TRANSPORT PROPERTIES OF STRONGLY COUPLED GAUGE THEORIES FROM HOLOGRAPHY

By

Sachin Jain

Institute of Physics, Bhubaneswar.

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Synopsis

Holography states that a $(d+1)-$dimensional gravity theory (bulk theory) has a description in terms of $d-$dimensional field theory (boundary theory), where extra dimension in the gravity side is identified with energy scale in the field theory side. A well understood example is AdS_5/CFT_4 duality, which arises in the study of $D3$ branes in type IIB string theory. According to this duality, type IIB string theory on $AdS_5 \times S^5$ is dual to four dimensional $N = 4,\ SU(N)$ super Yang-Mills theory. In the strong 't Hooft coupling and large N limit of the gauge theory, dual string theory can be approximated by supergravity in five dimension coupled with Kaluza-Klein (KK) modes (lowest lying modes) coming from S^5 compactification. Within this duality, one can ask questions such as whether it is possible to reconstruct bulk space time using conformal field theory (CFT) data or given a bulk space time, what properties of dual gauge theory can one read off? For example, heating up the above gauge theory implies that in the bulk we have black brane in AdS_5. Further more, adding some gauge charge to the black brane is equivalent to having some chemical potential for the gauge theory. Stated more precisely, solutions to bulk equations of motion determines the thermodynamic variables of the dual CFT at equilibrium. One can even introduce a small space time dependent perturbations about equilibrium which in the domain of linear response leads to important processes such as transport properties of CFT. The basic object of interest is that we want to compute are retarded Green's functions which captures response of the gauge theory to the external perturbation. To illustrate further, let us consider an interacting quantum field theory (QFT), in global thermal equilibrium at temperature T and chemical potentials (μ) for various conserved charges. Now if we perturb the system out of equilibrium by allowing system thermodynamic variables to fluctuate in a scale which is sufficiently large compared to scale set by temperature or energy density in equilibrium, one describes system in terms of hydrodynamics. Then we expect, around any given point a region where local temperature is roughly constant and one can use basic thermodynamic variables to describe the physical properties of the region. The role of hydrodynamics is to describe how these different regions exchange thermodynamic quantities among themselves. The dynamics in this regime is captured by conservation of energy momentum tensor and other conserved global charges. The fluid perturbed away from equilibrium, tries to equilibrate through dissipation and the response to these perturbations are encoded in transport coefficients such as shear viscosity, electrical conductivity, thermal conductivity etc. Using gauge/gravity duality one can compute retarded greens functions of dual gauge theory operators from gravity side and use Kubo formulas to relate it to the transport coefficients.

After its discovery, the AdS/CFT duality is generalized for many different situations such as the case of non conformal boundary theories which arises in the study of Dp branes $(p \neq 3)$. AdS/CFT duality also has been generalized for many other situations mostly based upon symmetry principles, not necessarily always they have a well understood string theory embedding. Some such examples are cases where boundary theory is not required to be relativistic invariant or boundary theory has Lifshitz like symmetry. In the following discussions we shall consider generic gravity set up assuming a field theory dual in a similar spirit.

Our motivation and goal can be summarized as follows. Given the fact that, number of models which exhibits such dualities are increasing rapidly, it is desirable to have some features which are independent (referred as universal) of particular model. For instance, it is well known that shear viscosity (η) to entropy density ratio is equal to $\frac{1}{4\pi}$, in the dimension less units for a large class of gauge theory having a gravity dual. Interestingly this falls with in the experimental range observed at RHIC. So, even though these theories in several ways are different from theories such as QCD, they seem to share qualitatively similar behavior. This motivates us to investigate possible universality of other transport coefficients which might shed some light into qualitative features of RHIC physics. We primarily focus on computation of electrical conductivity at finite chemical potential (μ) and temperature (T) which is related to current current correlator through Kubo formula. Assuming gravity theory has a gauge theory dual, under general assumptions in the gravity side we show that electrical conductivity at finite chemical potential is universal and can be expressed in terms of thermodynamic quantities of the dual gauge theory. We further propose a universality of thermal conductivity (κ_T) to viscosity ratio ($\frac{\kappa_T \mu^2}{\eta T}$). We also provide a proof of universality of electrical conductivity and shear viscosity to entropy density ratio at zero temperature.

