left(\frac{z_h}{z_0}\right)\cosh^4\left(\frac{z_h}{2z_0}\right)}{4g_g^2\ell^2} + 8\right)}{32\left(2\sinh^2\left(\frac{z_h}{2z_0}\right) + 3\right)}, \\
V_{12} &= \frac{\mu z_0\cosh^2\left(\frac{z_h}{2z_0}\right)}{2g_g\ell}.
\end{aligned} \tag{2.27}$$

We can obtain the temperature of the black hole by using Eq.(2.22). In figure (2.1), we show the temperature as a function of horizon radius z_h for three different chemical potentials. In this plot we take parameters $\ell = 1, z_0 = 1, g_g = 1$. We see from the figure (2.1) that the temperature with respect to horizon z_h is monotonic for a fixed chemical potential. A vanishing temperature means that the black hole is extremal with a smallest horizon radius. This smallest horizon radius increases with the chemical potential.

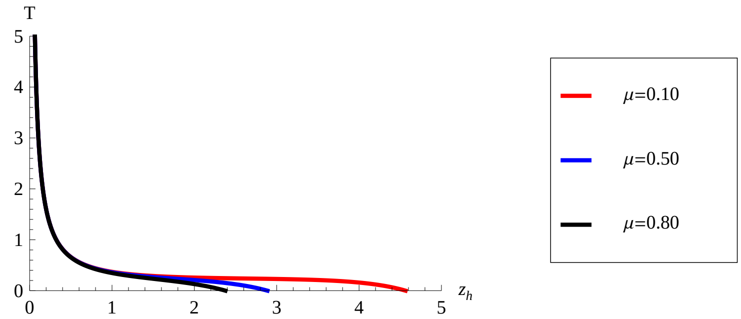


Figure 2.1: The temperature as a function of horizon radius z_h for the analytical black hole solution with parameters $\ell = 1, z_0 = 1, g_g = 1$.

2.2.1 The hQCD model

Based on the general solutions, in Ref. [178], a holographic QCD model has been proposed to realize the confinement/deconfinement phase transition of QCD. In passing we briefly review the main features of the model. Let us consider the warped factor $A_s(z)$ of the following form

$$A_s(z) = k^2 z^2, \tag{2.28}$$

where k is a constant. There are various phenomenological motivations for this choice. Following [178] we set $k = .3GeV$. With $A_s(z)$ as in eq (2.28), dilaton takes the form

$$\phi(z) \quad = \quad \frac{3}{4} k^2 z^2 (1 + H(z)), \tag{2.29}$$

where we have set the integration constant $\phi_0 = 0$, and $H(z)$ is given by

$$H(z) = {}_2F_2\left(1, 1; 2, \frac{5}{2}; 2k^2 z^2\right). \tag{2.30}$$

The characteristic function of the black hole background takes the form

$$\begin{aligned} f(z) = \quad & 1 + \frac{1}{4g_g^2 \ell^2}\left(\frac{\mu}{\int_0^{z_h} g(y)^{\frac{1}{3}} dy}\right)^2 \frac{\int_0^z g(x)\left(\int_0^{z_h} g(r)dr \int_r^x g(y)^{\frac{1}{3}} dy\right) dx}{\int_0^{z_h} g(x)dx} \\ & - \frac{\int_0^z g(x)dx}{\int_0^{z_h} g(x)dx}, \end{aligned} \tag{2.31}$$

where

$$g(x) = x^3 e^{\frac{3}{2}k^2 x^2 (1+H(x)) - 3k^2 x^2}. \tag{2.32}$$

One can clearly see that the second term in (2.31) comes from the contribution of electric field. If one turns off the electric field, one can reproduce the black hole solution in

Einstein-dilaton system [176]. In addition, the electric field $A_t(z)$ is given by

$$A_t(z) = \mu + \frac{\mu}{\int_0^{z_h} g(y)^{\frac{1}{3}} dy} \int_0^z x e^{\frac{1}{2}k^2 x^2(-1+H(x))} dx. \tag{2.33}$$

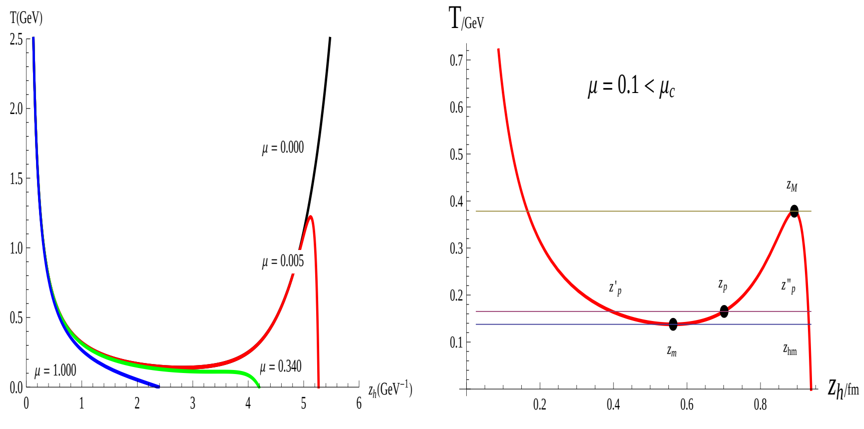


Figure 2.2: Plot (a): The black hole temperature as a function of horizon z_h with different chemical potentials. When $\mu > \mu_c$ the temperature monotonically decreases to zero with increase of z_h; when $0 < \mu < \mu_c$, the temperature decreases to a minimum at z_m and grows up to a maximum at z_M and then decreases to zero monotonically. When $\mu = \mu_c$, one has $z_m = z_M$. Plot (b): The temperature of the black hole with $\mu = 0.1$GeV. The three black hole solutions with horizon z'_p, z_p and z''_p have the same temperature. The black hole with $z_m < z_p < z_M$ is thermodynamically unstable. Here we take $g_g\ell = 1, k = 0.3$GeV. In this hQCD model, we always fix $k = 0.3$GeV and accordingly the critical chemical potential is $\mu_c = 0.34$GeV, which corresponds to the case $z_m = z_M$.

In the figure (2.2) we plot the temperature with respect to the horizon radius z_h for different chemical potentials. One can see clearly that the temperature crucially depends on the value of chemical potential: there is a critical chemical potential μ_c, beyond which the black hole is always thermodynamically stable, while when the chemical potential is less than the critical one, there is a region of horizon radius, where the black hole is thermodynamically unstable with negative heat capacity. To be more clear we plot in figure (2.2)(b) the temperature versus the horizon z_h in the case $\mu = 0.1\text{GeV} < \mu_c$ as an example. One can see from the figure that the black hole is thermodynamically unstable in the region $z_m < z_h < z_M$, where z_m and z_M are the black hole horizons corresponding to the minimal and maximal temperatures, respectively. In this region, the heat capacity of the black hole is negative. The black hole solutions in the regions $z_h < z_m$ and $z_h > z_M$ are thermodynamically stable. When $\mu \geq \mu_c$, z_m and z_M are degenerated to one point. In contrast to the case in the figure (2.1), there are local minimal and maximal values of temperature for small μ. This is crucial to realize the critical point in the $T-\mu$ phase diagram of the hQCD model [178].

2.3 Drag force

An external probe quark moving in the boundary plasma will experience a drag force. We now compute that force for the above model. We will follow the technique described in [95]. The drag force is a function of temperature and chemical potential. The boundary gauge theory we are considering is on $\mathcal{M}^4$ described by the boundary coordinates t, x^1, x^2, x^3. As mentioned in the previous chapter, to compute the drag force holographically, we need to consider a string in the bulk with appropriate boundary conditions. The dynamics of a fundamental string is specified by the Nambu-Goto action in the black hole backgroundhin (2.5). The world sheet action reads

$$S = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-\mathrm{det}g_{\alpha\beta}}, \quad g_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma_\alpha}\frac{\partial X^\mu}{\partial\sigma_\beta}G_{\mu\nu}, \tag{2.34}$$

where $g_{\alpha\beta}$ is the induced metric on the world sheet and $G_{\mu\nu}$ is the background metric. The equation of motion derived from (2.34) is given by

$$\Delta_\alpha P^\alpha_\mu = 0, \quad P^\alpha_\mu = -\frac{1}{2\pi\alpha'}G_{\mu\nu}\partial^\alpha X^\nu, \tag{2.35}$$

where Δ_α is the covariant derivative with respect to $g_{\alpha\beta}$ and P^α_μ is the world sheet current of space time energy-momentum of the test string. We consider the motion of the string along x^1. In the gauge, $\tau = t$ and $\sigma = z$, the string dynamics can be completely specified by the function $x^1(t,z)$. In this case, the Lagrangian reads

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{\frac{1}{H} + \frac{f(z)(\partial_z x^1)^2}{H} - \frac{(\partial_t x^1)^2}{Hf(z)}}, \tag{2.36}$$

where H is defined as

$$H = \sqrt{\frac{z^2}{\ell^2 e^{2A_s}}}. \tag{2.37}$$

To capture the dragged motion of the quark in the boundary theory we assume the following ansatz in the bulk [95]

$$x^1(t,z) = vt + \xi(z). \tag{2.38}$$

Here we have assumed only the late time behavior of the string motion. With this ansatz the Lagrangian reduces to

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\sqrt{\frac{1}{H} + \frac{f(z)(\partial_z\xi(z))^2}{H} - \frac{v^2}{Hf(z)}}. \tag{2.39}$$

The momentum which conjugates to $\xi(z)$ reads

$$\Pi_\xi = \frac{\partial \mathcal{L}}{\partial \xi'} = -\frac{\xi'}{2\pi\alpha'}\frac{f}{H}\sqrt{\frac{Hf}{f - v^2 + f^2\xi'^2}}. \tag{2.40}$$

For the sake of consistency, it is important to invert the equation (2.40) and write it in the following way

$$\xi' = \sqrt{\frac{{\Pi_\xi}^2(f - v^2)}{\frac{f^2}{H^2}[\frac{1}{4\pi^2\alpha'^2}Hf - {\Pi_\xi}^2 H^2]}}. \tag{2.41}$$

Here the positive sign is taken due to the trailing nature of the string profile [95]. To obtain the string profile we have to solve the differential equation (2.41). To have a real $\xi(z)$, we further impose the constraints

$$\begin{aligned} f(z)|_{z=z_v} &= v^2, \\ {\Pi_\xi}^2|_{z=z_v} &= \frac{1}{4\pi^2\alpha'^2}\frac{v^2}{H}, \end{aligned} \tag{2.42}$$

so that one has $\xi'|_{z=z_v} = v^2/f^2$, keeping finite. The profile of the string is defined in the region with $z < z_v$, That is, there is a maximal value $z_v < z_h$ for the string profile.

The constraints are very useful to figure out the final form of the drag force. Before to compute the drag force, we recapitulate the relation between the drag force in the boundary field theory and the dissipation of momentum flowing down the string, in the light of AdS/CFT correspondence. In the boundary theory, the presence of the thermal medium results into dissipation of energy and momentum of external quark until it reaches thermal equilibrium with the medium. In the bulk theory, the momentum flows the string from the boundary to the bulk and the change of momentum at a given spatial point on the world sheet for a given time interval can be calculated. The identifications of the endpoint of the string attached to the boundary with the quark and of the string in the bulk with the thermal medium around the quark suggest that the drag force can be realized in terms of the force imparted by the string on its boundary endpoint. To calculate the change of string momentum due to its motion along x_1 direction, we consider a closed curve on the world sheet and study how the momentum is conserved around this curve [199]. According to the conservation of world sheet current of space time energy-momentum of the test string, the total flux calculated around the path C must be zero,

$$\oint_{ABDEA}(P^\tau_\mu d\sigma - P^\sigma_\mu d\tau) = 0. \tag{2.43}$$

Note that one end of the string is attached to the boundary and the other end close to horizon is free. In the static gauge, Eq. (2.43) reduces to

$$p^{t_1}_{x_1} - p^{t_2}_{x_1} = -\int_{t_2}^{t_1}\sqrt{-g}P^z_{x_1}dt, \tag{2.44}$$

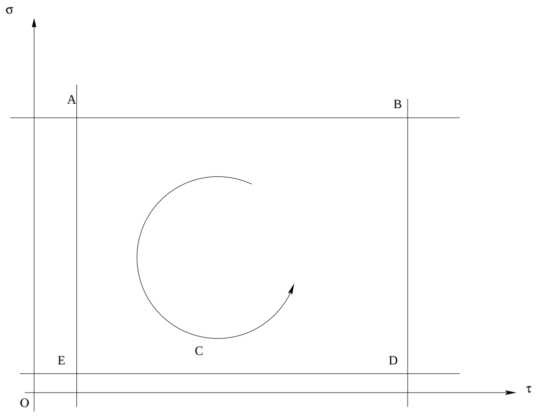


Figure 2.3: This plot shows a closed path in an anti-clockwise direction on a world sheet bounded by coordinates $[A = \tau_1, \sigma_2]$, $[B = \tau_2, \sigma_2]$, $[D = \tau_2, \sigma_1]$ and $[E = \tau_1, \sigma_1]$.

where $p^t_{x_1}$ is the x_1 component of the total momentum at time t. Consequently, the drag force is defined as

$$F_{drag} = \frac{dp_{x_1}}{dt} = -\sqrt{-g}P^z_{x_1} = -\frac{1}{2\pi\alpha'}\frac{\ell^2 e^{2A_s}}{{z_v}^2}v. \tag{2.45}$$

Finally we have to replace all the gravity parameters in terms of gauge theory parameters. Before doing that, we analyze the form of the constraints case by case. The exact forms of constraint for the solutions (2.25) and (2.31) are given respectively by

$$\begin{aligned} v^2 &= 1 - \tfrac{4V_{11}}{3}(3\sinh^4(\tfrac{z_v}{z_0}) + 2\sinh^6(\tfrac{z_v}{z_0})) + \tfrac{1}{8}V_{12}^2\sinh^4\left(\tfrac{z_v}{z_0}\right), \\ v^2 &= 1 + \tfrac{1}{4g_g^2\ell^2}\left(\frac{\mu}{\int_0^{z_h} g(y)^{\frac{1}{3}}dy}\right)^2 \frac{\int_0^{z_v} g(x)\left(\int_0^{z_h} g(r)dr \int_r^x g(y)^{\frac{1}{3}}dy\right)dx}{\int_0^{z_h} g(x)dx} - \frac{\int_0^{z_v} g(x)dx}{\int_0^{z_h} g(x)dx}. \end{aligned} \tag{2.46}$$

It is always desirable to express the drag force in a closed analytic form as a function of gauge theoretical variables. However, in the present case, due to the complexity of (2.46) and (2.22), analytic computations become difficult. Instead we solve them numerically and plot the drag force with respect to gauge theory parameters, e.g, temperature and chemical potential. The qualitative features of the drag force can be revealed from our results.

Although the analytic black hole solution (2.25) is not a dual to a QCD like theory. However, as a warm-up exercise, we plot the drag force in figure (2.4). We see that the drag force monotonically increases with temperature and for a fixed temperature, it increases with the chemical potential. These features are qualitatively expected in realistic QCD.

In figure (2.5), we plot the drag force for our hQCD model given by the solution (2.31) with different chemical potentials $\mu = 0.10$, 0.34, and 0.80. We see from the figure, that for fixed chemical potential and temperature, the drag force increases with the velocity of the quark, while for fixed chemical potential and velocity, the drag force increases with temperature. These are expected features in QCD. In particular, let us note that in the

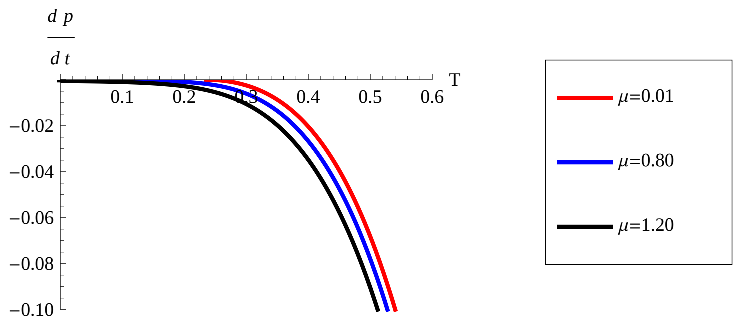
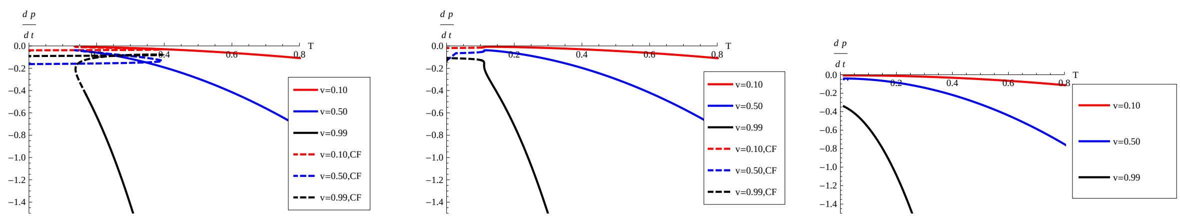


Figure 2.4: This plot shows the drag force from the analytic black hole solution as a function of T for chemical potential $\mu = 0.01, 0.80$, and 1.20 respectively. Here we take $v = 0.1$.



(a) (b) (c)

Figure 2.5: This figure shows the drag force as a function of T for the chemical potential $\mu = 0.10$ (plot a), 0.34 (plot b), and 0.80 (plot c), respectively, in the hQCD model. Here the dashed curves stand for the behavior of drag force in confined phase which is denoted by CF in the figure. In the confined phase, in fact the drag force is not well defined, meaning that the dashed curves do not make any sense here.

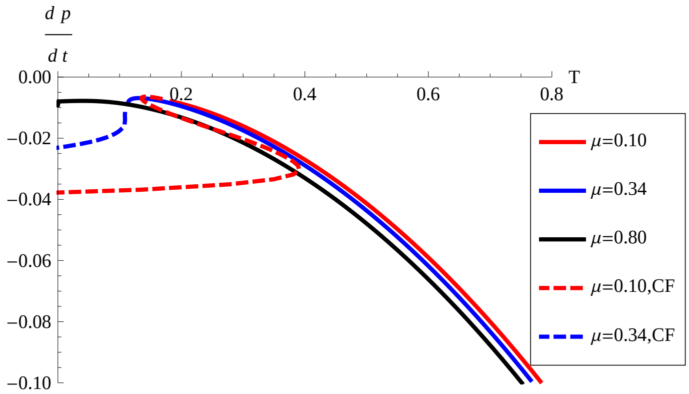


Figure 2.6: The figure shows the drag force as a function of T for three chemical potential $\mu = 0.10, 0.34$, and 0.80, respectively in the hQCD model. The dashed parts of the curves stand for the drag force in the confined phase which denoted by CF in the figure. Here we take $v = 0.1$.

low temperature region with small chemical potential, the drag force is a multi-valued function of temperature (see plot (a) and (b)). However it becomes a monotonic function when the chemical potential (see plot (c)) is large. This feature is closely related to the confinement/deconfinement phase transition in this hQCD model [178]. The dashed parts of curves in plot (a) and (b) denote the drag force in the confined phase and actually they do not make any sense here since drag force is not well-defined in the confined phase. Our result for the drag force in the deconfined phase is in agreement with the one in [26]. For comparison, in figure (2.6) we plot the drag force versus temperature with three different chemical potentials $\mu = 0.10$, 0.34 and 0.80, respectively. In this figure the velocity of quark is taken as $v = 0.1$.

2.4 Jet Quenching parameter

In this section, we use the AdS/CFT duality to compute the jet quenching parameter in our hQCD model. To employ the holographic principle in the computation of this parameter, the hQCD dual solution (2.31) as the bulk theory of gravity serves the purpose. The jet quenching parameter is related to the expectation value of light-like Wilson loop computed in the adjoint representation [101]. The gauge/gravity duality prescribes how to map the expectation value of light-like Wilson loop in fundamental representation $\langle W^F(\mathcal{C})\rangle$ into the exponential of the regularized extremal surface in the bulk with a boundary contour $\mathcal{C}$ located at $z = 0$.

$$\langle W^F(\mathcal{C})\rangle = \exp[-S(\mathcal{C})]. \tag{2.47}$$

In the planar limit, considering the fact $Tr_{Adj} = Tr^2_{Fund}$, the relation between $W^F(\mathcal{C})$ and $W^A(\mathcal{C})$ can be easily established as

$$\langle W^A(\mathcal{C})\rangle = \langle W^F(\mathcal{C})\rangle^2. \tag{2.48}$$

Now we start with the background black hole solution in string frame

$$ds_S^2 = \frac{\ell^2 e^{2A_s}}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^1dx^1 + dx^2dx^2 + dx^3dx^3\right). \tag{2.49}$$

By introducing the light cone coordinates defined as

$$x^{\pm} = \frac{t \pm x^1}{\sqrt{2}}, \tag{2.50}$$

the black hole metric (2.49) can be rewritten as

$$\begin{aligned} ds_S^2 &= \frac{\ell^2 e^{2A_s}}{z^2}\Bigg(\frac{(1-f(z))}{2}((dx^+)^2 + (dx^-)^2) - (1+f(z))(dx^+dx^-) \\ &\quad + \frac{dz^2}{f(z)} + dx^2dx^2 + dx^3dx^3). \end{aligned} \tag{2.51}$$

We take the gauge with $\tau = x^-(0 \le x^- \le L^-)$, $\sigma = x^2(-\frac{L_2}{2} \le x_2 \le \frac{L_2}{2})$, and set the pair of quarks at $x^2 = \pm\frac{L_2}{2}$ on $x^+ =$ constant, $x^3 =$ constant plane. In the limit with $L^- \gg L_2$ the string profile is completely specified by $z = z(\sigma)$. Following [101] and using (2.47), (2.48), we find

$$\langle W^A(\mathcal{C})\rangle = \exp(-\frac{1}{4\sqrt{2}}\hat{q}L^-L_2^2), \tag{2.52}$$

where the jet quenching parameter is defined as

$$\hat{q} = \frac{8\sqrt{2}(S-S_0)}{L^-\ell^2}, \tag{2.53}$$

where S is the Nambu-Goto action of the string and S_0 is the self energy from the mass of two quarks.

Substituting the induced metric of the fundamental string into the Nambu-Goto action (2.34), we get

$$\begin{aligned} S &= -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{-\mathrm{det}g_{\alpha\beta}}, \\ &= \frac{L^-\ell^2}{\sqrt{2}\pi\alpha'}\int_0^{\frac{L_2}{2}} d\sigma\frac{e^{2A_s}}{z^2}\sqrt{(1-f(z))(1+\frac{z'^2}{f(z)})}. \end{aligned} \tag{2.54}$$

Since the integrand in (2.54) does not explicitly depend on σ, one can regard σ as time and the integrand as a Lagrangian. In this case the corresponding Hamiltonian is conserved. That is, we can have

$$\frac{\partial\mathcal{L}}{\partial z'}z' - \mathcal{L} = E, \tag{2.55}$$

where E is a constant and $\mathcal{L}$ is the integrand in (2.54). From this relation we obtain the equation of motion for z as

$$z'^2 = f(z)(\frac{e^{4A_s}}{z^4}\frac{(1-f(z))}{E^2} - 1). \tag{2.56}$$

We choose the boundary conditions $z(\pm\frac{L_2}{2}) = 0$ and $z'(0) = 0$. In that case, the turning point z_T is determined by solving Eq.(2.56). Since $z'(\sigma)$ is a real function, so the square of it should be non-negative. The realization of boundary condition $z'(0) = 0$ at the turning point requires the proper choices of zeros and the positivity region of the right hand side of Eq.(2.56). From the boundary conditions of the black hole solution

$$\lim_{z\to z_h} f(z) = 0, \quad \lim_{z\to 0} f(z) = 1, \tag{2.57}$$

together with the fact that we are interested in the case with small E, it is clear that the factor $\frac{e^{4A_s}(1-f(z))-E^2z^4}{E^2z^4}$ is always positive near the black hole horizon and negative near the boundary. To remove the region with a negative z'^2, we consider a modified boundary at $z = \delta$. We assume that at $z = z_{min}$,

$$\frac{e^{4A_s(z_{min})}}{z_{min}^4}\frac{(1-f(z_{min}))}{E^2} - 1 = 0, \tag{2.58}$$

and $\delta > z_{min}$. In the region $\delta \le z \le z_h$, thus, the factor $[\frac{e^{4A_S}}{z^4}\frac{(1-f(z))}{E^2} - 1]$ is always positive. So only viable solution of $z'^2 = 0$ is

$$f(z) = 0 \Rightarrow z_T = z_h. \tag{2.59}$$

That is, the turning point is just at the horizon. The distance between two quarks can be determined by

$$\frac{L_2}{2} = \int_\delta^{z_h} dz \frac{E}{\sqrt{f[e^{4A_s}(1-f)z^{-4} - E^2]}}. \tag{2.60}$$

As we are interested in the small L_2 limit, considering the smallness of E, we can expand Eq.(2.60) in terms of E as

$$\frac{L_2}{2E} = \int_\delta^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}} + \frac{E^2}{2}\int_\delta^{z_h} dz \frac{e^{-6A_s}z^6}{\sqrt{f(1-f)^3}} + \mathcal{O}(E^4). \tag{2.61}$$

Inverting (2.61) suitably, we can obtain E up to the leading order of L_2 as

$$E = \frac{L_2}{2\int_\delta^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}}} + \mathcal{O}({L_2}^3). \tag{2.62}$$

Thus we can obtain the string action

$$S = \frac{L^- \ell^2}{\sqrt{2}\pi\alpha'}\int_\delta^{z_h} dz \frac{e^{4A_s}(1-f)}{z^2\sqrt{f(e^{4A_s}(1-f) - z^4E^2)}}. \tag{2.63}$$

Clearly this action is divergent. The divergence comes from the contribution of mass of two quarks. With the gauge $x^- = \tau$ and $z = \sigma$, the self energy of two free quarks reads

$$S_0 = \frac{L^- \ell^2}{\sqrt{2}\pi\alpha'} \int_\delta^{z_h} dz \frac{e^{2A_s}}{z^2} \sqrt{\frac{(1-f)}{f}}. \tag{2.64}$$

Thus the regularized action up to the leading order of L_2 is given by

$$S_I = S - S_0 = \frac{L^- L_2^2 \ell^2}{8\sqrt{2}\pi\alpha'} \frac{1}{\int_\delta^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}}} + \mathcal{O}({L_2}^4). \tag{2.65}$$

With the definition of the jet quenching parameter (2.53), we finally reach

$$\hat{q} = \frac{\ell^2}{\pi\alpha'} \frac{1}{\int_\delta^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}}}. \tag{2.66}$$

In fact the cutoff here can be removed by noting the fact that the integrand is regular inside the region $0 \leq z \leq z_h$, i.e, from the horizon to the real boundary,

$$\int_\delta^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}} = \int_0^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}} - \int_0^{\delta} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}}. \tag{2.67}$$

The second integral in the right hand side of the above equation smoothly vanishes in the limit $\delta \to 0$. So the final expression for the jet-quenching parameter is

$$\hat{q} = \frac{\ell^2}{\pi\alpha'} \frac{1}{\int_0^{z_h} dz \frac{z^2 e^{-2A_s}}{\sqrt{f(1-f)}}}. \tag{2.68}$$

Because the black hole metric is still too complicated it is hard to obtain an analytical expression of the jet-quenching parameter in terms of physical parameters. We rather solve it numerically and plot in figure (2.7) the jet-quenching parameter as a function of temperature in the hQCD model with three chemical potentials $\mu = 0.10$, 0.34 and 0.80, respectively. For large $\mu \geq \mu_c$ cases, the jet-quenching parameter decreases monotonically with temperature, which agrees with the one in [200] qualitatively. On the other hand, when $\mu < \mu_c$, the jet-quenching parameter is a multi-valued function of temperature in low temperature region and it decreases monotonically with respect to temperature in high temperature region. The multi-valued behavior of the jet-quenching parameter in low temperature region is clearly related to the first order phase transition between the hadron phase and the QGP phase. The jet-quenching parameter confirms the hydrodynamical description of QGP phase and agrees with the real QCD expectation at high temperature. Once again, as the drag force in the confined phase, the dashed parts of curves in figure (2.7) denote the jet-quenching parameter in the confined phase and thus they cease to make any sense.

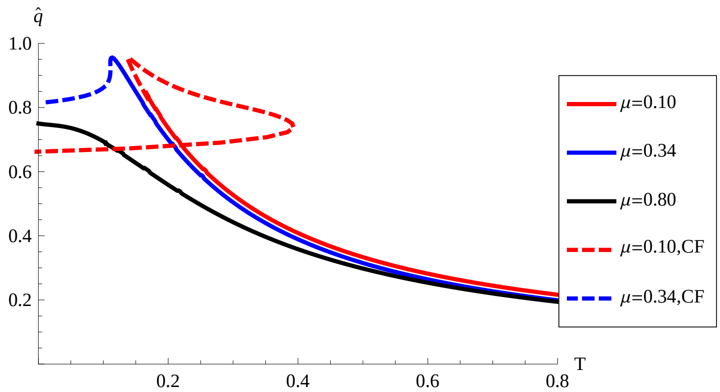


Figure 2.7: The figure shows the jet-quenching parameter as a function of T for three chemical potentials $\mu = 0.10, 0.34$, and 0.8, respectively, in the hQCD model. The dashed parts of curves stand for the jet-quenching parameter in the confined phase which are denoted by CF in the figure.

2.5 Hot plasma wind and screening length

The screening length is defined as the maximum length achieved by a quark-antiquark bound state at temperature $T > T_c$, beyond which the pair dissociates. For quark-antiquark pair, the energetically favorable configuration in the dual gravity theory is a fundamental string with both ends attached to the boundary. The attached endpoints correspond to the $q\bar{q}$ pair whereas being separated beyond the screening length, thus dissociated from each other, the pair maps into two separate strings hanging from the boundary. In [199], the screening length is computed in the rest frame of $q\bar{q}$ pair and the plasma wind flows at a constant speed v for the hot $\mathcal{N} = 4$ SYM plasma. This setup is identified with a quark-antiquark pair moving in hot $\mathcal{N} = 4$ SYM plasma. In this section, we compute the screening length for the hQCD model (2.31) in the same way as in [199].

In the static frame of $q\bar{q}$ pair, we assume that the hot plasma is moving with velocity v in the negative x_3 direction. The Wilson loop we are interested in lies in the $t - x_1$ plane specified by the length $\mathcal{T}$ and L respectively. We assume $\mathcal{T} \gg L$ such that the string world sheet is invariant under translation along the time direction. The boost we are considering is defined as

$$\begin{aligned} dt &= \cosh\eta dt' - \sinh\eta dx_3', \\ dx_3 &= -\sinh\eta dt' + \cosh\eta dx_3', \end{aligned} \tag{2.69}$$

where $\cosh\eta = \gamma$, $\sinh\eta = \gamma v$ and $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz boost factor. With the Lorentz transformation, we obtain the boosted black hole metric in string frame

$$\begin{aligned} ds_S^2 = \ & H(z)[-(1-(1-f)\cosh^2\eta)dt^2 + (1+(1-f)\sinh^2\eta)(dx^3)^2, \\ & -2(1-f)\cosh\eta\sinh\eta dt dx^3 + (dx^1)^2 + (dx^2)^2 + \frac{dz^2}{f(z)}], \end{aligned} \tag{2.70}$$

where $H(z) = \ell^2 e^{2A_s}/z^2$. We prefer to work in the static gauge

$$\tau = t, \sigma = x^1, x^2(\sigma) = x^3(\sigma) = \text{constant}, \tag{2.71}$$

with the following boundary conditions

$$z(\sigma = \pm\frac{L}{2}) = 0, z(\sigma = 0) = z_c, z^{'}(\sigma = 0) = 0. \tag{2.72}$$

Thus the world sheet metric induced on the boosted background is given as

$$\begin{aligned} g_{\tau\tau} &= -H(z)(1-(1-f)\cosh^2\eta), \\ g_{\tau\sigma} &= g_{\sigma\tau} = 0, \\ g_{\sigma\sigma} &= H(z)[1+(1+\frac{z^{'2}}{f(z)})]. \end{aligned} \tag{2.73}$$

Then the Nambu-Goto action for the string takes the form as

$$S = -\frac{\mathcal{T}}{\pi\alpha'}\int_0^{\frac{L}{2}} d\sigma H(z)\sqrt{(1-(1-f)\cosh^2\eta)(1+\frac{z^{'2}}{f})}. \tag{2.74}$$

As the Lagrangian $\mathcal{L}$ in (2.74) does not depend on σ explicitly, the corresponding Hamiltonian is conserved and can be viewed as a constant of motion

$$-q = \frac{\partial\mathcal{L}}{\partial z^{'}}z^{'} - \mathcal{L}. \tag{2.75}$$

With this we can cast the equation of motion in the form as

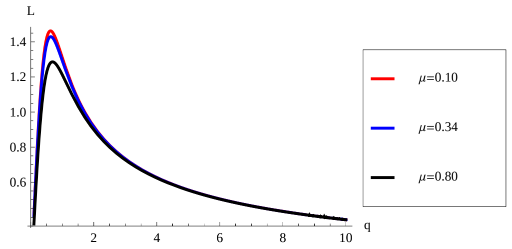
$$z' = \frac{\sqrt{f[H^2(1-(1-f)\cosh^2\eta) - q^2]}}{q}. \tag{2.76}$$

It is evident from the constraint (2.76) that at the horizon, $z = z_h$, where $f(z_h) = 0$, the factor $\frac{H^2}{q^2}(1-(1-f)\cosh^2\eta) - 1 = -\frac{H^2}{q^2}\sinh^2\eta - 1$ is always negative. At the boundary, $f(0) = 1$, the factor $\frac{H^2}{q^2}(1-(1-f)\cosh^2\eta) - 1 = \frac{H^2}{q^2} - 1$ is always positive for small values of $q < H$. Therefore in the range $0 < z < z_h$ there must be a location ($z = z_c$) where $\frac{H^2}{q^2}(1-(1-f)\cosh^2\eta) - 1$ switches its sign. Accordingly $z = z_c$ is the physical turning point of the string configuration. The string can not be stretched up to the horizon as z' is an imaginary quantity in the region $z_c < z < z_h$. By solving the equation

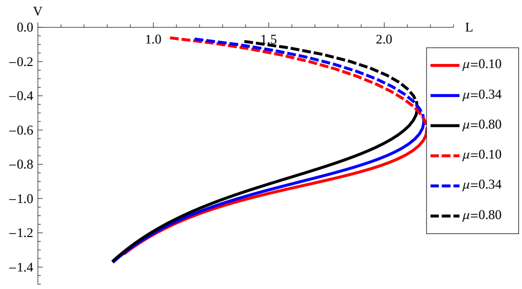
$$\frac{f(z_c)H^2(z_c)\cosh^2\eta}{q^2} - \frac{H^2(z_c)\sinh^2\eta}{q^2} - 1 = 0, \tag{2.77}$$

the turning point can be numerically determined. Then one can obtain the binding energy between the quark and antiquark pair through calculating the action (2.74) with constraint (2.76)

$$V = -\frac{S - S_0}{\mathcal{T}}, \tag{2.78}$$



(a)



(b)

Figure 2.8: Plot (a) shows the quark-antiquark distance as a function of q for a fixed rapidity, while plot (b) shows the binding energy with respect to the distance. In both plots we fix the chemical potential $\mu = 0.10, 0.34$, and 0.80, respectively. We have set a same temperature T to obtain these curves.

where S_0 is given by

$$S_0 = -\frac{\mathcal{T}}{\pi\alpha'}\int_0^{z_c} dz\sqrt{-G_{tt}G_{zz}}. \tag{2.79}$$

The distance between quark and antiquark can be calculated from (2.76) as

$$\frac{L}{2q} = \int_0^{z_c} dz\frac{1}{H\sqrt{f[(1-\cosh^2\eta(1-f)) - \frac{q^2}{H^2}]}}. \tag{2.80}$$

It is not possible to work out the integration in (2.80) explicitly. To determine the screening length, we plot the distance L with respect to the constant of motion q for a fixed rapidity η in figure (3.2 a). It turns out that for a fixed value of rapidity, there exists a maximum for L, which is regarded as the screening length $L_s = L_{max}(\eta)/(\pi T)$. Plot (b) in figure (3.2) shows the binding energy V given by (2.78) with respect to L. One can see from plot (a) that the quark-antiquark distance starts from zero when q is also zero, it increases sharply with respect to q, reaches its maximum at a certain q, and then decreases monotonically to zero at some finite q. In between these two zeros, there exists a single $L = L_{max}$ beyond which there is no solution of Eq.(2.80). This implies the quark-antiquark pair dissociates beyond $L = L_{max}$. We identify $L_{max}(\eta)/(\pi T)$ with the screening length L_s. For the $\mu = 0.1$ case, $L_{max} \simeq 1.4$ and $L_s \simeq 1.4/(\pi T) \simeq 0.45/T$, close to the lattice calculation $L_s \sim 0.5/T$ [179] of the static potential between heavy quark and antiquark in QCD. Plot (b) shows that there are two branches for the binding energy in the region $L < L_{max}$. The branch with dashed curves has a higher energy than the one with solid curves. This implies that the branch with dashed curves is physically disfavored.

The screening length $L_s(\eta)$ as a function of rapidity can be obtained numerically as illustrated in the figure(2.9). One finds that it decreases with the velocity which indicates

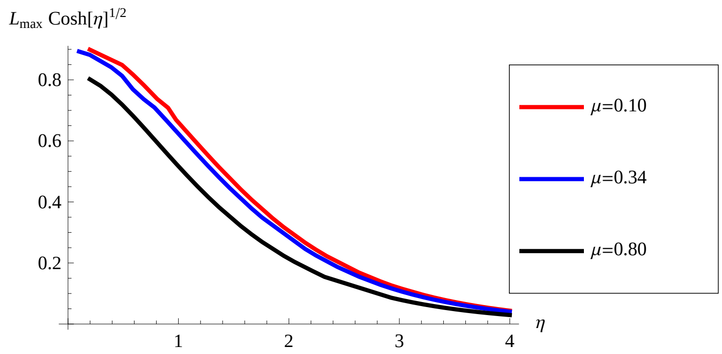


Figure 2.9: The screening length versus the rapidity η for the cases $\mu = 0.1, 0.34$, and 0.8, respectively. We have set a same temperature T to obtain these curves.

that the quark-antiquark pair dissociates at a lower temperature as it is moving. This behavior is also observed in [26]. If the qualitative behavior holds for QCD, it will have the consequence for quarkonium suppression in heavy ion collision. Additionally, our results show that the case with smaller chemical potential has a larger screening length.

2.6 Discussion

In this chapter, we studied a holographic QCD model proposed in [178], in an Einstein-Maxwell-Dilaton system. First, we have generalized the system by allowing a non-minimal coupling between between the Maxwell field and the dilaton field, and given an algorithm to generate a set of exact and asymptotic AdS black hole solutions in the EMD system. After briefly reviewing the main features of the hQCD model, we have studied some aspects of QGP phase of the hQCD model by calculating drag force, jet quenching parameter and screening length. The calculations show that the behaviors of those quantities are consistent with the expectation from the real QCD.

It is found that the drag force increases monotonically with the temperature which is consistent with the real QCD phenomenon in the large chemical potential region with $\mu \geq \mu_c$. In the small chemical potential region ($\mu < \mu_c$), the drag force monotonically increases at high temperature region, while at low temperature, it shows a multi-valued behavior. Note that, in the case $\mu < \mu_c$, the solution is dual to the confined phase of the QCD and the drag force is not well defined. Therefore the change from the multi-valued behavior to the monotonic behavior just manifests the existence of the first order phase transition. We further find that the jet quenching parameter has a monotonically decreasing behavior with the temperature This is also consistent with the QCD experiments in the region $\mu \geq \mu_c$. For $\mu < \mu_c$, the jet quenching parameter agrees with the real QCD expectation in the high temperature and once again, it shows the multi-valued behavior in the low-temperature region. As for drag force, the multi-valued behavior of the jet quenching parameter in the low temperature region is consistent with the existence of the first order phase transition in this hQCD model. For the screening length, we found the separation between the quark and anti-quark pair as a function of the constant of motion q. It is clear from the plot that for both cases $\mu \geq \mu_c$ and $\mu < \mu_c$, the dipole dissociates beyond a maximum separation distance (namely the screening length L_s). We have also calculated the binding energy as a function of the separation. In addition, we have presented $L_s(\eta)$ and found that there are qualitative consequences for quarkonium suppression in heavy ion collisions in this hQCD model.

3

Dissipative force on an external quark in heavy quark cloud

3.1 Introduction

In this chapter, we carry out a simple holographic calculation of the drag force for the thermal $\mathcal{N} = 4$ super Yang-Mills on R^3 in the following scenario. We consider an *uniformly* distributed heavy quark cloud in this hot plasma. We then ask: how does the drag force on an external heavy probe quark change with the density of the quark cloud? Our aim is to find how the cloud backreacts to the geometry and hence modifies the drag force on a moving heavy quark.