Our approach towards proving universality of electrical conductivity is as follows. First we compute electrical conductivity in the presence of one and more than one chemical potentials for several models [1]. What we observe is that, in the presence of multiple chemical potentials, there is a nontrivial mixing between current operators which, from the bulk point of view can be understood to be arising because of interaction through graviton. Then we compute thermal conductivity (defined as response to temperature gradient in the absence of electric current) and observe that thermal conductivity to shear viscosity ratio ($\frac{\kappa_T \sum_{i=1}^{n} \mu_i^2}{\eta T}$) is independent of how many chemical potential one turns on. This observation together with observation that at zero chemical potential as well the above ratio remains unchanged, lead us to conjecture that $\frac{\kappa_T \sum_{i=1}^{n} \mu_i^2}{\eta T}$, is universal and we showed that it can be expressed in terms of central charges of the dual conformal field theory (CFT) [2]. With the aim of proving above conjecture, in [3] we found out interesting connection between the membrane paradigm fluid which sits at the horizon effectively encoding the properties of the black brane to an external observer and fluid which sits at the boundary of the space time known from gauge/gravity duality. By exploiting the fact that changing radial position in the bulk corresponds to RG flow in the boundary fluid, in [4,5], authors proposed a number of relations and even interpolations between them. For example, radial independence of certain quantities is used to show that, the shear viscosity to entropy density ratio for both the fluids is the same, as well as the fact that at zero chemical potential, low frequency limit of electrical conductivities of these two distinct fluids are related. However the situation changes significantly at finite chemical potential in the boundary theory, where radial independence exploited earlier in relating electrical conductivity of these two fluids, gets completely destroyed. One needs to solve the flow equation in order to relate conductivities of the fluid at the boundary with the fluid at the horizon. In spite of this apparent difficulty, in [3], we observed that for charged Reissner-Nordström black brane in arbitrary dimension, there exist a simple relation between the conductivities of these two fluids. Further we computed electrical conductivity on hypothetical hyper surface at any radial position out side black brane horizon to show that there exist a smooth interpolation between conductivities of these two fluids. Based on these observations together with support from several other computation of electrical conductivity for asymptotically AdS spaces which corresponds to dual gauge theory to be CFT lead us to propose a form of electrical conductivity which is universal. We further checked that the proposed form of electrical conductivity holds for non conformal field theories, where the dual gravity theory is not asymptotically AdS (which arises in the study of Dp brane for $p \neq 3$), where as for asymptotically Lifshitz like gravity theories where boundary theory enjoy anisotropic scaling, it does not hold. This led us to ask, what is the most general gravity set up for which proposed form of electrical conductivity holds. In [6], we found that under general assumptions in the gravity side together with precise condition on the bulk stress tensor the electrical conductivity is the same as one we proposed. The condition on the bulk stress tensor may be related to the criteria for vacuum of dual gauge theory to be Lorentz invariant. This immediately explains why Lifshitz like theories does not have the form of conductivity as proposed since vacuum of dual gauge theory is not Lorentz invariant, where as for asymptotically AdS and some non AdS examples that we considered has dual gauge theory vacuum which are Lorentz invariant. Further we observed that thermal conductivity to viscosity ratio is again universal for non conformal theories.

We then turn our attention to study of transport coefficients of gauge theories at zero temperature which corresponds to extremal black hole in the bulk. In [1], we observed that for several examples the form of conductivity at zero temperature is same. Under the general assumption that extremal black brane has double pole structure at the horizon together with requirement that boundary theory vacuum has to be Lorentz invariant, we show that form of electrical conductivity is universal. Further in [7], we proved that shear viscosity to entropy density ratio is $\frac{1}{4\pi}$ even at zero temperature.

References:

[1] S. Jain, "Holographic electrical and thermal conductivity in strongly coupled gauge theory with multiple chemical potentials," JHEP **1003**, 101 (2010) [arXiv:0912.2228 [hep-th]].

[2] S. Jain, "Universal properties of thermal and electrical conductivity of gauge theory plasmas from holography," JHEP **1006**, 023 (2010) [arXiv:0912.2719 [hep-th]].

[3] S. Jain, "Universal thermal and electrical conductivity from holography," JHEP **1011**, 092 (2010) [arXiv:1008.2944 [hep-th]].

[4] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," Phys. Rev. **D79** (2009) 025023 [arXiv:0809.3808 [hep-th]].

[5] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, "Wilsonian Approach to Fluid/Gravity Duality," JHEP **1103** (2011) 141 [arXiv:1006.1902 [hep-th]].

[6] S. K. Chakrabarti, S. Chakrabortty, S. Jain, "Proof of universality of electrical conductivity at finite chemical potential," JHEP **1102**, 073 (2011) [arXiv:1011.3499 [hep-th]].

[7] S. K. Chakrabarti, S. Jain, S. Mukherji, "Viscosity to entropy ratio at extremality," JHEP **1001**, 068 (2010) [arXiv:0910.5132 [hep-th]].