As previously discussed within the framework of AdS/CFT, the heavy probe quark is modeled by a fundamental string attached to the boundary of an AdS black hole. For the $\mathcal{N} = 4$ super Yang-Mills, the end point of the string carry a fundamental $SU(N)$ charge. The string extends along the radial direction of the AdS-Schwarzschild metric. This external quark, with a mass proportional to the length of the string, loses its energy as the string trails back imparting a drag force. The gravity dual of the quark cloud represents a black hole in the presence of a string cloud. These strings, assumed to be non-interacting, are aligned along the radial direction of the bulk geometry and are distributed uniformly over R^3.

The usual probe approximation one normally uses is justified because the free energy of the external quark goes as $\mathcal{O}(N_c)$ whereas the plasma, being in adjoint representation, contributes $\mathcal{O}(N_c^2)$ to the free energy. So, in this sense, in the large color limit, with $N_c \to \infty$, external quark can be treated as a probe. However, when large number of external quarks are introduced, the background geometry may get modified. And it is this effect that we would like to incorporate in our computation. This chapter is organized as follows.

In section 2 we discuss the construction of gravity dual for external quark cloud. We solve the Einstein equation of motion and obtain a AdS black hole solution by the presence of the cloud. We also discuss the thermodynamical properties of the black hole solution. In section 3, we check the gravitational stability of the black hole solution. Our stability analysis is restricted for the tensor and vector modes of the background solution. In section 4, using gauge/gravity dictionary, we calculate the drag force on an external quark moving in the heavy quark cloud. We conclude this chapter with a discussion.

3.2 Gravity dual for external quark cloud

We consider the $(n+1)$ dimensional gravitational action given by

$$\mathcal{S} = \frac{1}{16\pi G_{n+1}} \int dx^{n+1}\sqrt{-g}(R-2\Lambda) + S_m, \tag{3.1}$$

where S_m represents the matter part of the action. We represent the matter part as

$$S_m = -\frac{1}{2}\sum_i \mathcal{T}_i \int d^2\xi\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}, \tag{3.2}$$

where we considered $g^{\mu\nu}$ and $h^{\alpha\beta}$ are the space-time metric and world-sheet metric respectively with μ, ν represents space-time directions and α, β stands for world sheet coordinates. S_m is a sum over all the string contributions with i'th string having a tension $\mathcal{T}_i$. The integration in (3.2) is over the two dimensional string coordinates.

Varying this action with respect to the space-time metric leads to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{n+1}T_{\mu\nu}, \tag{3.3}$$

with

$$T^{\mu\nu} = -\sum_i \mathcal{T}_i \int d^2\xi \frac{1}{\sqrt{|g_{\mu\nu}|}}\sqrt{|h_{\alpha\beta}|}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \delta_i^{n+1}(x-X). \tag{3.4}$$

In the above, the delta function represents the source divergences due to the presence of the strings. In the following, we will consider the space-time metric of the form

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2\delta_{ab}dx^a dx^b, \tag{3.5}$$

where (a, b) run over $n-1$ space directions. We will further consider strings with uniform tensions T and use the static gauge $t = \xi^0, r = \xi^1$. The non vanishing components of $T^{\mu\nu}$, following from (3.4), are

$$T^{tt} = -\frac{ag^{tt}}{r^{n-1}}, \quad T^{rr} = -\frac{ag^{rr}}{r^{n-1}}. \tag{3.6}$$

Here we have assumed that the strings are uniformly distributed over $n-1$ directions such that the density is[1]

$$a(x) = T\sum_i \delta_i^{(n-1)}(x-X_i), \quad \text{with } a > 0. \tag{3.7}$$

[1] To define this properly, we need to think of an IR cutoff in $n-1$ directions.

For negative a, $T_{\mu\nu}$ will cease to satisfy the weak and the dominant energy conditions[2]. We look for a solution of (3.3) in AdS space and parametrize the metric acccordingly treating a as a constant[3],

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 h_{ij} dx^i dx^j. \tag{3.9}$$

Here h_{ij} is the metric on the $(n-1)$ dimensional boundary. As for the matter part we will focus on to the string cloud for which the nonzero $T_{\mu\nu}$ components are[4] given by

$$T^t_t = T^r_r = -\frac{a}{r^{n-1}}, \text{ with } a > 0. \tag{3.10}$$

The solution which satisfy the Einstein's equation can be easily constructed. It is given by[5]

$$V(r) = K + \frac{r^2}{l^2} - \frac{2m}{r^{n-2}} - \frac{2a}{(n-1)r^{n-3}}. \tag{3.11}$$

Here $K = 0, 1, -1$ depending on whether the $(n-1)$ dimensional boundary is flat, spherical or hyperbolic respectively, having curvature $(n-1)(n-2)K$ and volume V_{n-1}. In writing down $V(r)$ we have also parametrized cosmological constant as $\Lambda = -n(n-1)/(2l^2)$. With equation (3.11), the metric (3.9) represents a black hole with singularity at $r = 0$ and the horizon is located at $V(r) = 0$. The horizon has a topology of flat, spherical or hyperbolic depending on the value of K. However, our interest in this work, lies in the $K = 0$ case. In this case of flat horizon, the integration constant m is related to the ADM (M) mass of the black hole as follows,

$$M = \frac{(n-1)V_{n-1}m}{8\pi G_{n+1}}. \tag{3.12}$$

The horizon radius, denoted by r_+, satisfies the following equation

$$\frac{r_+^2}{l^2} - \frac{2m}{r_+^{n-2}} - \frac{2a}{(n-1)r_+^{n-3}} = 0. \tag{3.13}$$

This allow us to write m in terms of horizon radius as

$$m = \frac{(n-1)r_+^n - 2al^2 r_+}{2(n-1)l^2}. \tag{3.14}$$

[2]For earlier discussions on string cloud/fluid models see [203–205].

[3]Clearly a in (3.7) depends on x. However, in equations (3.9), (3.10) and in (3.11), a is treated as constant. To do this, we have replaced $a(x)$ by an average density as

$$a = \frac{1}{V_{n-1}} \int a(x) d^{n-1}x = \frac{T}{V_{n-1}} \sum_{i=1}^{N} \int \delta_i^{(n-1)}(x - X_i) d^{n-1}x = \frac{T}{V_{n-1}} \sum_{i=1}^{N} 1 = \frac{TN}{V_{n-1}}. \tag{3.8}$$

Here, V_{n-1} is the volume in $n-1$ dimensional space after imposing an IR cut-off. Now we consider the limit V_{n-1} going to infinity along with the number of strings N, keeping N/V_{n-1} constant.

[4]It turns out that replacement of δ_{ij} by h_{ij} in (3.5) keep the components of the stress-tensor same.

[5]This is a slight generalization of the metric in [206].

The temperature of the black hole is given by

$$T = \frac{\sqrt{g^{rr}}\partial_r \sqrt{g_{tt}}}{2\pi}|_{r=r_+} = \frac{n(n-1)r_+^{n+2} - 2al^2 r_+^3}{4\pi(n-1)l^2 r_+^{n+1}}. \tag{3.15}$$

Note that the zero mass black hole has a non-zero temperature and is given by

$$T_0 = \frac{a}{2\pi}\Big(\frac{n-1}{2al^2}\Big)^{\frac{n-2}{n-1}}. \tag{3.16}$$

The black hole temperature increases with the horizon size and for large r_+, it behaves as $T \sim r_+/l^2$. The entropy is defined as

$$S = \int T^{-1} dM, \tag{3.17}$$

leading to the entropy density [6]

$$s = \frac{r_+^{n-1}}{4G_{n+1}}. \tag{3.18}$$

Note that s is finite even for black hole with zero mass. The specific heat associated with the black hole is

$$C = \frac{\partial M}{\partial T} = \frac{V_{n-1}(n-1)r_+^{n-1}(n(n-1)r_+^n - 2al^2 r_+)}{4G_{n+1}(n(n-1)r_+^n + 2(n-2)al^2 r_+)}. \tag{3.19}$$

Now we have a detail look at the thermodynamic quantities just evaluated. First of all, if we restrict the temperature to be non-negative, the black hole can have minimum radius

$$r_+^{\min} = \Big(\frac{2al^2}{n(n-1)}\Big)^{\frac{1}{n-1}}. \tag{3.20}$$

It can easily be checked that if we focus on to the region $T \geq 0$, there is a single positive real solution of (3.13). We also note from (3.20) and (3.14) that the mass becomes negative at zero temperature

$$m^{\min} = -\frac{a}{n} r_+^{\min}. \tag{3.21}$$

This is somewhat similar to the AdS-Schwarzschild with negative curvature horizon [207]. We note that, for mass $m \geq m^{\min}$, the specific heat (3.19) continues to be positive and is continuous as a function of r_+. This suggests thermodynamical stability of the black hole. Finally, we write down the free energy of this black hole

$$\mathcal{F} = -\frac{(n-1)r_+^n + 2al^2(n-2)r_+}{16\pi(n-1)}. \tag{3.22}$$

[6] Due to the nature of a dependent term in (3.11), our definition of ADM mass is perhaps ambiguous. However, with this definition, entropy of the black hole comes out to be one quarter of the horizon area, provided we assume that the equation $dS = MdT$ holds for this black hole.

Before we go on to analyze gravitational stability of the black hole, we would like to make the following comment. Quite naturally, one may wonder if this black hole has a higher dimensional origin. In particular, can this arise, in some near horizon limit, from some brane configuration in ten or eleven dimensions after compactifying on spheres (with the cloud smeared over the compact manifold)? We indeed tried to get this from some bound state configurations of D-branes and strings but have not succeeded yet.

3.3 Stability of the flat black hole

We now study the stability of the $K = 0$ black hole geometry using the gravitational perturbation in a gauge invariant way [209, 210, 212–214]. We consider perturbation on a background space time M^{2+p}

$$M^{2+p} = \mathcal{N}^2 \times \mathcal{K}^p, \tag{3.23}$$

where the space time metric is,

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\delta_{ij}dx^i dx^j, \\ f(r) &= \frac{r^2}{l^2} - \frac{2m}{r^{p-1}} - \frac{2a}{pr^{p-2}}. \end{aligned} \tag{3.24}$$

We identify $\mathcal{N}^2$ as a two dimensional space time coordinatized by t and r, whereas $\mathcal{K}^p$ is a p dimensional maximally symmetric space coordinatized by $x^i s$. Each perturbed tensor realized on $\mathcal{K}^p$ can be grouped into scalar, vector, and tensor components such that Einstein equations of motion respect the decomposition. Here we do stability analysis for tensor and vector perturbations. Scalar perturbation is somewhat more involved and will be reported else where.

3.3.1 Tensor perturbation

In the case of the tensor perturbation, the metric tensor and energy momentum tensor are decomposed in scalar, vector, tensor part with respect to $\mathcal{K}^p$ in the following manner [210],

$$\begin{aligned} h_{ab} = 0, h_{ai} = 0, h_{ij} = 2r^2 H_T \mathbb{T}_{ij} \\ \delta T_{ab} = 0, \delta T^a_i = 0, \delta T^i_j = \tau_T \mathbb{T}^i_j, \end{aligned} \tag{3.25}$$

$\mathbb{T}_{ij}$ is the tensor harmonic function defined on $\mathcal{K}^p$. It satisfies the following properties,

$$\begin{aligned} (\hat{\triangle} + k_T^2)\mathbb{T}_{ij} = 0 \\ \mathbb{T}^i_i = 0, \hat{D}_j \mathbb{T}^j_i = 0. \end{aligned} \tag{3.26}$$

Here we note that in $\mathcal{K}^p$ space, $\hat{\triangle}$ and $\hat{D}_j$ are realized as the Laplace-Beltrami self-adjoint operator and the covariant derivative respectively. For $K = 0$, k_T^2 can take non-negative

real continuous values [212]. Gauge invariant quantities like H_T and τ_T are function of variables belong to $\mathcal{N}^2$ spacetime. [213].
Now substituting all the variations in the perturbed Einstein equation, we get the master equation of tensor perturbation [210].

$$\Box H_T + \frac{p}{r} Dr \cdot DH_T - \frac{k_T^2}{r^2} H_T = -\kappa^2 \tau_T \tag{3.27}$$

We introduce a new variable Φ,

$$\Phi = r^{p/2} H_T, \tag{3.28}$$

and substitute it into the master equation. It takes following canonical form,

$$\Box \Phi - \frac{V_T \Phi}{f} = 0, \tag{3.29}$$

where V_T is defined as,

$$V_T = \frac{f}{r^2}[{k_T}^2 + \frac{prf'}{2} + \frac{p(p-2)f}{4}]. \tag{3.30}$$

According to (3.4) the energy momentum tensor is constructed with the spacetime vector $X^\mu(t,r)$ which does not contribute to the linear order of gauge invariant tensor perturbation. Therefore we set τ_T to be zero in (3.29). It is clear in plot-1 of figure (3.2) that for higher dimensions V_T is always positive beyond horizon. So the black hole geometry is stable under tensor perturbation.

3.3.2 Vector perturbation

In the case of vector perturbation, the metric and the energy momentum tensor are decomposed in terms of vector harmonics V_i as well as vector harmonic tensor V_{ij} [210].

$$\begin{aligned} h_{ab} = 0, h_{ai} = r f_a V_i, h_{ij} = 2r^2 H_T V_{ij} \\ \delta T_{ab} = 0, \delta T_i^a = r\tau^a V_i, \delta T_j^i = \tau_T V_j^i \end{aligned} \tag{3.31}$$

The vector harmonics are defined as

$$(\hat{\Delta} + {k_V}^2) V_i = 0, \hat{D}_i V^i = 0 \tag{3.32}$$

>From vector harmonics we can construct vector type harmonic tensor,

$$\begin{aligned} V_{ij} = -\frac{1}{2k_V}(\hat{D}_i V_j + \hat{D}_j V_i), \\ (\hat{\Delta} + k_V^2) V_{ij} = 0, V_i^i = 0, \hat{D}_j V_i^j = \frac{k_V}{2} V_i. \end{aligned} \tag{3.33}$$

The gauge invariant parameters for K $= 0$ are given by

$$F_a = f_a + rD_a(\frac{H_T}{k_V}), \tau_T, \tau^a. \tag{3.34}$$

Upon considering the perturbations of the Einstein equation and the conservation law of energy momentum tensor, master equation arising from the gravitational perturbation with the source term takes the following form [212],

$$r^p D_a(\frac{1}{r^p} D^a \Omega) - \frac{{k_V}^2}{r^2}\Omega = -\frac{2\kappa^2}{{k_V}^2} r^p \epsilon^{ab} D_a(r\tau_b) \tag{3.35}$$

where,

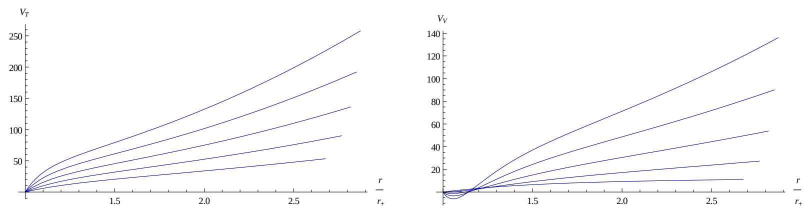
$$r^{p-1}F^a = \epsilon^{ab} D_b \Omega + \frac{2\kappa^2}{{k_V}^2} r^{p+1}\tau^a. \tag{3.36}$$

Now introducing the change of variable

$$\Phi = r^{-p/2}\Omega \tag{3.37}$$

we recast the master equation in a canonical form, where the effective potential for vector perturbation comes out as [214],

$$V_V = \frac{f}{r^2}[{k_V}^2 + \frac{p(p+2)}{4} f - \frac{pr}{2} f'] \tag{3.38}$$



(a) Plot-1 (b) Plot-2

Figure 3.1: Plot 1 shows, for various dimensions, the effective potential V_T in tensor perturbation is positive beyond horizon radius . Plot 2 shows the effective potential V_V in vector perturbation is not always non-negative for $p > 3$. In both cases horizontal axis is normalized with respect to black hole horizon radius r_+.

The plot-2 in figure (3.2) implicates that beyond horizon, V_V is not always non-negative for $p > 3$. We follow $S - deformation$ method [210] to construct modified effective potential

$$\tilde{V_V} = V_V + f\frac{dS}{dr} - S^2, \tag{3.39}$$

where S is an arbitrary function of r. If we choose $S = \frac{pf}{2r}$, we get the modified effective potential,

$$\tilde{V}_V = \frac{f {k_V}^2}{r^2} > 0 \tag{3.40}$$

Once again ${k_V}^2$ is the eigenvalue of a positive operator. So the above form of $\tilde{V}_V$ furnishes the sufficient condition for the stability of the black hole.
Having constructed this black hole geometry we compute the drag force on an external quark moving in external quark cloud

3.4 Dissipative force on an external quark moving in the heavy quark cloud

We now like to calculate the dissipative force experienced by the external heavy quark moving in the cloud of heavy quarks. Our aim is to study the force as a function of the cloud density. Calculational procedure to evaluate drag force can be found, for example, in [94,95,101]. As in the previous chapter we follow the notation of [95]. The drag force on a very massive quark with fundamental $SU(N)$ charge at finite temperature is calculated holographically by studying the motion of a string whose end point is on the boundary. This end point represents the massive quark whose mass is proportional to the length of the string. We will consider here the gauge theory on R^3 coordinatized by x^1, x^2, x^3. This means, for the purpose of this computation, we only consider $K = 0, n = 4$ case of the black holes discussed previously.

Let us consider the motion of a string only in one direction, say x^1. In static gauge, $t = \xi^0, r = \xi^1$, the embedding of the world-sheet is given by the function $x^1(t, r)$. The induced action of the string in our case follows from a straightforward computation

$$S = -\frac{1}{2\pi\alpha'} \int dtdr \sqrt{1 + \frac{3r^4 - 2al^2r - 6ml^2}{3l^4}(\partial_r x^1)^2 - \frac{3r^4}{3r^4 - 2al^2r - 6ml^2}(\partial_t x^1)^2}, \tag{3.41}$$

where we have scaled x^1 by l.

The ansatz that describes the behaviour of the string with attached quark moving with constant speed v along x^1 is given by [95]

$$x^1(r, t) = vt + \xi(r), \tag{3.42}$$

for which (3.41) simplifies to

$$S = -\frac{1}{2\pi\alpha'} \int dtdr \sqrt{1 + \frac{3r^4 - 2al^2r - 6ml^2}{3l^4}(\partial_r \xi)^2 - \frac{3r^4}{3r^4 - 2al^2r - 6ml^2}v^2}. \tag{3.43}$$

The momentum conjugate to $\xi(r)$ is

$$\pi_\xi = -\frac{1}{2\pi\alpha'} \frac{(3r^4 - 2al^2r - 6ml^2)\partial_r \xi}{3l^4 \sqrt{1 + \frac{3r^4 - 2al^2r - 6ml^2}{3l^4}(\partial_r \xi)^2 - \frac{3r^4}{3r^4 - 2al^2r - 6ml^2}v^2}}. \tag{3.44}$$

Equation of motion can be obtained by inverting this equation for $\partial_r \xi$. However, as in [95], to get a real ξ, the constant of motion π_ξ has to be set to

$$\pi_\xi = -\frac{1}{2\pi\alpha'}\frac{v r_v^2}{l^2} \tag{3.45}$$

where r_v is the real positive solution of the equation

$$3(1-v^2)r_v^4 - 2al^2 r_v - 6ml^2 = 0. \tag{3.46}$$

Though this equation can be solved explicitly, the solutions are not very illuminating. However, it is easy to check that there is only one real positive solution. Substituting this solution of r_v in (3.45), we get π_ξ. The dissipative force is then given by [95]

$$F = -\frac{1}{2\pi\alpha'}\frac{v r_v^2}{l^2}, \tag{3.47}$$

with r_v given by the positive real solution of (3.46). Now we wish to rewrite the expression of the dissipative force in terms of gauge theory parameters. Along this line, we solve (3.15) for r_+,

$$r_+ = \frac{l^2}{6}A(T,b), \tag{3.48}$$

where b is the scaled quark cloud density, $b = a/l^4$ and A is given by,

$$\begin{aligned} A(T,b) \quad &= \quad [\{2(9b+4\pi^3T^3)+6\sqrt{b(9b+8\pi^3T^3)}\}^{1/3} + \\ & 2\pi T\{1+\frac{2^{2/3}\pi T}{\{(9b+4\pi^3T^3)+3\sqrt{b(9b+8\pi^3T^3)}\}^{1/3}}\}] \end{aligned} \tag{3.49}$$

Substituting (3.48) and the following useful relation

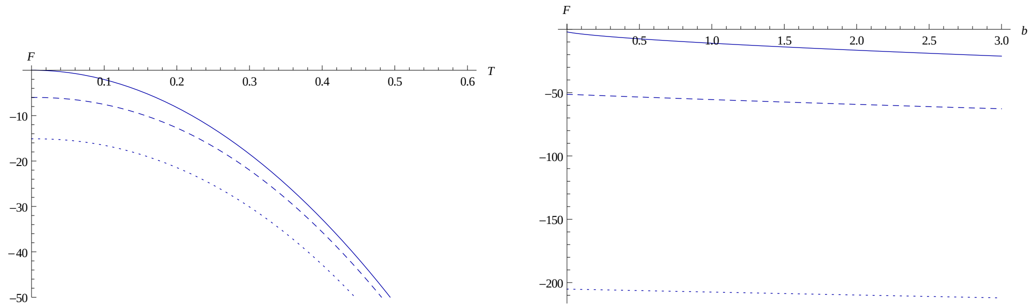
$$\frac{l^4}{\alpha'^2} = {g_{YM}}^2 N, \tag{3.50}$$

in the expression of the dissipative force (3.47), we get the modified form

$$F = -\frac{A^2}{72\pi}\sqrt{{g_{YM}}^2 N} v \frac{{r_v}^2}{{r_+}^2}. \tag{3.51}$$

Here g_{YM} is the Yang-Mills(YM) gauge coupling and N is the order of the gauge group $SU(N)$. We are able to solve the ratio ${r_v}^2/{r_+}^2$ in a closed form by substituting (3.13) into (3.46). The relevant equation takes the following form,

$$(1-v^2)(\frac{{r_v}^4}{{r_+}^4}) - \frac{144b}{(A(T,b))^3}((\frac{r_v}{r_+})-1)-1 = 0, \tag{3.52}$$



(a) Plot 1 (b) Plot 2

Figure 3.2: Plot 1 shows the variation of F as a function of T for the values of quark density $b = 0$ (solid), 0.5 (dashed), 2 (dotted) respectively. Plot 2 shows the variation of F as a function of b for the values of $T = .1$ (solid), .5 (dashed), 1 (dotted) respectively. We see in both cases the larger the quark cloud density as well as temperature, the more is the dissipative force.

It turns out that the real positive solution of (3.52) is expressible in terms of $A(T, b)$ and b itself. Denoting the solution as $f(A, b)$ and plugging it back into (3.51) we achieve the form of dissipative force expressible in terms of gauge theory parameters

$$F = -\frac{A^2}{72\pi}\sqrt{g_{YM}{}^2 N} v f(A, b)^2. \tag{3.53}$$

We note here that $f(A, b)$ is an explicitly computable function.

We would now study (3.53) for different values of heavy quark density and for fixed T. As for an example, it is interesting to check that if the temperature is fixed at the value T_0 as mentioned in (3.16) the dissipative force behaves as $F \sim -b^{2/3}$, where b is now the density of the quark cloud. Also for $T = 0$, $A(T, b)$ in (3.49) simplifies significantly resulting the dissipative force to vary as $F \sim -b^{2/3}$. For generic temperature and small b, it is possible to have a power series solution of (3.53) in b. However, for appreciable density, we find it more suitable to analyze F in terms of plots. In figure (3.2) plot 1 shows the behaviour of the drag force as a function of T for different b. For fixed T, we clearly see that the force becomes stronger with the quark density[7]

3.5 Conclusion

In this chapter we have computed the dissipative force experienced by an external heavy quark with fundamental $SU(N)$ charge moving in the heavy quark cloud at finite temperature.

[7]Note that the free energy (3.22) is perfectly well behaved at $T = 0$. Infact, it is $\mathcal{F} = -\frac{3ar_+}{32\pi}$. Substituting r_+, we find $\mathcal{F} \sim -b^{\frac{4}{3}}$. Furthermore computation of the drag force leads to $F \sim -b^{\frac{2}{3}}$ in this limit.

To conclude we address the relevance of this work in the context of quark gluon plasma. Let us consider the dynamics of a heavy quark (say charm) passing through the plasma. The dynamics is encoded in it's interaction with the medium and the resulting energy loss can be calculated. In such calculations, any possible effects of other heavy quarks due to the back-reaction of the plasma are neglected. In the context of $\mathcal{N} = 4$ SYM, we have computed such back-reaction effects. Within the gauge/gravity correspondence, such effects can be modeled in terms of the deformation of the geometry due to finite density string cloud. The back-reacted dual gravity background is explicitly computable. The black hole is parametrized by it's mass and the string cloud density a. The black hole was found to be thermodynamically stable. We further checked the gravitational stability of the geometry for tensor and vector perturbations. We then holographically computed the drag force exerted on the external quark. The most general form of the dissipative force turned out to be a complicated function of the temperature of the boundary theory T and the re-scaled quark cloud density b. The nature of the function shows the enhancement of the drag force due to the presence of the quark cloud.

4

Correlators of Giant Gravitons from dual ABJ(M) Theory

4.1 Introduction

In the previous chapters, we used the gravity duals to find certain features of the strongly coupled plasma. Here we will exploit the other side of the duality. In particular, using the strong/weak duality of the AdS_4/CFT_3 correspondence [120], we study some aspects of the strongly coupled gravity via their boundary duals.

In the introduction, while reviewing the AdS_4/CFT_3 correspondence, we noted that the M/$typeIIA$ string theory contains giant gravitons which are non-perturbative in nature. Our aim in this chapter is to study the transitions between giant gravitons and also between the giants and the ordinary gravitons. We address this issue within the ABJ theory. This requires the construction of gauge invariant operators by generalizing the Schur polynomial constructed in the ABJM theory [166]. We then find out the realization of the duality between the giant gravitons and the Schur polynomials for both ABJM and ABJ theory. Further we study the transition probabilities among the giants as well as between the giants and the ordinary gravitons by analyzing the corresponding gauge theory correlators. For this purpose one needs the appropriate normalization of the gauge theory operators. Here, we consider two normalization prescriptions, namely overlap-of-states normalization and multi-particle normalization. Both normalization procedures have been extensively discussed in the literature [159, 167]. Both types of normalization factor depend on the boundary coordinates as well as the gauge indices. With these proper normalizations one can identify the gauge theory correlators with the probability amplitudes associated with the transition among giant graviton states or between the giant graviton states and ordinary graviton states in the dual gravity theory.

This chapter is organized in the following way: in section 2, we discuss the Schur polynomials for ABJ theory and also identify them with the gravitons and giant gravitons of the dual gravity theory. In section 3 we elaborate on the normalization of the boundary CFT correlators. In section 4, 5 and 6 we provide various examples of computing the normalized boundary correlators for ABJ theory. In section 7, we compute the normalized gauge theory correlators for ABJM theory. Section 8 is devoted for studying the modification of the large

N expansions in the ABJ theory due to the presence of a non-trivial background in the dual gravity theory.

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4.2 Schur Polynomial

In this section we generalize the Schur polynomial of the ABJM theory to the one for the ABJ theory. With this goal in mind, we closely follow the derivation of the Schur polynomial of ABJM theory keeping track of whether the gauge indices are associated with the gauge groups, $U(N_1)$ or $U(N_2)$. Depending on the choice of the gauge group, contraction of the gauge indices gives rise to the factors associated with either N_1 or N_2. Following the logic as described in [166], we can write down the simplest gauge invariant half-BPS operator as $\prod_{n_i}[\text{Tr}((AB^\dagger)^l)]^{n_i}$. We consider the compact notation A and B to depict the four complex scalars A_1, A_2 and B_1, B_2 respectively. Since these complex scalar fields transform under the bi-fundamental representation of the gauge group, therefore in the matrix notation, we can write A and $B^\dagger$ in the following way

$$A^i_j \text{ and } (B^\dagger)^j_i.$$

Unlike ABJM now i and j are gauge indices of $U(N_1)$ and $U(N_2)$ respectively. Here l counts the amount of $\mathcal{R}$-charge of the operator. According to [217,218], these operators are represented by Young tableaux consisting of boxes equal to the number of $(AB^\dagger)$ fields and at most the smallest of (N_1, N_2) rows. In [166] we have shown that single trace operators are not valid basis to study the ABJM gauge theory in the large $\mathcal{R}$-charge limit. The correct description of the gauge theory operators are Schur polynomials. Therefore to check the validity of the single trace operator in this theory, following [159, 166], we compute the 3-point function of two different operators. In the leading order the result is

$$\frac{\left\langle \mathcal{O}_1 \mathcal{O}_2^\dagger \right\rangle}{\sqrt{\left\langle \mathcal{O}_1 \mathcal{O}_1^\dagger \right\rangle}\sqrt{\left\langle \mathcal{O}_2 \mathcal{O}_2^\dagger \right\rangle}} \sim \frac{\sqrt{l l_1 l_2}}{N_1} + \frac{\sqrt{l l_1 l_2}}{N_2}. \tag{4.1}$$

Where

$$\begin{aligned} \mathcal{O}_1 &= \text{Tr}\big((AB^\dagger)^l\big) \\ \text{and} \quad \mathcal{O}_2 &= \text{Tr}\big((AB^\dagger)^{l_1}\big)\text{Tr}\big((AB^\dagger)^{l_2}\big) \qquad \text{with} \qquad l_1 + l_2 = l. \end{aligned}$$

To compute this three point function we use the basic Wick contraction between two fields. One simple example of this contraction is

$$\left\langle \text{Tr}\big(AB^\dagger\big)\text{Tr}\big(A^\dagger B\big) \right\rangle = \left\langle A^{i_1}_{j_1} B^{\dagger j_1}_{\ i_1} A^{\dagger j_2}_{\ i_2} B^{i_2}_{j_2} \right\rangle = \delta^{i_1}_{i_2}\delta^{j_2}_{j_1}\delta^{j_1}_{j_2}\delta^{i_2}_{i_1} = N_1 N_2.$$

Since i and j are the gauge indices of $U(N_1)$ and $U(N_2)$ group respectively, the sum over i gives N_1 and the same over j gives N_2. However, we drop the space-time dependence in this computation, we can easily bring them back when needed. The result of the three point function says that at large (N_1, N_2) it vanishes if the factor $\sqrt{ll_1l_2}$ is less than (N_1, N_2) and diverges otherwise. Therefore like ABJM model, in this case also, the trace operator will be the gauge invariant operator if $\mathcal{R}$-charge l is less than the smaller one between $N_1^{2/3}$ or $N_2^{2/3}$. However, we should remember that we have just computed one specific three point function and not the most general correlator of the theory. The BMN type analysis in $\mathcal{N} = 4$ super Yang-Mills theory as well as ABJM theory is more exhaustive to conclude about the limit of the $\mathcal{R}$-charge [160, 219]. These study suggest that it must be less than $\sqrt{N}$ to suppress the non-planar corrections those are important even before $\mathcal{R}$-charge gets to $N^{2/3}$. Thus we rather say that if $\mathcal{R}$-charge of the operator in the ABJ(M) theory is greater than the smallest of $(\sqrt{N_1}, \sqrt{N_2})$, for the correct description, we need a new basis and our natural choice is Schur polynomial given by

$$\chi_R(AB^\dagger) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \mathrm{Tr}\big(\sigma(AB^\dagger)\big) \tag{4.2}$$

where R is the representation of a specific Young diagram with n boxes. This Young diagram labels both the representation of unitary gauge group and symmetric group S_n. $\chi_R(\sigma)$ is the character or trace of the element $\sigma \in S_n$ in the representation R.

The two point function for this theory can easily be calculated using the same procedure of [166, 167]. The result of the two point function of our interest is

$$\Big\langle \chi_R(AB^\dagger)\chi_S(A^\dagger B) \Big\rangle = \Big(n! \frac{Dim_{N_1}(S)}{d_S}\Big)\Big(n! \frac{Dim_{N_2}(S)}{d_S}\Big)\delta_{RS} = f_S^{N_1} f_S^{N_2} \delta_{RS}. \tag{4.3}$$

Here R and S represent the Young diagrams with n number of boxes for symmetric group S_n. d_S is the dimension of a representation S of the permutation group S_n. $Dim_{N_1}(S)$ and $Dim_{N_2}(S)$ are the dimensions of the representation of the unitary group $U(N_1)$ and $U(N_2)$ respectively. $f_S^{N_1}$ and $f_S^{N_2}$ are the product of the weights of the same Young diagram but for the gauge group $U(N_1)$ and $U(N_2)$ respectively. In this calculation we suppress the space-time dependence. The presence of delta function says that Schur polynomials satisfy the orthogonality condition in the free field limit and therefore these Schur polynomials are the correct gauge invariant operators to study the ABJ(M) theory for large $\mathcal{R}$-charge. Recall that for usual Schurs in $\mathcal{N} = 4$ super Yang-Mills theory, if we have more than N boxes in a column the product of weights vanishes [167]; this is the stringy exclusion principle of AdS_5 [157]. Equation(4.3) shows a similar behavior for ABJ(M) theory. When the number of boxes is more than the smallest of (N_1, N_2) the two point correlator and, hence, the operator vanish. Thus it is the smallest of (N_1, N_2) which sets the bound on the stringy exclusion principle.

In order to bring back the space-time dependence of the two point function we first find out the Green's function $G(x-y)$ for three dimensional gauge theory. This Green's function is the solution of the differential equation

$$\Delta_x G(x-y) = -\delta^3(x-y). \tag{4.4}$$

The solution is given by

$$G(x-y) = \frac{1}{4\pi|x-y|}. \tag{4.5}$$

Therefore the above two point function, with space-time dependence included, takes the form

$$\Big\langle \chi_R(AB^\dagger)(x)\chi_S(A^\dagger B)(y) \Big\rangle = \frac{f_S^{N_1} f_S^{N_2} \delta_{RS}}{\big(4\pi|x-y|\big)^{2\Delta_R}} \tag{4.6}$$

where Δ_R is the conformal dimension of the operator $\chi_R(AB^\dagger)$.

The three point and multi point functions can easily be calculated from this two point function by using the product rule of Schur polynomials. The product rule of Schur polynomials is as follows

$$\chi_{R_1}(AB^\dagger)\chi_{R_2}(AB^\dagger) = \sum_S g(R_1, R_2; S)\chi_S(AB^\dagger). \tag{4.7}$$

Here the Littlewood-Richardson number $g(R_1, R_2; S)$ counts the number of times irreducible representation S appears in the direct product of irreducible representations R_1 and R_2. By repeated use of this product rule we can write a general direct product $\prod_{i=1}^{l} \chi_{R_i}(AB^\dagger)$ as,

$$\begin{aligned}\prod_{i=1}^{l} \chi_{R_i}(AB^\dagger) &= \sum_{S_1, S_2 \cdots S_{l-2}, S} g(R_1, R_2; S_1) g(S_1, R_3; S_2) \cdots g(S_{l-2}, R_l; S)\chi_S(AB^\dagger) \\ &= \sum_S g(R_1, R_2 \cdots R_l; S)\chi_S(AB^\dagger). \end{aligned} \tag{4.8}$$

By using this product rule we can have the three point function as

$$\Big\langle \chi_{R_1}(AB^\dagger)(x_1)\chi_{R_2}(AB^\dagger)(x_2)\chi_S(A^\dagger B)(y) \Big\rangle = \frac{g(R_1, R_2; S) f_S^{N_1} f_S^{N_2}}{\{4\pi(y-x_1)\}^{2\Delta_{R_1}} \{4\pi(y-x_2)\}^{2\Delta_{R_2}}} \tag{4.9}$$

where Δ_{R_1} and Δ_{R_2} are conformal dimensions of the operator $\chi_{R_1}(AB^\dagger)$ and $\chi_{R_2}(AB^\dagger)$. Combination of these two conformal dimensions gives the conformal dimension of $\chi_S(A^\dagger B)$. Similarly we can find out the multi point function as

$$\begin{aligned}&\Big\langle \chi_{R_1}(AB^\dagger)(x_1)\chi_{R_2}(AB^\dagger)(x_2) \cdot \cdot \chi_{R_l}(AB^\dagger)(x_l)\chi_{S_1}(A^\dagger B)(y)\chi_{S_2}(A^\dagger B)(y) \cdot \cdot \chi_{S_k}(A^\dagger B)(y) \Big\rangle \\ &= \sum_S \frac{g(R_1, R_2 \cdots R_l; S)\, f_S^{N_1} f_S^{N_2}\, g(S_1, S_2 \cdots S_k; S)}{\{4\pi(y-x_1)\}^{2\Delta_{R_1}} \cdots \{4\pi(y-x_l)\}^{2\Delta_{R_l}}}. \end{aligned} \tag{4.10}$$

The class of correlators that we study here are the analog of the extremal correlators of the $\mathcal{N} = 4$ super Yang-Mills theory. Motivated by our experience with $\mathcal{N} = 4$ super Yang-Mills theory, we make a guess that these correlators are the exact answer i.e. they will not receive any higher loop corrections in the 'tHooft coupling. We need to compute the Feynman graphs to confirm the guess. As before, $\Delta_{R_1} \cdots \Delta_{R_l}$ are the conformal dimensions of $\chi_{R_1}(AB^\dagger) \cdots \chi_{R_l}(AB^\dagger)$ and the sum of these conformal dimensions gives the total conformal dimension of the representation S.

According to [217,218,228] trace operators are identified with the giant gravitons of the dual gravity theory and the $\mathcal{R}$-charge of the operators with the angular momentum of the giant gravitons. The trace operator represented by a single row Young diagram is mapped into the giant graviton which grows within the AdS part of the geometry and is called as AdS giant and the operator corresponding to Young diagram of single column with maximum number of boxes equal to the smallest of (N_1, N_2) is mapped into the giant graviton which wraps in S^7 or $\mathbb{CP}^3$, and is called as sphere giant. Similarly we can also map this Schur polynomial with the giant graviton of the dual gravity theory. The Schur polynomial associated with Young diagram of single column i.e. fully antisymmetric representation of symmetric group S_n with at most the smallest of (N_1, N_2) number of boxes is mapped into a sphere giant and the operator represented by single row Young diagram i.e. fully symmetric representation of the same group corresponds to an AdS giant. In this chapter following [216], we use the notation $\chi_{[L]}$ for the AdS giant and $\chi_{[1^L]}$ for the sphere giant with angular momentum of L units. If the sphere giant brane wraps k times, the corresponding Schur polynomial will have Young diagram of k number of columns whereas the number of boxes associated with each of them is of the same order as the smaller one between N_1 and N_2. For the k number of wrapping of AdS giant within the circle of AdS the corresponding Young diagram will have k number of rows where the maximum value of k can be the smallest of (N_1, N_2). Schur polynomial represented by Young Diagram R associated with small $\mathcal{R}$-charge, i.e conformal dimension $\Delta_R = O(1)$ is associated with KK state of the gravity. They can be written as sums of products of small numbers of traces.

By using this mapping we can compute the gravity correlators between KK states, giant gravitons and among KK states and giant gravitons from the corresponding gauge theory correlators. These correlators give the probability for the state created by the operator at a particular point of the space-time to evolve into the state created by another operator at different point of the space-time by proper normalization. Therefore, it is very crucial to normalize the gauge theory correlators in a consistent way to get the probability less than 1. In the literature, there exist two types of normalization [216], namely the overlap of state normalization and the multi particle normalization. Both prescriptions consist of two parts, one depends on gauge indices and other is a function of space-time coordinates of involved operators. Without the space-time dependence, the first scheme gives the probabilities within one but the second procedure suffers from growth in N. However by including the space-time part, the problem is resolved in [216]. So to figure out the normalization factor, we need to know the topology of the space on which the operators live. In conformally invariant field theories, the analysis of correlators on different topologies and the relation between them is generically known as CFT factorization leading to factorization equations and inequalities in the specific dimension. In the next section we briefly review the CFT factorization for different topologies and interpret a normalized version of this as the probability.

4.3 From CFT factorization to probability interpretation

This section generalizes the discussion of [216] to ABJ(M) theory. Factorization can be explained in the following intuitive way. We consider a manifold M with two operator insertions and compute the corresponding correlator Z_M. Now we cut the manifold M along a boundary $\mathbb{B}$ into two constituent manifolds, M_1, M_2 with one operator insertion in each and compute the correlators Z_{M_1} and Z_{M_2} accordingly, constrained to the fact that all possible boundary configuration should be taken into account. The CFT factorization suggests,

$$Z_M = \sum_B Z_{M_1}(B) \times Z_{M_2}(B). \tag{4.11}$$

In the context of overlap state normalization we consider a manifold S^3 with two operator insertions. Then we cut it into two manifolds with one boundary $\mathbb{B}$ having one operator in each as depicted in fig 4.1.