List of Publications

***[1]** *"Proof of universality of electrical conductivity at finite chemical potential"*,
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JHEP **1011**, 092 (2010) [arXiv:1008.2944 [hep-th]].

***[3]** *"Universal properties of thermal and electrical conductivity of gauge theory plasmas from holography"*,
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***[4]** *"Holographic electrical and thermal conductivity in strongly coupled gauge theory with multiple chemical potentials"*,
Sachin Jain,
JHEP **1003**, 101 (2010) [arXiv:0912.2228 [hep-th]].

***[5]** *"Viscosity to entropy ratio at extremality"*,
Sayan K. Chakrabarti,**Sachin Jain**, Sudipta Mukherji,
JHEP **1001**, 068 (2010) [arXiv:0910.5132 [hep-th]].

[6] *" Notes on R-charged black holes near criticality and gauge theory"*,
Sachin Jain, Sudipta Mukherji, Subir Mukhopadhyay,
JHEP **0911:051**, (2009) arXiv:0906.5134 [hep-th].

[7] *" Spiky Strings in AdS_4 x CP^3 with Neveu-Schwarz Flux"*,
Sachin Jain, Kamal L. Panigrahi,
JHEP **0812:064**, (2008) arXiv:0810.3516 [hep-th].

[8] *"w_∞ 3-algebra "*,
Shankhadeep Chakrabortty, Alok Kumar, **Sachin Jain**,
JHEP **0809:091**, (2008) arXiv:0807.0284 [hep-th].

A (*) indicates papers on which this thesis is based.

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1 Introduction

1.1 Overview

Quantum chromodynamics (QCD) is a theory of strong interactions-one of the fundamental forces in nature that describes the interactions between quarks and gluons making up the hadrons. QCD enjoys two special properties. First is asymptotic freedom-at very high energy, quarks and gluons interact very weakly. Second is confinement-forces between quarks increase with their separations. Indirect support of the confinement comes from the fact that so far no free quarks have been experimentally observed. Lattice calculations suggest, confinement to deconfinement transition in QCD occurs at a temperature around $T \simeq 175$Mev.

Recently, in the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory, a new phase called Quark-Gluon Plasma (QGP) was believed to have been created by colliding gold nuclei at energies of order 100 GeV per nucleon. Estimation suggests that the temperature of this phase created at RHIC is about two times the deconfinement temperature. In this phase, quarks and gluons behave like a near perfect strongly coupled fluid. The fact that, QGP at RHIC is strongly coupled gets support from some unexpected features observed in the experiment. These include but not limited to the observation of strong collective behaviour (elliptic flow), large energy loss of high energy particles moving in this medium (jet quenching). Clearly this indicates that the plasma, in fact, interacts very strongly with itself and is thus referred to as strongly coupled. Most of our knowledge of QCD, however, is not applicable in this regime. Known calculational techniques involve a perturbative expansion of the theory in terms of the coupling constant and, therefore, it breaks down when the coupling becomes large.

One of the remarkable developments during the late last century was to provide a framework where we could make distinct progress in understanding strongly coupled gauge theories. This goes by the name of gauge/gravity duality. According to this conjectured duality, there is a correspondence between certain strongly coupled gauge theories with the weakly coupled string theories. By this it is meant that both the theories describe same physics. However, calculations become easier in one theory than the other[1]. This immediately opens up a possibility of an application: what is the dual of QCD? If we find one, we can carry out relevant computations in the dual theory to gain insights into QCD itself.

Unfortunately, till to date, gauge/gravity duality is well developed only for a certain class of theories which excludes QCD. These gauge theories share some properties with QCD, but differ from QCD in many essential ways. Nevertheless, we can still look for some universal features of these strongly coupled gauge theories. Our hope is that these results might be useful if a dual of QCD is discovered. This will be the *central theme* of this thesis.

As a concrete illustration, let us consider one of the most well understood examples of the gauge/gravity correspondence. It states that $\mathcal{N}=4$, four dimensional $SU(N)$ super Yang-Mills (SYM), at finite temperature is dual to type IIB string theory on AdS_5 - Schwarzschild black hole times S^5. Both $\mathcal{N}=4$ and QCD have gluons but they differ in their other ingredients and properties. On the gravity side of this duality, AdS_5 represents the five dimensional anti-de Sitter space which has a constant negative curvature. Finite temperature is introduced by adding a black hole into this background. One can reach this correspondence by studying non-extremal $D3$ -branes in IIB string theory and we will defer this discussion for the later sections. The strong coupling behavior of this gauge theory at finite temperature is captured by studying weakly coupled string theory on AdS_5 - Schwarzschild black hole times S^5 background.