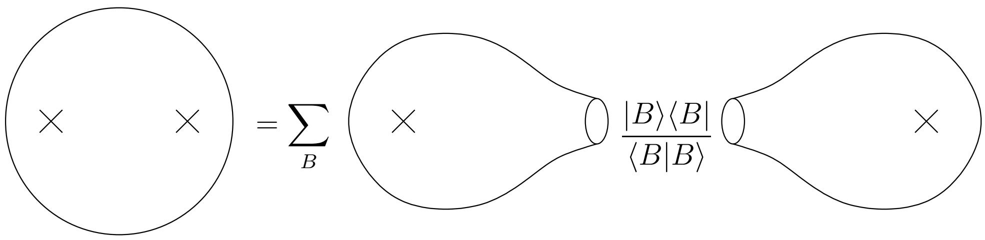

Figure 4.1: In this figure the operator insertions are represented by cross marks.

Now the factorization in conformal field theory relates this n-point correlator to lower point correlator as

$$\Big\langle \mathcal{O}^\dagger(x^*)\mathcal{O}(x)\Big\rangle = \sum_B \frac{\big\langle \mathcal{O}^\dagger(x^*)B(y)\big\rangle\big\langle B^\dagger(y^*)\mathcal{O}(x)\big\rangle}{\big\langle B^\dagger(y^*)B(y)\big\rangle} \tag{4.12}$$

where $\mathcal{O}'$s and B's are the local operators defined on the manifold of interest and on the boundary cut respectively. While defining equation(4.12) we assume that we are working in a basis which diagonalizes the metric on the space of local operators and the conjugation operation is executed by reversing the Euclidean time coordinate. This can be generalized for the extremal correlators where the operators are localized at number of different points:

$$\Big\langle \mathcal{O}^\dagger{}_1(x^*{}_1)\mathcal{O}^\dagger{}_2(x^*{}_2)\cdots\mathcal{O}^\dagger{}_k(x^*{}_k)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)\Big\rangle = \sum_B \frac{\Big\langle \mathcal{O}^\dagger{}_1(x^*{}_1)\mathcal{O}^\dagger{}_2(x^*{}_2)\cdots\mathcal{O}^\dagger{}_k(x^*{}_k)B(y)\Big\rangle\Big\langle B^\dagger(y^*)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)\Big\rangle}{\big\langle B^\dagger(y^*)B(y)\big\rangle} \tag{4.13}$$

Now it is straightforward to promote equation (4.13) into the probability interpretation by dividing both sides of (4.13) by the left hand side of the same equation and thus we get

$$1 = \sum_B \frac{\Big|\langle \mathcal{O}^\dagger{}_1(x^*{}_1)\mathcal{O}^\dagger{}_2(x^*{}_2)\cdots\mathcal{O}^\dagger{}_k(x^*{}_k)B(y)\rangle\Big|^2}{\big\langle \mathcal{O}^\dagger{}_1(x^*{}_1)\mathcal{O}^\dagger{}_2(x^*{}_2)\cdots\mathcal{O}^\dagger{}_k(x^*{}_k)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)\big\rangle\big\langle B^\dagger(y^*)B(y)\big\rangle} \tag{4.14}$$

We call P as the probability for $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)$ to evolve into B.

$$P\Big(\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k) \to B(y)\Big) = \frac{\Big|\mathcal{O}^\dagger{}_1(x^*{}_1)\mathcal{O}^\dagger{}_2(x^*{}_2)\cdots\mathcal{O}^\dagger{}_k(x^*{}_k)B(y)\rangle\Big|^2}{\big\langle \mathcal{O}^\dagger{}_1(x^*{}_1)\mathcal{O}^\dagger{}_2(x^*{}_2)\cdots\mathcal{O}^\dagger{}_k(x^*{}_k)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)\big\rangle\big\langle B^\dagger(y^*)B(y)\big\rangle} \tag{4.15}$$

The above formula (4.15) for probability is based on the notion of overlap state normalization. If we replace the state $|B(y)\rangle$ by $|B(y_1)B(y_2)\rangle$, the probability will not correspond to the separate measurement of the operators $B(y_1)$ and $B(y_2)$.

The extension of this idea for the probability interpretation of separate measurements or multi particle normalization needs correlator on higher topology. For two separate measurements, unlike the case of overlap state normalization we take a manifold $S^1 \times S^2$ with genus-1 and $2k$ number of operator insertions. Then cut out two boundaries $\mathbb{B}_1$ and $\mathbb{B}_2$ in such a way that both manifolds have k number of operator insertions as in fig 4.2.

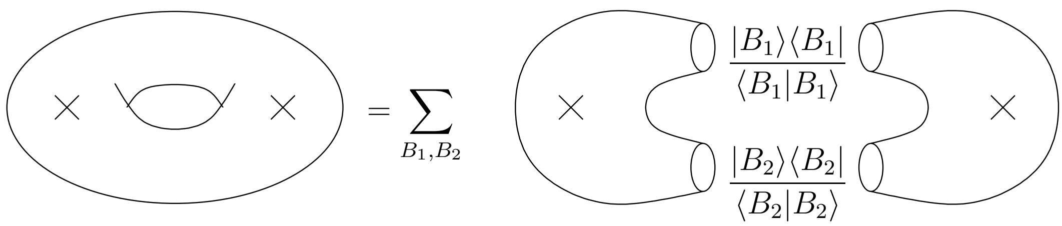

Figure 4.2: In this figure k number of operator insertions are represented by a single mark.

Now for the n-point correlator, the factorization equation takes the following form

$$\Big\langle \mathcal{O}^{\dagger}{}_1(x^*{}_1)\mathcal{O}^{\dagger}{}_2(x^*{}_2)\cdots\mathcal{O}^{\dagger}{}_k(x^*{}_k)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)\Big\rangle_{G=1} = \sum_{B_1,B_2}\frac{\Big\langle \mathcal{O}^{\dagger}{}_1(x^*{}_1)\cdots\mathcal{O}^{\dagger}{}_k(x^*{}_k)B_1(y_1)B_2(y_2)\Big\rangle\Big\langle B^{\dagger}{}_2(y_2^*)B^{\dagger}{}_1(y_1^*)\mathcal{O}_1(x_1)\cdots\mathcal{O}_k(x_k)\Big\rangle}{\big\langle B^{\dagger}{}_1(y_1^*)B_1(y_1)\big\rangle\big\langle B^{\dagger}{}_2(y_2^*)B_2(y_2)\big\rangle} \tag{4.16}$$

where $\mathcal{O}$'s are again the local operators defined on the genus-1, $S^1\times S^2$ manifold of interest whereas B_1's and B_2's are the local operators defined on the boundary cut $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively. Then the probability interpretation arises in the same fashion

$$P\Big(\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_k(x_k)\to B_1(y_1)B_2(y_2)\Big) = \sum_{B_1,B_2}\frac{\Big|\mathcal{O}^{\dagger}{}_1(x^*{}_1)\cdots\mathcal{O}^{\dagger}{}_k(x^*{}_k)B_1(y_1)B_2(y_2)\Big|^2}{\big\langle \mathcal{O}^{\dagger}{}_1(x^*{}_1)\cdot\cdot\mathcal{O}^{\dagger}{}_k(x^*{}_k)\mathcal{O}_1(x_1)\cdot\cdot\mathcal{O}_k(x_k)\big\rangle_{G=1}\big\langle B^{\dagger}{}_1(y_1^*)B_1(y)\big\rangle\big\langle B^{\dagger}{}_2(y_2^*)B_2(y_2)\big\rangle} \tag{4.17}$$

It is interesting to note that to formulate the probability interpretation for multiple number of separate measurements we need higher genus factorization.

Therefore depending on the number of measurements or number of out going states or operators we have to consider the appropriate topology of the space on which the operators live to find out the transition probability among the states of the gravity. Keeping this lesson in our mind we compute few probabilities in the forthcoming sections.

4.4 Sphere Factorization

In this section, we want to find out the probability of getting one state from the different number of states of the gravity. Thus we should use the genus zero factorization. To do that first we consider two S^3 manifolds. By cutting out a 3-ball of unit radius around the origin of each one, we map them in two separate $\mathbb{R}^3$ spaces described by the set of coordinates $(r,\Omega_{2,r})$ and $(s,\Omega_{2,s})$ respectively. The metrics in the $\mathbb{R}^3$ spaces take the following forms,

$$ds_r^2 = dr^2 + r^2 d\Omega_{2,r}^2 \qquad \text{and} \qquad ds_s^2 = ds^2 + s^2 d\Omega_{2,s}^2. \tag{4.18}$$

Finally, we glue these two manifolds by using $rs = 1$ to get the genus zero manifold of our interest. In this manifold, the most general formula for the probability can be written as

$$P\Big(R_1(r=e^{x_1})\cdots R_k(r=e^{x_k})\to R(r=0)\Big) = \frac{\big\langle R_1^{\dagger}(s=e^{x_1})\cdots R_k^{\dagger}(s=e^{x_k})R(r=0)\big\rangle\big\langle R^{\dagger}(s=0)R_k(r=e^{x_k})\cdots R_1(r=e^{x_1})\big\rangle}{\big\langle R_k^{\dagger}(s=e^{x_1})\cdots R_1^{\dagger}(s=e^{x_k})R_1(r=e^{x_k})\cdots R_k(r=e^{x_1})\big\rangle\big\langle R^{\dagger}(s=0)R(r=0)\big\rangle} \tag{4.19}$$

Notice that in this formula we have considered final operator at $r, s = 0$ instead of $r = s = 1$. To consider that we first replaced the operators at $r = s = 1$ by their corresponding states via operator/state correspondence. We then evolved the states up to $r, s = 0$ by doing the path integral over the unit disc. Since in these regions there are no operator insertions, in the end result states differ by a scale factor only. The scaling factor of state in r space is exactly canceled by the scale factor arises by evolving state in s space. Finally we again replaced the states as their dual operators at $r, s = 0$ (for details see [216]).

Before going to find out the probability for a specific case, we first compute few useful correlators which will be used in our later computations. From now we suppress the angular dependence part in most of the computations since the angles, in all of the gluings, are identified trivially. Most of the time, we also abbreviate $\chi_R(AB^\dagger)$ and $\chi_R(A^\dagger B)$ as R and $R^\dagger$. First we like to find out the two point function of two Schur polynomials, one in each S^3 with a cut-out of 3-ball those are glued together to construct the manifold of our interest. The two point function of interest is

$$\langle R^\dagger(s=0)R(r=0)\rangle \tag{4.20}$$

with $s = 1/r$ and $R^\dagger$ is a conformal primary operator. To calculate the correlator we should bring the operator $R^\dagger$ in the r coordinate frame. Under this coordinate change, the metric of $\mathbb{R}^3_s$ changes as

$$ds^2 + s^2 d\Omega^2_{2,s} \to \frac{1}{r^4}(dr^2 + r^2 d\Omega^2_{2,r}) \tag{4.21}$$

and so the operator $R^\dagger$ transform as

$$R^\dagger(s) \to \Omega(r)^{-\Delta/2} R^\dagger(r) = r^{2\Delta} R^\dagger(r) \tag{4.22}$$

where $\Omega(r) = 1/r^4$ and Δ is the conformal dimension of the operator $R^\dagger$. Now we are ready to compute the above correlator which is also called Zamalodchikov metric of the conformal field theory. The result is

$$\begin{aligned}
\langle R^\dagger(s=0)R(r=0)\rangle &= \lim_{r_0\to\infty} \langle r_0^{2\Delta} R^\dagger(r=r_0) R(r=0)\rangle \\
&= \lim_{r_0\to\infty} \left[\frac{r_0^{2\Delta} f_R^{N_1} f_R^{N_2}}{(4\pi|0-r_0|)^{2\Delta}}\right] \\
&= \frac{f_R^{N_1} f_R^{N_2}}{(4\pi)^{2\Delta}}.
\end{aligned} \tag{4.23}$$

We would also like to find out this correlator:

$$A^1_M = \langle \mathrm{Tr}(AB^\dagger)^M \mathrm{Tr}(A^\dagger B)^M \rangle. \tag{4.24}$$

We find out this by writing down a recursion relation. Choose one trace from $\mathrm{Tr}(AB^\dagger)^M$ and contract the involved fields A and $B^\dagger$ with the fields $A^\dagger$ and B of the other trace

Tr$(A^\dagger B)^M$. We can do this in two ways: firstly, A and $B^\dagger$ contract with $A^\dagger$ and B fields of the same trace of Tr$(A^\dagger B)^M$ and give N_1N_2. There lies M number of choices since there is M number of traces. In the second way, $B^\dagger$ does not contract with B field of trace where $A^\dagger$ is already contracted with A. So for this way, A field has again M number of choices but for the $B^\dagger$ there are $(M-1)$ choices. For both ways, at the end $(M-1)$ number of traces will be left out. So we can write the above correlator through this recursion relation

$$\begin{aligned} A^1_M &= \Big(MN_1N_2 + M(M-1)\Big)A_{M-1} \\ &= M(N_1N_2+M-1)A_{M-1} \\ &= M!\frac{(N_1N_2+M-1)!}{(N_1N_2-1)!}. \end{aligned} \tag{4.25}$$

Finally, before going to the main computation, we compute the correlator like

$$A^J_M = \langle \text{Tr}((AB^\dagger)^J)^M \text{Tr}((A^\dagger B)^J)^M \rangle. \tag{4.26}$$

Since exact computation is difficult, we restrict the analysis up to leading order where $JM << N_1N_2$ and the result is

$$A^J_M = \langle \text{Tr}((AB^\dagger)^J)^M \text{Tr}((A^\dagger B)^J)^M \rangle = J^M M!(N_1N_2)^{JM}. \tag{4.27}$$

By using the above correlators, we first calculate the probability to get one giant graviton from two giant gravitons of angular momentum $N_1/2$. The formula in equation(4.19) reduces to

$$\begin{aligned} &P\Big(R_1(r=e^x), R_2(r=e^x) \to R(r=0)\Big) \\ &= \frac{\Big|\langle R_1^\dagger(r=e^x)R_2^\dagger(r=e^x)R(r=0)\rangle\Big|^2}{\langle R_2^\dagger(s=e^x)R_1^\dagger(s=e^x)R_1(r=e^x)R_2(r=e^x)\rangle\langle R^\dagger(s=0)R(r=0)\rangle} \\ &= \frac{g\left(R_1,R_2;R\right)^2\left[f_R^{N_1}f_R^{N_2}\right]^2 e^{-4N_1x}(4\pi)^{-4N_1}}{\sum_S g\left(R_1,R_2;S\right)^2 f_S^{N_1}f_S^{N_2}e^{-2N_1x}(4\pi)^{-2N_1}(e^x-e^{-x})^{-2N_1}f_R^{N_1}f_R^{N_2}(4\pi)^{-2N_1}} \end{aligned} \tag{4.28}$$

Here we have considered R_1 and R_2 at the same position, so that the normalization factor in the denominator is an extremal correlator. We also consider the large x limit to maximize the distance between the operators R_1 and R_2 from R. It gives the probability which is space-time independent. Thus, at large x limit, we get

$$P(R_1,R_2\to R) = \frac{f_R^{N_1}f_R^{N_2}}{\sum_S g\left(R_1,R_2;S\right)^2 f_S^{N_1}f_S^{N_2}}. \tag{4.29}$$

The fusion of the two sphere giants (two vertical Young diagrams of length $N_1/2$) gives a sum of representations, with column lengths $(N_1/2+i, N_1/2-i)$. Hence the denominator can be written as

$$\sum_{i,j=0}^{N_1/2} \frac{N_1!(N_1+1)!}{(N_1/2-i)!(N_1/2+i+1)!} \frac{N_2!(N_2+1)!}{(N_2-N_1/2-j)!(N_2-N_1/2+j+1)!}. \tag{4.30}$$

By straightforward simplification we arrives at the final expression for the probability of sphere giant

$$\sim e^{-\left[(2N_2+\frac{3}{2})\log(1+\frac{1}{N_2})+(N_2-N_1+\frac{1}{2})\log(1-\frac{N_1}{N_2})+(N_1-2N_2-\frac{3}{2})\log(1-\frac{N_1}{2N_2})+2N_1\log(2)-2\right]}. \tag{4.31}$$

Similarly for AdS giant, the denominator will be

$$\sum_{i,j=0}^{N_1/2} \frac{(3N_1/2+i-1)!(3N_1/2-i-2)!}{(N_1-1)!(N_1-2)!} \frac{(N_2+N_1/2+j-1)!(N_2+N_1/2-j-2)!}{(N_2-1)!(N_2-2)!}. \tag{4.32}$$

Since, in the sum the $f_R^{N_1} f_R^{N_2}$ is included, the probability becomes less than 1.

Let us consider one sphere giant graviton produced by the combination of N_1 number of KK gravitons of angular momentum 1. From now, we assume that N_1 is less than N_2. Since these KK gravitons are with angular momentum 1, we should write them in terms of trace operators. Thus we write the probability as

$$\begin{aligned} &P\Big(\text{Tr}(AB^\dagger)^{N_1}(r=e^x) \to \chi_{[1^{N_1}]}(AB^\dagger)(r=0)\Big) \\ &= \frac{\Big|\langle \text{Tr}(A^\dagger B)^{N_1}(r=e^x)\chi_{[1^{N_1}]}(AB^\dagger)(r=0)\rangle\Big|^2}{\langle \text{Tr}(A^\dagger B)^{N_1}(s=e^x)\text{Tr}(AB^\dagger)^{N_1}(r=e^x)\rangle\langle \chi_{[1^{N_1}]}(A^\dagger B)\chi_{[1^{N_1}]}(AB^\dagger)\rangle}. \end{aligned} \tag{4.33}$$

Since in the numerator one side of the correlator is in Schur basis, we should change our trace operator in to Schur polynomial and we do that by the formula derived in [229]

$$\text{Tr}((A^\dagger B)^n) = \sum_R \chi_R(\sigma)\chi_R(A^\dagger B) \tag{4.34}$$

where R is the representation of the symmetric group S_n and $\chi_R(\sigma)$ is the character of a cycle of length n [168]. By doing this one can write the trace operator with unit $\mathcal{R}$-charge that corresponds to a single box Young diagram as

$$\text{Tr}(A^\dagger B) = \chi_R(A^\dagger B).$$

Therefore we can write N_1 number of KK graviton with angular momentum 1 as

$$\Big(\text{Tr}(A^\dagger B)\Big)^{N_1} = \Big(\chi_R(A^\dagger B)\Big)^{N_1} = \sum_S g(R_1 \cdots R_{N_1}; S)\chi_S(A^\dagger B). \tag{4.35}$$

Finally, by replacing trace basis as Schur basis in the numerator, we have the probability

$$
\begin{aligned}
&P\Big(\mathrm{Tr}(AB^\dagger)^{N_1}(r=e^x)\to\chi_{[1^{N_1}]}(AB^\dagger)(r=0)\Big)\\
&= \frac{\Big|\langle\sum_S g(R_1\cdots R_{N_1};S)\chi_S(A^\dagger B)(r=e^x)\chi_{[1^{N_1}]}(AB^\dagger)(r=0)\rangle\Big|^2}{\langle\mathrm{Tr}(A^\dagger B)^{N_1}(s=e^x)\mathrm{Tr}(AB^\dagger)^{N_1}(r=e^x)\rangle\langle\chi_{[1^{N_1}]}(A^\dagger B)\chi_{[1^{N_1}]}(AB^\dagger)\rangle}\\
&= \frac{\left[f^{N_1}_{[1^{N_1}]}f^{N_2}_{[1^{N_1}]}\right]^2 e^{-4N_1x}(4\pi)^{-4N_1}}{N_1!\frac{(N_1N_2+N_1-1)!}{(N_1N_2-1)!}e^{-2N_1x}(4\pi)^{-2N_1}(e^x-e^{-x})^{-2N_1}f^{N_1}_{[1^{N_1}]}f^{N_2}_{[1^{N_1}]}(4\pi)^{-2N_1}}.
\end{aligned}
$$

In the second line we use the fact that two point function is finite when representation S is same with the representation of the sphere giant. Thus the contributing Littlewood-Richardson number is only $g(R_1\cdots R_{N_1};[1^{N_1}])$ which has value 1. Again at large x limit probability reduces to

$$
\begin{aligned}
&P\Big(\mathrm{Tr}(AB^\dagger)^{N_1}(r=e^x)\to\chi_{[1^{N_1}]}(AB^\dagger)(r=0)\Big)\\
&= \frac{\left[f^{N_1}_{[1^{N_1}]}f^{N_2}_{[1^{N_1}]}\right]}{N_1!\frac{(N_1N_2+N_1-1)!}{(N_1N_2-1)!}} = \frac{\frac{N_1!N_2!}{(N_2-N_1)!}}{N_1!\frac{(N_1N_2+N_1-1)!}{(N_1N_2-1)!}} = \frac{N_2!}{(N_2-N_1)!}\frac{(N_1N_2-1)!}{(N_1N_2+N_1-1)!}\\
&\sim e^{-[N_1+\frac{N_1}{N_2}-\frac{1}{2N_2}+N_1\log(N_1)+(N_2-N_1+\frac{1}{2})\log(1-\frac{N_1}{N_2})]}. \qquad (4.36)
\end{aligned}
$$

Similarly to produce an AdS giant from N_1 number of KK gravitons with angular momentum 1 we will have the probability

$$
\begin{aligned}
&P\Big(\mathrm{Tr}(AB^\dagger)^{N_1}(r=e^x)\to\chi_{[N_1]}(AB^\dagger)(r=0)\Big)\\
&= \frac{\frac{(2N_1-1)!(N_2+N_1-1)!}{(N_1-1)!(N_2-1)!}}{N_1!\frac{(N_1N_2+N_1-1)!}{(N_1N_2-1)!}} = \frac{(2N_1-1)!(N_2+N_1-1)!(N_1N_2-1)!}{(N_1-1)!(N_2-1)!N_1!(N_1N_2+N_1-1)!}\\
&\sim \pi^{-\frac{1}{2}}e^{-\left[N_1+\frac{N_1}{N_2}-\frac{1}{2N_2}+(N_1+\frac{1}{2})\log(N_1)-(N_1+N_2-\frac{1}{2})\log(1+\frac{N_1}{N_2})-(2N_1-1)\log(2)\right]}. \qquad (4.37)
\end{aligned}
$$

Now consider a combination of L/J number of gravitons with angular momentum $J<\sqrt{N_1}$, resulting into a sphere giant of angular momentum L. So the probability for this transition will be

$$
\begin{aligned}
&P\Big(\mathrm{Tr}((AB^\dagger)^J)^{L/J}(r=e^x)\to\chi_{[1^L]}(AB^\dagger)(r=0)\Big)\\
&= \frac{\Big|\langle\mathrm{Tr}((A^\dagger B)^J)^{L/J}(r=e^x)\chi_{[1^L]}(AB^\dagger)(r=0)\rangle\Big|^2}{\langle\mathrm{Tr}((A^\dagger B)^J)^{L/J}(s=e^x)\mathrm{Tr}((AB^\dagger)^J)^{L/J}(r=e^x)\rangle\langle\chi_{[1^L]}(A^\dagger B)\chi_{[1^L]}(AB^\dagger)\rangle}\\
&= \frac{\sum_{R_1..R_{L/J}}g(R_1,..,R_{L/J};[1^L])^2\left[\chi_{R_1}(J)..\chi_{R_{L/J}}(J)\right]^2\left[f^{N_1}_{[1^L]}f^{N_2}_{[1^L]}\right]^2 e^{-4Lx}(4\pi)^{-4L}}{J^{L/J}(L/J)!(N_1N_2)^L e^{-2Lx}(4\pi)^{-2L}(e^x-e^{-x})^{-2L}f^{N_1}_{[1^L]}f^{N_2}_{[1^L]}(4\pi)^{-2L}}. \qquad (4.38)
\end{aligned}
$$

In this calculation we consider the leading order value of the first correlator of the denominator. Since the final state is sphere giant, which is an antisymmetric representation, the only allowed representations of the gravitons are antisymmetric so that combined representation gives the antisymmetric representation. Therefore, we do not need the sum over Littlewood-Richardson number and it gives 1 only and also $\chi_R(J)$, the characters of a cycle of length J, always be ± 1 and due to square over the product of the characters the total contribution of them will be 1 only. Thus at the large x limit, the probability reduces to

$$\frac{\left[f^{N_1}_{[1^L]} f^{N_2}_{[1^L]}\right]}{J^{L/J}(L/J)!(N_1N_2)^L} = \frac{N_1!N_2!}{J^{L/J}(L/J)!(N_1N_2)^L(N_1-L)!(N_2-L)!}$$
$$\sim \quad \pi^{-\frac{1}{2}} e^{-[2L-\frac{L}{J}-\frac{1}{2}\log(J)+(\frac{L}{J}+\frac{1}{2})\log(L)+(N_1-L+\frac{1}{2})\log(1-\frac{L}{N_1})+(N_2-L+\frac{1}{2})\log(1-\frac{L}{N_2})+\frac{1}{2}\log(2)]} \tag{4.39}$$

Similarly, to create an AdS giant from L/J number of gravitons with angular momentum $J < \sqrt{N_1}$, the probability will be

$$P\Big(\mathrm{Tr}((AB^\dagger)^J)^{L/J}(r = e^x) \to \chi_{[L]}(AB^\dagger)(r = 0)\Big) = \frac{(N_1+L-1)!(N_2+L-1)!}{J^{L/J}(L/J)!(N_1N_2)^L(N_1-1)!(N_2-1)!}$$
$$\sim \quad \pi^{-\frac{1}{2}} e^{-[2L-\frac{L}{J}-\frac{1}{2}\log(J)+(\frac{L}{J}+\frac{1}{2})\log(L)-(N_1+L-\frac{1}{2})\log(1+\frac{L}{N_1})-(N_2+L-\frac{1}{2})\log(1+\frac{L}{N_2})+\frac{1}{2}\log(2)]} \tag{4.40}$$

All these correlators are always less than 1 and decay exponentially with N_1 and N_2.

If L number of gravitons with J amount of angular momentum interact and produce single graviton of angular momentum (LJ) which is less than $\sqrt{N_1}$, the probability can be written as

$$P\Big(\mathrm{Tr}((AB^\dagger)^J)^L(r = e^x) \to \mathrm{Tr}((AB^\dagger)^{LJ})(r = 0)\Big)$$
$$= \frac{\Big|\langle \mathrm{Tr}((A^\dagger B)^J)^L(r{=}e^x)\mathrm{Tr}((AB^\dagger)^{LJ})(r{=}0)\rangle\Big|^2}{\langle \mathrm{Tr}((A^\dagger B)^J)^L(s{=}e^x)\mathrm{Tr}((AB^\dagger)^J)^L(r{=}e^x)\rangle\langle \mathrm{Tr}((A^\dagger B)^{LJ})\mathrm{Tr}((AB^\dagger)^{LJ})\rangle}$$
$$= \frac{\sum_{R_1\cdots R_L;S} g(R_1,\ldots,R_L;S)^2\Big[\chi_{R_1}(J)\cdots\chi_{R_L}(J)\chi_S(LJ)\Big]^2\Big[f^{N_1}_S f^{N_2}_S\Big]^2 e^{-4LJx}(4\pi)^{-4LJ}}{J^L L!(N_1N_2)^{LJ}e^{-2LJx}(4\pi)^{-2LJ}(e^x-e^{-x})^{-2LJ}(LJ)!\frac{(N_1N_2+LJ-1)!}{(N_1N_2-1)!}(4\pi)^{-2LJ}}. \tag{4.41}$$

Since we know $\chi_R(I)$ will only be non-zero for hooks $\chi_{[(R-r),1^r]}(I) = (-1)^r$. Therefore the contribution of the each character is ± 1. Thus the total contribution of the characters is only 1. Then the above probability reduces at large x limit as

$$P\Big(\mathrm{Tr}((AB^\dagger)^J)^L(r = e^x) \to \mathrm{Tr}((AB^\dagger)^{LJ})(r = 0)\Big)$$
$$= \frac{\sum_{r_1\cdots r_L,s} g\Big([(R_1-r_1),1^{r_1}],\ldots,[(R_L-r_L),1^{r_L}];[(S-s),1^s]\Big)^2\Big[f^{N_1}_{[(S-s),1^s]}f^{N_2}_{[(S-s),1^s]}\Big]^2}{J^L L!(N_1N_2)^{LJ}(LJ)!\frac{(N_1N_2+LJ-1)!}{(N_1N_2-1)!}}. \tag{4.42}$$

Using this method one can find out other type of correlators those will produce only one final state. Like large number of gravitons with different $\mathcal{R}$-charge producing one sphere giant, AdS giant or graviton. However we are not going to find out those in this chapter. We close this section with these above correlators. In the next section we compute the correlators for two out going states.

4.5 The genus one factorization

To obtain the probability of finding two states from many states we need to compute correlators on genus one manifold. Particularly, we consider here the manifold $S^1 \times S^2$. We construct this manifold by gluing two cylinders $I \times S^2$ described by coordinates $(r, \Omega_{2,r})$ and $(s, \Omega_{2,s})$ in the two different $\mathbb{R}^3$ spaces. The radial variables are bounded by these ranges

$$1 \leq r \leq e^T \qquad \text{and} \qquad 1 \leq s \leq e^T. \tag{4.43}$$

We also introduce the coordinates $r' = 1/r$ and $s' = 1/s$. Now to produce the manifold $S^1 \times S^2$, we glue the two cylinders at the inner ends $r = 1$, $s = 1$ with $rs = 1$ and outer ends at $r = e^T$, $s = e^T$ with $r's' = e^{-2T}$(i.e $rs = e^{2T}$).

As earlier, we can now define the probability of one giant graviton going to two smaller giant gravitons as follows

$$\begin{aligned} & P\Big(R(r = e^x) \to R_1'(r' = 0)R_2(r = 0)\Big) \\ = & \frac{\left|\left\langle R^\dagger(r = e^x)R_1'(r' = 0)R_2(r = 0)\right\rangle\right|^2}{\left\langle R^\dagger(s = e^x)R(r = e^x)\right\rangle_{G=1}\left\langle R_1^{\dagger'}(s' = 0)R_1'(r' = 0)\right\rangle\left\langle R_2^\dagger(s = 0)R_2(r = 0)\right\rangle} \end{aligned} \tag{4.44}$$

Following the same logic as described in the previous section we have again considered the operators at $r, r' = 0$. To compute this probability we study term by term. Lets first work out the three point function of the numerator

$$\begin{aligned} & \langle R^\dagger(r = e^x)R_1'(r' = 0)R_2(r = 0)\rangle \\ = & \lim_{r_0 \to \infty} \langle R^\dagger(r = e^x) r_0^{2\Delta_1} R_1(r = r_0)R_2(r = 0)\rangle \\ = & (4\pi)^{-2(\Delta_1 + \Delta_2)} e^{-2x\Delta_2} g(R_1, R_2; R) f_R^{N_1} f_R^{N_2}. \end{aligned} \tag{4.45}$$

Then we compute second two point function of the denominator

$$\langle R_1^{\dagger'}(s' = 0)R_1'(r' = 0)\rangle = (4\pi)^{-2\Delta_1} e^{2T\Delta_1} f_{R_1}^{N_1} f_{R_1}^{N_2}. \tag{4.46}$$

Similarly for the third two point function of the denominator we get

$$\langle R_2^\dagger(s = 0)R_2(r = 0)\rangle = f_{R_2}^{N_1} f_{R_2}^{N_2} (4\pi)^{-2\Delta_2}. \tag{4.47}$$

In addition to these three correlators we need to know one more correlator which is on $S^2 \times S^1$, seating in the denominator . For the space-time dependent part of the correlator, we have to know the Green's function for this manifold. To find out Green's function we start with the associated metric of this space

$$ds^2 = d\tau^2 + d\theta^2 + \sin^2\theta d\phi^2 \tag{4.48}$$

with the proper range of coordinates, $\tau \in [0, 2T]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$.

Sturm-Liouville theory suggests that, if the eigenvectors $\Psi_n(x)$ of a hermitian operator $\mathcal{L}$ span a basis, the Green's function of interest, $G(x, y)$ is expressible as a linear combination of the $\Psi_n(x)$

$$G(x, y) = \sum_{n|\lambda_n \neq 0} \frac{\Psi_n^*(x)\Psi_n(x)}{\lambda_n}. \tag{4.49}$$

As we are analysing three dimensional conformal filed theory of scalar field on 3-sphere , while defining the differential operator $\mathcal{L}$ it is necessary to consider coupling to the 3-dimensional curvature. In general $\mathcal{L}$ takes the following form

$$\mathcal{L} = \Delta - \frac{R}{8} \tag{4.50}$$

where Δ is Laplacian and not conformal dimension. R is the Ricci scalar. More specifically, considering the space $S^1 \times S^2$ with unit radii and also noting that only the curvature of S^2 contributes, the Ricci scalar comes out as $R = 2$. The differential operators $\mathcal{L}$ modifies into a particular form

$$\mathcal{L} = \Delta_{Euclidean} - \frac{1}{4}. \tag{4.51}$$

The form of differential operator $\mathcal{L}$ leads to the identification of its eigenvectors as a complete set of spherical harmonics on $S^2 \times S^1$

$$\Psi_n = \varsigma_m(\tau) Y_J^M(\theta, \phi) \tag{4.52}$$

where $n = (m, J, M)$. The explicit form of S^2 harmonics is given by

$$Y_J^M(\theta, \phi) = \sqrt{\frac{(2J+1)}{4\pi}\frac{(J-M)!}{(J+M)!}}(-1)^M \sin^M\theta \Big(\frac{d^{J+M}}{d^{J+M}\cos\theta}\Big)\frac{(\cos^2\theta - 1)^J}{2^J J!} \tag{4.53}$$

where the quantum numbers J and M take values as

$$\begin{aligned} J &= 0, 1, 2, 3 \cdots \\ M &= -J, -J+1, \cdots J. \end{aligned} \tag{4.54}$$

The harmonics on S^1 are

$$\varsigma_m(\tau) = \frac{1}{\sqrt{2T}} e^{\frac{im\pi\tau}{T}}, \tag{4.55}$$

with m takes the values as, $m = 0, 1, 2 \cdots$.
The spherical harmonics of S^2 and S^1 satisfy the following differential equations respectively

$$\begin{aligned}\Delta_{S^2} Y_J^M(\theta,\phi) &= -J(J+M)Y_J^M(\theta,\phi),\\ \Delta_{S^1} &= -\Big(\frac{m\pi}{T}\Big)^2 \varsigma_m(\tau).\end{aligned} \tag{4.56}$$

Now with all the eigenfunctions of Euclidean Laplacian operators defined on $S^2 \times S^1$ in hand, we are able to write the differential equation for operator $\mathcal{L}$

$$\mathcal{L}\Psi_n = \big(\Delta_{S^2\times S^1} - \frac{1}{4}\big)\Psi_n = \Big[- J(J+M) - \Big(\frac{m\pi}{T}\Big)^2 - \frac{1}{4}\Big]\Psi_n. \tag{4.57}$$

In the 3-dimensional space-time of our interest we define the action of operator $\mathcal{L}$ on corresponding Green's function defined in conformity with $\mathbb{R}^3$ correlator.

$$\mathcal{L}G(x,y) = -\delta^3(x-y) \tag{4.58}$$

and the Green's function in terms of the spherical harmonics is

$$G(x,y) = \sum_{J,M,m} \frac{\varsigma_m{}^*(\tau) Y_J^{M*}(\theta,\phi)\varsigma_m(\tau)Y_J^M(\theta,\phi)}{J(J+M) + (\frac{m\pi}{T})^2 + \frac{1}{4}}. \tag{4.59}$$

Now we are ready to compute the two point correlator on $S^1 \times S^2$

$$\langle R^\dagger(s=e^x)R(r=e^x)\rangle_{G=1}. \tag{4.60}$$

Since the associated metric of $S^1 \times S^2$ involves the coordinate τ, θ, ϕ and we suppress angular coordinates in our calculation, we should bring the correlator in τ coordinate instead of r and s. We do that by changing the coordinates $s = e^{-\tau},\ r = e^{\tau}$ and finally we get

$$\langle R^\dagger(s=e^x)R(r=e^x)\rangle_{G=1} = e^{-2x\Delta}\langle R^\dagger(\tau=-x)R(\tau=x)\rangle_{G=1} \tag{4.61}$$

with $\Delta = \Delta_1 + \Delta_2$ by charge conservation. Finally in terms of gauge index and Green's function we have the correlator as

$$\langle R^\dagger(s=e^x)R(r=e^x)\rangle_{G=1} = e^{-2x\Delta} f_R^{N_1} f_R^{N_2}\Big[\sum_{J,M,m}\frac{\varsigma_m{}^*(0)Y_J^{M*}(\theta,\phi)\varsigma_m(2x)Y_J^M(\theta,\phi)}{J(J+M)+(\frac{m\pi}{T})^2+\frac{1}{4}}\Big]^{2\Delta}. \tag{4.62}$$

As we suppress angular part, the final result should be independent of the choice of θ and ϕ. We thus choose $\theta = 0$, so that the sum simplifies significantly. Equation(4.53) demands for $\theta = 0$ the only non-zero term contributing corresponds to $J = 0, M = 0$ and the above correlator reduces to

$$\langle R^\dagger(s=e^x)R(r=e^x)\rangle_{G=1} = e^{-2x\Delta} f_R^{N_1} f_R^{N_2}\Big[\frac{1}{2T}\sum_m e^{\frac{im2\pi x}{T}}\frac{Y_0^{0*}(\theta,\phi)Y_0^0(\theta,\phi)}{(\frac{m\pi}{T})^2+\frac{1}{4}}\Big]^{2\Delta}. \tag{4.63}$$

With the value of harmonic function $Y_0^0 = 2^{-1}\pi^{-\frac{1}{2}}$ we get the final expression

$$
\begin{aligned}
\langle R^\dagger(s=e^x)R(r=e^x)\rangle_{G=1} &= \Big[\frac{1}{8\pi T}[2\sum_{m>0}\frac{\cos\frac{m2\pi x}{T}}{(\frac{m\pi}{T})^2+\frac{1}{4}}+4]\Big]^{2\Delta}e^{-2x\Delta}f_R^{N_1}f_R^{N_2} \\
&= \Big[\frac{1}{8\pi T}[8\sum_{m>0}\frac{(-1)^m}{(\frac{2m\pi}{T})^2+1}+4]\Big]^{2\Delta}e^{-2x\Delta}f_R^{N_1}f_R^{N_2} \\
&= \Big[\frac{1}{8\pi T}[-4+T(\coth\frac{T}{4}-\tanh\frac{T}{4})+4]\Big]^{2\Delta}e^{-2x\Delta}f_R^{N_1}f_R^{N_2} \\
&= \Big[\frac{1}{4\pi}\mathrm{cosech}\frac{T}{2}\Big]^{2\Delta}e^{-2x\Delta}f_R^{N_1}f_R^{N_2} \\
&= \Big[\frac{1}{4\pi}(2e^{-T/2})\Big]^{2\Delta}e^{-2x\Delta}f_R^{N_1}f_R^{N_2}. \qquad (4.64)
\end{aligned}
$$

To get the last line we have used the large T limit where $\mathrm{cosech}\frac{T}{2}\to 2e^{-T/2}$.

Thus combining all four separate correlators we get the probability of one giant graviton goes to two giant gravitons as

$$
\begin{aligned}
& P\Big(R(r=e^{T/2})\to R_1(r=e^T)R_2(r=0)\Big) \\
= \;& \frac{g(R_1,R_2;R)^2 f_R^{N_1}f_R^{N_2}}{2^{2(\Delta_1+\Delta_2)}f_{R_1}^{N_1}f_{R_1}^{N_2}f_{R_2}^{N_1}f_{R_2}^{N_2}} \qquad (4.65)
\end{aligned}
$$

in the $T\to\infty$ limit and R at $r=e^{T/2}$, which maximize the distance of the operators R_1 and R_2 from R and suppress the space-time dependence in the probability as earlier.

In particular, the probability of the transition of an AdS giant with angular momentum N_1 into to two smaller AdS giants with angular momentum $N_1/2$ is given by

$$
\begin{aligned}
& \frac{1}{2^{2N_1}}\frac{f_{[N_1]}^{N_1}f_{[N_1]}^{N_2}}{\Big[f_{[N_1/2]}^{N_1}f_{[N_1/2]}^{N_2}\Big]^2} = \frac{1}{2^{2N_1}}\frac{(2N_1-1)!(N_1-1)!(N_2+N_1-1)!(N_2-1)!}{[(3N_1/2-1)!(N_2+N_1/2-1)!]^2} \\
\sim\;& e^{-\Big[(2N_2+N_1-1)\log(1+\frac{N_1}{2N_2})-(N_2+N_1-\frac{1}{2})\log(1+\frac{N_1}{N_2})+(3N_1-1)\log(3)-(3N_1-\frac{3}{2})\log(2)\Big]}. \qquad (4.66)
\end{aligned}
$$

For a sphere giant $[1^N]$ evolving into two smaller sphere giants $[1^{\frac{N}{2}}]$ this probability becomes

$$
\begin{aligned}
& \frac{1}{2^{2N_1}}\frac{f_{[1^{N_1}]}^{N_1}f_{[1^{N_1}]}^{N_2}}{\Big[f_{[1^{N_1/2}]}^{N_1}f_{[1^{N_1/2}]}^{N_2}\Big]^2} = \frac{1}{2^{2N_1}}\frac{\Big[(N_1/2)!(N_2-N_1/2)!\Big]^2}{N_1!N_2!(N_2-N_1)!} \\
\sim\;& \pi^{\frac{1}{2}}e^{-\Big[(N_2-N_1+\frac{1}{2})\log(1-\frac{N_1}{N_2})-(2N_2-N_1+1)\log(1-\frac{N_1}{2N_2})-\frac{1}{2}\log(N_1)+(3N_1+\frac{1}{2})\log(2)\Big]}. \qquad (4.67)
\end{aligned}
$$

We can also compute the transition probability of one sphere giant with angular momentum N_1 going into two gravitons with angular momentum $N_1/2$. Where $N_1/2$ is less than $\sqrt{N_1}$. In this case we can consider the trace basis for gravitons as genus zero case and then transform as Schur basis by using the equation(4.34). Then the probability reduces to

$$
\begin{aligned}
& P\Big(\chi_{[1^{N_1}]}(AB^\dagger)(r=e^{T/2}) \rightarrow \mathrm{Tr}((AB^\dagger)^{N_1/2})(r=e^T)\mathrm{Tr}((AB^\dagger)^{N_1/2})(r=0)\Big) \\
= \;& \frac{\sum_{R_1,R_2} g(R_1,R_2;[1^{N_1}])^2 \big[\chi_{R_1}(N_1/2)\,\chi_{R_2}(N_1/2)\big]^2 \Big[f^{N_1}_{[1^{N_1}]} f^{N_2}_{[1^{N_1}]}\Big]}{2^{2N_1}(N_1/2)^2(N_1N_2)^{N_1}} \\
= \;& \frac{N_1!N_2!}{2^{2N_1}(N_1/2)^2(N_1N_2)^{N_1}(N_2-N_1)!} \\
\sim \;& \pi^{\frac{1}{2}} e^{-\Big[2N_1+\frac{3}{2}\log(N_1)+(N_2-N_1+\frac{1}{2})\log(1-\frac{N_1}{N_2})+(2N_1-\frac{5}{2})\log(2)\Big]}.
\end{aligned} \tag{4.68}
$$

To get the 3rd line from 2nd line we have again used the fact that we can only have the antisymmetric representations of the gravitons from a sphere giant and in this condition sum over Littlewood-Richardson number takes the value 1. Same computation can be done for the AdS giant also. In that case probability becomes

$$
\begin{aligned}
& P\Big(\chi_{[N_1]}(AB^\dagger)(r=e^{T/2}) \rightarrow \mathrm{Tr}((AB^\dagger)^{N_1/2})(r=e^T)\mathrm{Tr}((AB^\dagger)^{N_1/2})(r=0)\Big) \\
= \;& \frac{\Big[f^{N_1}_{[N_1]} f^{N_2}_{[N_1]}\Big]}{2^{2N_1}(N_1/2)^2(N_1N_2)^{N_1}} \\
= \;& \frac{(2N_1-1)!(N_2+N_1-1)!}{2^{2N_1}(N_1/2)^2(N_1N_2)^{N_1}(N_2-1)!(N_1-1)!} \\
\sim \;& e^{-\Big[2N_1+2\log(N_1)-(N_2+N_1-\frac{1}{2})\log(1+\frac{N_1}{N_2})-\frac{3}{2}\log(2)\Big]}.
\end{aligned} \tag{4.69}
$$

Again all correlatots are less than 1 and decaying exponentially with N_1 and N_2.

If L number of gravitons with J amount of angular momentum interact and produce two gravitons of angular momentum L_1 and L_2 which are less than $\sqrt{N_1}$, the probability takes the following form

$$
\begin{aligned}
& P\Big(\mathrm{Tr}((AB^\dagger)^J)^L(r=e^x) \rightarrow \mathrm{Tr}((AB^\dagger)^{L_1})(r=e^T)\mathrm{Tr}((AB^\dagger)^{L_2})(r=0)\Big) \\
= \;& \frac{\Big|\langle \mathrm{Tr}((A^\dagger B)^J)^L(r=e^x)\mathrm{Tr}'((AB^\dagger)^{L_1})(r'=0)\mathrm{Tr}((AB^\dagger)^{L_2})(r=0)\rangle\Big|^2}{\langle \mathrm{Tr}((A^\dagger B)^J)^L(s=e^x)\mathrm{Tr}((AB^\dagger)^J)^L(r=e^x)\rangle_{G=1}} \\
\times \;& \frac{1}{\langle \mathrm{Tr}'((A^\dagger B)^{L_1})(s'=0)\mathrm{Tr}'((AB^\dagger)^{L_1})(r'=0)\rangle\langle \mathrm{Tr}((A^\dagger B)^{L_2})(s=0)\mathrm{Tr}((AB^\dagger)^{L_2})(r=0)\rangle}.
\end{aligned}
$$

Following the same procedure of previous section, the above probability can be written at large x limit as

$$P\Big(\mathrm{Tr}((AB^\dagger)^J)^L(r=e^x)\to\mathrm{Tr}((AB^\dagger)^{L_1})(r=e^T)\mathrm{Tr}((AB^\dagger)^{L_2})(r=0)\Big)$$
$$=\frac{\sum_{r_1\cdots r_L,s,s_1,s_2}g\Big([(R_1-r_1),1^{r_1}],\ldots,[(R_L-r_L),1^{r_L}];[(S-s),1^s]\Big)^2\Big[f^{N_1}_{[(S-s),1^s]}f^{N_2}_{[(S-s),1^s]}\Big]^2}{2^{2LJ}J^LL!(N_1N_2)^{2LJ}L_1L_2}$$
$$\times\quad g\Big([(S_1-s_1),1^{s_1}],[(S_2-s_2),1^{s_2}];[(S-s),1^s)]\Big)^2.$$

It is needless to say, one can compute other different types of correlators those have two final states like large number of gravitons creating two giant gravitons. However we close this section having these five above specific examples and in the next section we consider correlators involving more than two out going states.

4.6 Higher genus factorization

To get the large number of final states we should consider the higher genus $G=n-1$ factorization. Following [216], we also guess the probability for this condition as

$$P(R\to R_1,R_2,\ldots,R_n)=\frac{1}{k_n^{2(\Delta_1+\Delta_2+\cdots+\Delta_n)}}\frac{g(R_1,R_2,\ldots,R_n;R)^2f_R^{N_1}f_R^{N_2}}{f^{N_1}_{R_1}f^{N_2}_{R_1}f^{N_1}_{R_2}f^{N_2}_{R_2}\cdots f^{N_1}_{R_n}f^{N_2}_{R_n}}. \tag{4.70}$$

Here k_n is a constant and it takes value 1 for genus zero and 2 for genus one. We are again computing the probability at long-distance limit, that is the operators are in a symmetric configuration far apart from each other.

We find the probability for AdS giant with angular momentum N_1 going to n number of smaller giants is,

$$P\Big([N_1]\to n\times[N_1/n]\Big)$$
$$=\frac{1}{{k_n}^{2N_1}}\frac{f^{N_1}_{[N_1]}f^{N_2}_{[N_1]}}{[f^{N_1}_{[\frac{N_1}{n}]}]^n[f^{N_2}_{[\frac{N_1}{n}]}]^n}$$
$$\sim\frac{1}{\sqrt{2}}\Big(\frac{n+1}{n}\Big)^{\frac{n}{2}}\Big[\frac{4n^{n+1}}{k_n^2(n+1)^{n+1}}\Big]^{N_1}\Big(1+\frac{N_1}{N_2}\Big)^{N_1+N_2-\frac{1}{2}}\Big(1+\frac{N_1}{nN_2}\Big)^{\frac{n}{2}-nN_2-N_1}. \tag{4.71}$$

As a special case, we explicitly calculate the higher genus amplitude for 3 outgoing smaller AdS giants,

$$P\Big([N_1]\to 3\times[N_1/3]\Big)$$
$$=\frac{e^{-\Big[(3N_2+N_1-\frac{3}{2})\log(1+\frac{N_1}{3N_2})-(N_2+N_1-\frac{1}{2})\log(1+\frac{N_1}{N_2})+(6N_1-\frac{5}{2})\log(2)-(4N_1-\frac{3}{2})\log(3)\Big]}}{k_3^{2N_1}} \tag{4.72}$$

In the same way the transition of a sphere giant to smaller giants is given by,

$$
\begin{aligned}
& P\Big([1^{N_1}] \to n \times [1^{N_1/n}]\Big) \\
&= \frac{1}{{k_n}^{2N_1}} \frac{f^{N_1}_{[1^{N_1}]} f^{N_2}_{[1^{N_1}]}}{[f^{N_1}_{[1^{\frac{N_1}{n}}]}]^n [f^{N_2}_{[1^{\frac{N_1}{n}}]}]^n} \\
&\sim \frac{\sqrt{2\pi N_1}}{k_n^{2N_1}} \Big(\frac{n-1}{n}\Big)^{\frac{n}{2}+nN_1-N_1} \Big(1-\frac{N_1}{N_2}\Big)^{N_1-N_2-\frac{1}{2}} \Big(1-\frac{N_1}{nN_2}\Big)^{\frac{n}{2}+nN_2-N_1} .
\end{aligned} \tag{4.73}
$$

Again in the case of sphere giant we calculate the transition amplitude for 3 outgoing smaller giants.

$$
\begin{aligned}
& P\Big([1^{N_1}] \to 3 \times [1^{N_1/3}]\Big) \\
&= \tfrac{4}{k_3^{2N_1}} \sqrt{\tfrac{\pi}{27}} e^{-\Big[(N_2-N_1+\frac{1}{2})\log(1-\frac{N_1}{N_2})-(3N_2-N_1+\frac{3}{2})\log(1-\frac{N_1}{3N_2})-\frac{1}{2}\log(N_1)+2N_1(\log 3-\log 2)\Big]}
\end{aligned} \tag{4.74}
$$

The transition of an AdS giant carrying $\mathcal{R}$-charge Δ_R into n number of KK gravitons is given by,

$$
\begin{aligned}
& P\Big([\Delta_R] \to \mathrm{Tr}((AB^\dagger)^{\Delta_1}) \cdots \mathrm{Tr}((AB^\dagger)^{\Delta_n})\Big) \\
&= \frac{1}{k_n^{2\Delta_R}} \frac{f^{N_1}_{[\Delta_R]} f^{N_2}_{[\Delta_R]}}{\langle \mathrm{Tr}((A^\dagger B)^{\Delta_1}) \mathrm{Tr}((AB^\dagger)^{\Delta_1})\rangle \cdots \langle \mathrm{Tr}((A^\dagger B)^{\Delta_n}) \mathrm{Tr}((AB^\dagger)^{\Delta_n})\rangle} \\
&\sim \frac{e^{-\Big[2\Delta_R-(N_1+\Delta_R-\frac{1}{2})\log(1+\frac{\Delta_R}{N_1})-(N_2+\Delta_R-\frac{1}{2})\log(1+\frac{\Delta_R}{N_2})\Big]}}{k_n^{2\Delta_R}\Delta_1\cdots\Delta_n} .
\end{aligned} \tag{4.75}
$$

The transition of an sphere giant carrying $\mathcal{R}$-charge Δ_R into n number of KK gravitons is given by,

$$
\begin{aligned}
& P\Big([1^{\Delta_R}] \to \mathrm{Tr}((AB^\dagger)^{\Delta_1}) \cdots \mathrm{Tr}((AB^\dagger)^{\Delta_n})\Big) \\
&= \frac{1}{k_n^{2\Delta_R}} \frac{f^{N_1}_{[1^{\Delta_R}]} f^{N_2}_{[1^{\Delta_R}]}}{\langle \mathrm{Tr}((A^\dagger B)^{\Delta_1}) \mathrm{Tr}((AB^\dagger)^{\Delta_1})\rangle \cdots \langle \mathrm{Tr}((A^\dagger B)^{\Delta_n}) \mathrm{Tr}((AB^\dagger)^{\Delta_n})\rangle} \\
&\sim \frac{e^{-\Big[2\Delta_R+(N_1-\Delta_R+\frac{1}{2})\log(1-\frac{\Delta_R}{N_1})+(N_2-\Delta_R+\frac{1}{2})\log(1-\frac{\Delta_R}{N_2})\Big]}}{k_n^{2\Delta_R}\Delta_1\cdots\Delta_n} .
\end{aligned} \tag{4.76}
$$

4.7 Transition probability in ABJM theory

In this section we carry on our computation on transition probabilities among giant gravitons or from giant gravitons to ordinary gravitons for ABJM theory where $N_1 = N_2 = N$.

We will only enumerate the main results.

The transition amplitude between two sphere giants of angular momentum $\frac{N}{2}$ to a single sphere giant of angular momentum N is given as

$$P\Big(\Big[1^{\frac{N}{2}}\Big],\Big[1^{\frac{N}{2}}\Big]\to\Big[1^{N}\Big]\Big)=2^{-2N}. \tag{4.77}$$

We extend the result for same type of transition occurring between AdS giants.

$$P\Big(\Big[\frac{N}{2}\Big],\Big[\frac{N}{2}\Big]\to\Big[N\Big]\Big)=\frac{\Big[(2N-1)!(N-2)!\Big]^2}{\Big(\sum_{i=0}^{N_1/2}(\frac{3N}{2}+i-1)!(\frac{3N}{2}-i-2)!\Big)^2}. \tag{4.78}$$

We calculate transition probability of the process depicting a sphere giant graviton is produced by the combination of N number of KK gravitons of angular momentum 1.

$$P\Big(\text{Tr}(AB^\dagger)^N(r=e^x)\to\chi_{[1^N]}(AB^\dagger)(r=0)\Big)$$
$$\sim\pi^{\frac{1}{2}}e^{-\Big[(N-\frac{1}{2})\log(N)+(N^2+N-\frac{1}{2})\log(1+\frac{1}{N})-\frac{1}{2}\log(2)\Big]}. \tag{4.79}$$

The transition probability to go from N number of KK gravitons with angular momentum 1 to an AdS giant is

$$P\Big(\text{Tr}(AB^\dagger)^N(r=e^x)\to\chi_{[N]}(AB^\dagger)(r=0)\Big)$$
$$\sim\pi^{-\frac{1}{2}}e^{-\Big[N-\frac{1}{2N}+1+(N+\frac{1}{2})\log(N)-(4N-\frac{3}{2})\log(2)\Big]}. \tag{4.80}$$

The transition probability of the process where L/J number of gravitons with angular momentum $J<\sqrt{N}$ combining and giving a sphere giant of angular momentum L is given by

$$P\Big(\text{Tr}((AB^\dagger)^J)^{L/J}(r=e^x)\to\chi_{[1^L]}(AB^\dagger)(r=0)\Big)$$
$$\sim\pi^{-\frac{1}{2}}e^{-\Big[2L-\frac{L}{J}-\frac{1}{2}\log(J)+(\frac{L}{J}+\frac{1}{2})\log(L)+2(N-L+\frac{1}{2})\log(1-\frac{L}{N})+\frac{1}{2}\log(2)\Big]}. \tag{4.81}$$

Similarly for AdS giant the result changes as,

$$P\Big(\text{Tr}((AB^\dagger)^J)^{L/J}(r=e^x)\to\chi_{[L]}(AB^\dagger)(r=0)\Big)$$
$$\sim\pi^{-\frac{1}{2}}e^{-\Big[2L-\frac{L}{J}-\frac{1}{2}\log(J)+(\frac{L}{J}+\frac{1}{2})\log(L)-2(N+L-\frac{1}{2})\log(1+\frac{L}{N})+\frac{1}{2}\log(2)\Big]}. \tag{4.82}$$

Now we calculate the transition probability from an AdS giant with angular momentum N to two smaller AdS giants with angular momentum $N/2$. The probability is

$$\sim e^{-\Big[2(3N-1)\log(3)-(8N-3)\log(2)\Big]}. \tag{4.83}$$

For a sphere giant $[1^N]$ evolving into two smaller sphere giants $[1^{\frac{N}{2}}]$ probability becomes

$$\sim \pi e^{-\Big[(4N+1)\log(2)-\log(N)\Big]}. \tag{4.84}$$

The transition probability of evolving from one sphere giant with angular momentum N to two gravitons with angular momentum $N/2$ with the restriction $N/2$ is less than $\sqrt{N}$ is

$$\sim \pi e^{-\Big[2N+\log(N)+(2N-3)\log(2)\Big]}. \tag{4.85}$$

We also calculate the transition probability of evolving from one AdS giant with angular momentum N to two gravitons with angular momentum $N/2$ where we assume again the fact that $N/2$ is less than $\sqrt{N}$ and the probability is

$$\sim e^{-2\Big[N+\log(N)-(N+1/2)\log(2)\Big]}. \tag{4.86}$$

Finally we calculate the higher genus transition probability for AdS giants. With the choice of $N_1 = N_2$, (4.71) gives the appropriate higher genus correlator for the ABJM theory

$$2^{2N-1}\Big(\frac{n+1}{n}\Big)^{n-nN-N}\Big[\frac{4n^{n+1}}{k_n^2(n+1)^{n+1}}\Big]^N. \tag{4.87}$$

Again for the case of sphere giants equation (4.73) modifies as,

$$\frac{1}{k_n^{2N}}(2\pi N)\Big(\frac{n-1}{n}\Big)^{n(1+2N)-2N}. \tag{4.88}$$

Having these discussion on transition probability in the last four sections we are going to find out the large N expansion of the theory in the non-trivial background.

4.8 Large N expansion in non-trivial background

We know that the large N expansion of $\mathcal{N}=4$ SYM theory as well as of ABJM theory is replaced by $1/(N+M)$ in the presence of non-trivial background created by Young diagram of N number of rows and M number of columns of the order of N. Therefore for ABJ theory it is natural to expect the same. To verify that we compute the amplitude of the multi trace operator in the non-trivial background as ABJM theory. Following [**?**, 166, 230, 231], we first calculate the amplitude without the presence of background and the result is

$$\begin{aligned}\mathcal{A}\Big(\{n_i;m_j\},N_1,N_2\Big) &\equiv \Big\langle\prod_{ij}\mathrm{Tr}\Big((AB^\dagger)^{n_i}\Big)\mathrm{Tr}\Big((A^\dagger B)^{m_j}\Big)\Big\rangle\\ &= \sum_{R,S}\alpha_R\beta_S\Big\langle\chi_R(AB^\dagger)\chi_S(A^\dagger B)\Big\rangle\\ &= \sum_R \alpha_R\beta_R f_R^{N_1}f_R^{N_2}. \end{aligned}\tag{4.89}$$

Here we have rewritten the multi trace operator in terms of Schur polynomials as

$$\prod_i \mathrm{Tr}\Big((AB^\dagger)^{n_i}\Big) = \sum_R \alpha_R \chi_R(AB^\dagger), \qquad \prod_j \mathrm{Tr}\Big((A^\dagger B)^{m_j}\Big) = \sum_R \beta_R \chi_R(A^\dagger B). \quad (4.90)$$

Coefficients α_R and β_R are independent of N_1 and N_2. The amplitude can also be calculated in the presence of non-trivial background and the result is

$$\begin{aligned} \mathcal{A}_B\Big(\{n_i;m_j\},N_1,N_2\Big) &\equiv \left\langle \prod_{ij} \mathrm{Tr}\Big((AB^\dagger)^{n_i}\Big)\mathrm{Tr}\Big((A^\dagger B)^{m_j}\Big)\right\rangle_B \\ &= \sum_R \alpha_R\beta_R \left(\frac{f_{+R}^{N_1}}{f_B^{N_1}}\right)\left(\frac{f_{+R}^{N_2}}{f_B^{N_2}}\right). \end{aligned} \quad (4.91)$$

Here f_B is the product of weights of the background Young diagram B and f_{+R} is the product of the weights of the Young diagram $+R$ produced by the product of background Young diagram and Young diagram representing multi trace operator. All the weights of the diagram B are repeated in the diagram $+R$. Therefore all weights of f_B will be canceled by the weights of the f_{+R} and the remaining weights of the f_{+R} contribute to find out the amplitude of the multi trace operator in presence of non-trivial background. However it seems that the remaining weights turn out the weights of the diagram R which represent the multi trace operator, corresponding to the gauge group $U(N_1+M)$ or $U(N_2+M)$. Thus the amplitude with background can easily be calculated from the amplitude without background just by replacing the gauge group $U(N_1)$ and $U(N_2)$ as $U(N_1+M)$ and $U(N_2+M)$ respectively. Therefore we can write

$$\mathcal{A}_B\Big(\{n_i;m_j\},N_1,N_2\Big) = \mathcal{A}\Big(\{n_i;m_j\},N_1+M,N_2+M\Big). \quad (4.92)$$

Since $\mathcal{A}\Big(\{n_i;m_j\},N_1,N_2\Big)$ admits expansions $1/N_1$ and $1/N_2$, so we can expect that $\mathcal{A}\Big(\{n_i;m_j\},N_1+M,N_2+M\Big)$ should have expansions $1/(N_1+M)$ and $1/(N_2+M)$.

4.9 Conclusion:

Here we have shown that in $U(N_1)\times U(N_2)$ ABJ theory ($N_1 < N_2$), the correct $\frac{1}{2}$ BPS gauge invariant operators carrying a $\mathcal{R}$ charge greater than $\sqrt{N_1}$, are not the single trace operators. Rather, one needs to consider the Schur operators. We then identified the corresponding states in the dual gravity theory. Subsequently, we computed the two, three and multi-point correlation functions involving these operators. Our computations show, for large N_1 and N_2, all of the correlators with proper normalization converge to the values less than unity – a fact that is consistent with the probability interpretation of the correlators. We also have seen that the two point correlators show a stringy exclusion principle.

While calculating the correlators, we have used the overlap state and the multi-particle normalization; both were found to depend on the gauge indices as well as the boundary space time. We have further found that the correlators of the gravity state have an exponential decay. However, owing to the parity non-invariance of the ABJ theory, the results are not symmetric under the exchange of N_1 and N_2. Finally, we have considered a particular gravity background dual to an operator with a $\mathcal{R}$-charge of $\mathcal{O}(N^2)$ in ABJ gauge theory. As a result, due to the non-planar contributions, the large N_1 and N_2 expansions get replaced by $1/(N_1+M)$ and $1/(N_2+M)$ respectively where M is the number of extra columns added to the Young diagram.

5

w_∞ 3-algebra

5.1 Introduction

ABJM theory is a $U(N)_k \times U(N)_{-k}$, $\mathcal{N} = 6$ Chern-Simons-matter theory, where the two Chern-Simons gauge fields have the levels k and $-k$. In the limit, $N = 2$, supersymmetry of ABJM theory enhances from $\mathcal{N} = 6$ to $\mathcal{N} = 8$. Finally, in the special limit $N = 2$ and $k = 2$, ABJM theory becomes equivalent to another independent world volume theory of two M-2 branes known as BLG theory. The gauge field as well as the matter fields in BLG theory are valued in a completely anti-symmetric ternary product satisfying the so-called fundamental identity and a Euclidean metric. This ternary product is also known as 3-algebra. Although this 3-algebra plays an important role in the formulation of multiple M-2 brane theory, its rich mathematical structure makes the algebra very important to its own right. The consistent generalization of Kac-Moody and (centerless) Virasoro 2-algebras into respective 3-algebras motivates us to construct a further extension.

In this chapter, we explicitly obtain a classical w_∞ 3-algebra and show that our relation satisfies the 3-algebra "Fundamental Identity" (FI). Our construction is based on the earlier work on W_∞ and $W_{1+\infty}$ symmetries (see [240, 241] *and references therein*). Using the 'lone-star' product of $W_{1+\infty}$ generators and their commutation relations we write down a antisymmetric 3-algebra relation. This 3-algebra relation formed by $W_{1+\infty}$ generators simplifies significantly in a double scaling limit. The resulting mathematical structure still maintains a 3-algebra relation formed by w_∞ generators. This procedure of taking the double scaling limit is a generalization to that of taking a single scaling limit to obtain the w_∞ 2-algebra from the W_∞ 2-algebra. We explicitly check that the w_∞ 3-algebra satisfy the FI

We organize the chapter in the following way. In section 2, we introduce the generators of $W_{1+\infty}$ algebra which happens to be the building blocks of our w_∞ algebra. In section 3, we define the "lone-star' product and construct an antisymmetric ternary bracket involving the generators of $W_{1+\infty}$ algebra. In section 4, we take double scaling limit of this ternary bracket and obtain the w_∞ 3-algebra. In section 5, we give a geometric interpretation of the w_∞ 3-algebra. Lastly we end this chapter by a discussion.

5.2 Generators of $W_{1+\infty}$ algebra

We now start with the commutation relations defining $W_{1+\infty}$ algebra written in terms of generators $\tilde{V}^i_m$ [240]:

$$[\tilde{V}^i_m, \tilde{V}^j_n] = \sum_{l\geq 0} q^{2l} \tilde{g}^{ij}_{2l}(m,n) \tilde{V}^{i+j-2l}_{m+n} + q^{2i} \tilde{c}_i(m) \delta^{ij} \delta_{m+n,0}, \tag{5.1}$$

where superscripts i, j, l, representing the conformal spin of the generators, are in general integers: $-1, 0, 1, \cdots$ etc. whereas integer subscripts m, n can take arbitrary positive or negative values. We also have:

$$\tilde{g}^{ij}_l(m,n) \equiv g^{ij}_l(m,n,-\frac{1}{2}) \tag{5.2}$$

given by an expression:

$$g^{ij}_l(m,n,s) = \frac{1}{2(l+1)!} \phi^{ij}_l(s) N^{ij}_l(m,n). \tag{5.3}$$

Explicitly, $\phi^{ij}_l(s)$ are given by a generalized hypergeometric function:

$$\phi^{ij}_l(s) = {}_4F_3 \left[\begin{array}{cccc} -\frac{1}{2}-2s, & \frac{3}{2}+2s, & -\frac{l}{2}-\frac{1}{2}, & -\frac{l}{2} \\ & & & ; 1 \\ -i-\frac{1}{2}, & -j-\frac{1}{2}, & i+j-l+\frac{5}{2} & \end{array} \right] \tag{5.4}$$

and

$$N^{ij}_l(m,n) = \sum_{k=0}^{l+1} (-1)^k \binom{l+1}{k} (2i+2-l)_k [2j+2-k]_{l+1-k} [i+1+m]_{l+1-k} [j+1+n]_k, \tag{5.5}$$

where $[a]_n$ stands for $\frac{a!}{(a-n)!}$ and $(a)_n$ stands for $\frac{(a+n-1)!}{(a-1)!}$. Also, in eq. (5.1) q is an arbitrary scaling parameter, which we will fix later on through a double scaling process. Finally, the central term of the algebra in eq. (5.1) can be consistently set to zero and corresponds to the analysis of classical symmetries.

5.3 Lone-star product

Another property of interest for us will be the 'lone-star' product of the $W_{1+\infty}$ generators:

$$\tilde{V}^i_m \star \tilde{V}^j_n = \sum_{l\geq -1} q^l \tilde{g}^{ij}_l(m,n) \tilde{V}^{i+j-l}_{m+n}. \tag{5.6}$$

This star product is classical, since it does not contain information about the central term. As in the following we make use of the relation (5.6) to construct our 3-algebra, this analysis therefore holds for the classical case only. Note also that the commutation relation (5.1) follows from the 'lone-star' product eq. (5.6) (in absence of central term) by realizing that coefficients $\tilde{g}_l^{ij}(m,n)$ are symmetric under the simultaneous interchange of i,j and m,n for odd l's whereas they are antisymmetric for even l's. We now restrict ourselves to the case when the central term is absent.

Now, using the definition of the 3-algebra relation:

$$[A,B,C] = A[B,C] + B[C,A] + C[A,B] \tag{5.7}$$

and the commutation relation (5.1) as well as the star product (5.6), we can write the 3-algebra relation:

$$[\tilde{V}_m^a, \tilde{V}_n^b, \tilde{V}_p^c] = \sum_{l\geq 0, r\geq -1} q^{2l+r}[\tilde{g}_{2l}^{b,c}(n,p)\tilde{g}_r^{a,b+c-2l}(m,n+p) +$$
$$\tilde{g}_{2l}^{c,a}(p,m)\tilde{g}_r^{b,c+a-2l}(n,m+p) + \tilde{g}_{2l}^{a,b}(m,n)\tilde{g}_r^{c,a+b-2l}(p,m+n)]\tilde{V}_{m+n+p}^{a+b+c-2l-r}, \tag{5.8}$$

where the index r, for a given l, runs over indices $r = -1, 0, \cdots, (a+b+c-2l+1)$, whereas the running of index l in the three terms in rhs of eq. (5.8) is from zero upto $b+c$, $c+a$ and $a+b$ respectively.

The 3-algebra relation in eq. (5.8) may be of interest in its own right, however in the following we present a simpler situation by using a double scaling limit on the above relation. We also recall that a similar procedure (but with a single scaling parameter q) was used earlier to obtain the w_∞-algebra from W_∞. Relationship between w_∞-algebra and area preserving reparameterizations of 2-torus are also well known [242]. We observe an interesting relation at the 3-algebra level by comparing the structure constants of the 3-algebra emerging from the 3-bracket given in eq. (5.8), after taking the double scaling limit, with the one for the classical Nambu 3-brackets of globally defined functions f, g, h on T^2.

5.4 Double scaling limit and w_∞ 3-algebra

Now, to apply our double scaling, we scale all the generators $\tilde{V}_m^a$ in eq. (5.8) by a parameter β. Note that such a scaling is in addition to the one given in [240] which lead to the powers of q^{2l} in the commutation relation (5.1). We also note that the smallest power of q in eq. (5.8) corresponds to $l = 0$ and $r = -1$. In order to keep only this term, after the double scaling, we take the limits: $q \to 0$, $\beta \to \infty$ such that $\beta^2 q = 1$. We then obtain the simplified 3-algebra in terms of the rescaled generators w_m^a's:

$$[w_m^a, w_n^b, w_p^c] = [c(n-m) + b(m-p) + a(p-n)]w_{m+n+p}^{a+b+c+1}, \tag{5.9}$$

where we have also made use of the fact that

$$\tilde{g}_{-1}^{ab}(m,n) = 1, \quad \tilde{g}_0^{ab}(m,n) = (b+1)m - (a+1)n. \tag{5.10}$$

We now verify that w_∞ 3-algebra satisfies the FI, written in the present case as:

$$[w_m^a, w_n^b, [w_p^c, w_q^d, w_r^e]] = [[w_m^a, w_n^b, w_p^c], w_q^d, w_r^e]] + [w_p^c, [w_m^a, w_n^b, w_q^d], w_r^e]] + [w_p^c, w_q^d, [w_m^a, w_n^b, w_r^e]] \tag{5.11}$$

A discussion on the necessity of the FI's defining the Leibniz rule for the action of 3-brackets, as well as an analysis of the associativity constraints in such cases, is presented in [237]. In our case, evaluating the four terms we obtain:

$$\begin{aligned}[w_m^a, w_n^b, [w_p^c, w_q^d, w_r^e]] = {} & [e(q-p)+d(p-r)+c(r-q)]\times \\ & [(c+d+e+1)(n-m)+b(m-p-q-r)+a(p+q+r-n)]w_{m+n+p+q+r}^{a+b+c+d+e+2}\end{aligned} \tag{5.12}$$

$$\begin{aligned}[[w_m^a, w_n^b, w_p^c], w_q^d, w_r^e]] = {} & [c(n-m)+b(m-p)+a(p-n)]\times \\ & [e(q-m-n-p)+d(m+n+p-r)+(a+b+c+1)(r-q)]w_{m+n+p+q+r}^{a+b+c+d+e+2}\end{aligned} \tag{5.13}$$

$$\begin{aligned}[w_p^c, [w_m^a, w_n^b, w_q^d], w_r^e]] = {} & [d(n-m)+b(m-q)+a(q-n)]\times \\ & [e(m+n+q-p)+(a+b+d+1)(p-r)+c(r-m-n-q)]w_{m+n+p+q+r}^{a+b+c+d+e+2}\end{aligned} \tag{5.14}$$

$$\begin{aligned}[w_p^c, w_q^d, [w_m^a, w_n^b, w_r^e]] = {} & [e(n-m)+b(m-r)+a(r-n)]\times \\ & [(a+b+e+1)(q-p)+d(p-m-n-r)+c(m+n+r-q)]w_{m+n+p+q+r}^{a+b+c+d+e+2}\end{aligned} \tag{5.15}$$

Using eqs. (5.12), (5.13), (5.14) and (5.15), it can now be checked directly that the 3-algebra in eq. (5.9) satisfies the FI in eq. (5.11).

We have therefore obtained a 3-algebra generalization of the w_∞-algebra. Note that our double scaling is such that it gives a nontrivial 3-algebra in terms of w_∞ generators. This double scaling would however make the original commutation relations [240] of w_∞ generators trivial. There is, however, no inconsistency with our analysis above, since the 'lone-star' product also goes to infinity in this limit, thus giving us a well defined 3-algebra with finite coefficients. We have also analyzed the expressions for the (totally antisymmetrized) 4-brackets involving the generators w_m^a, using the relation (5.9):

$$[w_m^a, w_n^b, w_p^c, w_q^d] = w_m^a[w_n^b, w_p^c, w_q^d] - w_n^b[w_p^c, w_q^d, w_m^a] + w_p^c[w_q^d, w_m^a, w_n^b] - w_q^d[w_m^a, w_n^b, w_p^c]. \tag{5.16}$$

By explicit calculation we find that it is identically zero, a result similar to the one [237] for the Virasoro 3-algebra.

5.5 Geometric realization of w_∞ 3-algebra

As already pointed out before, above results can also be reinterpreted in terms of the algebraic structure of the reparameterizations of 2-torus through the evaluation of the classical Nambu 3-brackets (3CNB) of globally defined functions f, g, h on a 2-torus. 3CNB of functions f, g, h, that are completely antisymmetrized, are defined as the Jacobian of the transformation from (x, y, z) to $(f(x, y, z), g(x, y, z), h(x, y, z))$:

$$\{f, g, h\} \equiv \frac{\partial(f, g, h)}{\partial(x, y, z)} \equiv \ f\{g, h\} + g\{h, f\} + h\{f, g\}$$
$$= \frac{\partial f}{\partial x}(\frac{\partial g}{\partial y}\frac{\partial h}{\partial z} - \frac{\partial h}{\partial y}\frac{\partial g}{\partial z}) + \frac{\partial g}{\partial x}(\frac{\partial h}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial y}\frac{\partial h}{\partial z}) + \frac{\partial h}{\partial x}(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}). \qquad (5.17)$$

Now, to establish the connection with our results given above, we note that by choosing:

$$f \equiv w^a_m = \sqrt{z}\exp\left((a + \frac{1}{2})x + my\right),$$
$$g \equiv w^b_n = \sqrt{z}\exp\left((b + \frac{1}{2})x + ny\right),$$
$$h \equiv w^c_p = \sqrt{z}\exp\left((c + \frac{1}{2})x + py\right), \qquad (5.18)$$

we obtain the 3CNB of generators $\{w^a_m, w^b_n, w^c_p\}$, which matches with the 3-bracket given in eq. (5.9) (by a constant scaling of the generators), with structure constant:

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ m & n & p \end{vmatrix}. \qquad (5.19)$$

We also note that a somewhat similar structure appeared in the Moyal (sine) brackets of [242] and its correspondence to 3-algebra structure constants in our case will be of interest to examine. Also, it is noticed from eq. (5.18) that the 3-algebra generators of eq. (5.9) can be identified with the modes of the deformations of 2-torus [242]. In the present case, however, one also needs to multiply the exponential functions in eqs. (5.18) by an extra factor $\sqrt{z}$ common to all three generators in the 3CNB. The geometric interpretation of such an extra factor may be possible by identifying the complete geometry as a direct product of 2-torus with a point, since the deformation mode along the z direction is frozen.

5.6 Discussion

In this chapter, we establish the emergence of a novel classical 3-algebra formed by the generators of w_∞ symmetry group. Like $SO(4)$ 3-algebra described earlier in BLG theory, w_∞ 3-algebra is also completely antisymmetric with respect to the indices of its elements and satisfies FI. Using Nambu 3-bracket, we achieve the geometrical realization of this w_∞ 3-algebra. We also check the validity of the 3-bracket expression (before taking the scaling limit), i.e. eq. (5.8), as a proper 3-algebra relation. Our study shows that the ternary bracket of $W_{\infty+1}$ generators does not satisfy FI except for few low lying modes.

6
Summary

Gauge/gravity duality provides a systematic prescription to explore some strongly coupled theories. On one hand, by exploiting strong/weak nature of this duality, we can uncover features of strongly coupled gauge theories by studying dual weakly coupled gravity. On the other hand, physics of non-perturbative string/M theory becomes accessible by analysing weakly coupled boundary duals. The thesis attempts to explore both sides of this duality.

While studying the strong coupling limit of gauge theory, we focused on a holographic QCD model and the $\mathcal{N} = 4$ SYM plasma at finite temperature with additional heavy fundamental static quarks. We use the holographic techniques to calculate the drag force, jet quenching parameter and screening length of the quark and the antiquark. These quantities are sensitive to the strong coupling nature of gauge theory in the context of quark gluon plasma. In the case of holographic QCD model, we found that the drag force increases with temperature in the large chemical potential region ($\mu \geq \mu_c$). When the chemical potential is small ($\mu < \mu_c$) the drag force also increases for high temperature region, but it shows a multi-valued behavior at low temperature region. The jet quenching parameter has a monotonically decreasing behavior with the temperature in the region $\mu \geq \mu_c$. But for $\mu < \mu_c$, it shows similar multi-valued behavior in the low-temperature region. These multi-valued behaviors of the drag force and the jet quenching parameter, for $\mu < \mu_c$ and in the low temperature region, is consistent with the existence of first order phase transition in this hQCD model. We also find that, for all values of μ, the quark-antiquark pair dissociates beyond a characteristic screening length L_s of the system. Moreover, we calculated the binding energy as a function of the separation between a quark and antiquark. In another model, we discussed the computation of the dissipative force experienced by an external heavy quark with fundamental $SU(N)$ charge moving through a collection of other heavy static quarks uniformly distributed over $\mathcal{N} = 4$ SYM plasma at finite temperature. We explicitly calculated the effect of the back reaction of plasma on an external probe quark due to the presence of other heavy quarks. To realize this back reaction, we introduced a uniform distribution of fundamental strings embedded in a dual AdS black hole background. Correspondingly, the back reaction on the AdS black hole geometry due to the presence of the string cloud was computed. The back reacted black hole is parametrized by it's mass and the string cloud density. We have verified the thermodynamical stability as well as the gravitational stability (up to the tensor and the vector perturbation) of this new black hole geometry in a systematic way. Finally, we computed the drag force on a heavy probe quark in this new black hole background. Drag force enhances with the temperature and with the density of heavy quark distribution.

In the second part of the thesis, we explored the other side of the gauge/gravity correspondence to reveal the strong coupling side of string/M theory via the analysis in dual weakly coupled field theory. In particular, by exploiting the AdS_4/CFT_3 correspondence, we studied a class of semiclassical objects in gravity theory known as the giant graviton and the dual giant graviton. These objects are nonperturbative in nature and can be thought as either spherical M-2/M-5 branes living in $AdS_4 \times S^7/\mathbb{Z}_k$ geometry or spherical D-2/D-4 branes living in $AdS_4 \times \mathbb{CP}_3$ geometry. Using the operator state correspondence prescribed by the AdS_4/CFT_3 dictionary, we mapped these gravity states in to the $\frac{1}{2}$ BPS gauge-invariant chiral primary operators in the dual weakly coupled $\mathcal{N} = 6$, $U(N_1) \times U(N_2)$ ABJ gauge theory. We showed the correct gauge invariant operators dual to the giants with large angular momentum are the Schur polynomial operators with the large $\mathcal{R}$ charges. Using the combinatorics of Young tableaux representation of these Schur polynomials, we calculated various gauge theory correlators represnting transitions among the giants or between the giants and the ordinary gravitons. Our analysis in the weakly coupled gauge theory respects the orthogonality condition maintained by the giant graviton states in gravity Fock space and the probability interpretation of the transition amplitudes among those gravity states. Moreover, we considered a particular gravity background dual to an operator with a $\mathcal{R}$-charge of $\mathcal{O}(N^2)$ in ABJ gauge theory. This type of oprators are constructed by adding M number of extra columns to the Young diagrams representing the relevent Schur polynomials, where M is of the order of the smallest one among N_1 and N_2. We showed due to the non-planar contributions, the large N_1 and N_2 expansions get replaced by $1/(N_1 + M)$ and $1/(N_2 + M)$ respectively. We noted that apart from the ABJ(M) theory, there is an alternative world volume theory of multiple M-2 branes known as BLG theory. Motivated by the importance of 3-algebraic structure in this BLG theory we constructed a novel 3-algebra consisted of w_∞ generators. We explicitly showed this algebraic structure satisfies the FI. We also gave a geometrical interpretation of this w_∞ 3-algebra.

We hope that our study will be useful to understand some generic features of a strongly coupled system.

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