In [1], using this duality, Policastro, Son and Starinets performed an elegant and striking calculation of the shear viscosity of strongly coupled $\mathcal{N}=4$ theory with the result

$$\frac{\eta}{s}=\frac{1}{4\pi}\frac{\hbar}{k_B}\sim 0.08\frac{\hbar}{k_B}. \tag{1.1}$$

Here, s is the entropy density and k_B is the Boltzmann constant. Interestingly enough, RHIC data suggests that QGP has very low viscosity and the estimated value is

$$\frac{\eta}{s}\sim 0.1\frac{\hbar}{k_B}. \tag{1.2}$$

The proximity of these two results initiated major activities in this area. This is not only because of its calculational simplicity but also for the universal nature of this result. Indeed, this ratio of shear viscosity to the entropy density is found to be same for all gauge theories with an Einstein gravity dual in the $N\to\infty$ and large t'Hooft coupling limit. In fact, it was further conjectured by Kovtun,

[1] Such an equivalence is possible in string theory because of the existence of the Dirichlet branes or the D-branes in short. These are the solitons in string theory which admit descriptions in terms of both open or closed strings. While the low energy dynamics of the open strings with their ends attached to D-branes (due to the Dirichlet condition) represent a gauge theory, the closed string description surely contains gravity. In the later sections, we will have occasions to further elaborate upon this idea.

Son, Starinets in[2], that this number is a universal lower bound for all materials including water and liquid helium!

A very natural question is, therefore, to ask if there are other universal quantities associated with the strongly coupled gauge theories which have a gravity dual. Indeed, as discussed in [3], the R-charge conductivity (σ) to the charge susceptibility ratio (χ) at zero chemical potential is expected to be another universal quantity. The ratio is given by

$$\frac{\sigma}{\chi} \geq \frac{\hbar c^2}{4\pi T}\frac{d}{d-2}. \tag{1.3}$$

Here c is the velocity of light, d represents the dimension of the gauge theory at temperature T.

One of the primary aims of this thesis is to study the universality in electrical conductivity for gauge theories at *finite* chemical potentials. The presence of chemical potential introduces another scale in the theory (besides the temperature) and, consequently, complicates the matters in several ways. Let us pause for a moment and discuss this here. As will be explained later in greater detail, the gauge/gravity correspondence suggests that gauge theory fluctuations at large length scales are dictated by the behavior of the near horizon geometry of the gravity dual. In the absence of chemical potentials, boundary transport coefficients such as shear viscosity or electrical conductivity can be computed solely in terms of horizon data. This is because the response function in the low frequency limit evolves in a very simple manner as we go away from the horizon along the radial direction[4, 5]. It is here that the introduction of a chemical potential primarily brings in non-trivialities. Although shear viscosity can still be computed solely in terms of horizon data, for the computation of electrical conductivity, horizon data is not enough. The reason is that the evolution of the response function does no longer remain trivial as above. Rather, it is given by a complicated *flow* equation. Nevertheless, our analysis reveals that if the stress-energy tensor related to the matter content of the bulk satisfies a compact relation among its space and time components, the boundary conductivity at low frequencies is universal. In the same spirit, we also discuss the universality of thermal conductivity to viscosity ratio at both zero and finite chemical potentials. Furthermore, this thesis also addreses the issue of transport coefficients of gauge theories at zero temperature, where the gravity dual is represented by extremal black holes. Though it is not immediately obvious, we show that the universality relation in Eq.(1.1) continues to hold at extremality. Finally, we also elaborate upon the universal nature of electrical conductivity at $T = 0$.

Before we go on to present our results in the later chapters, in the next section, we give a brief introduction to D-branes in string theory and their complementary descriptions in terms of open and closed strings. Subsequent sections concern the gauge/gravity correspondence, the mapping of operators in the gauge theory to the fields in the bulk dual. This chapter also includes a description of hydrodynamics and the techniques for the computation of the hydrodynamic response functions.

1.2 Strings and D-branes

The fundamental constituents in string theory are the *strings*[2] which can be either *closed*, or *open* and are characterized by a string tension $T_{\rm str}$ which is related to string length l_s by

$$T_{\rm str} \equiv \frac{1}{2\pi\alpha'} \quad \text{with} \quad \alpha' \equiv \mathrm{l}_{\mathrm{s}}^2. \tag{1.4}$$

In addition, the interactions between strings are controlled by a dimensionless coupling constant g_s, related to the expectation value of a dilaton; a field that appears in the massless spectrum of the string. Different vibrational modes of the strings give rise to different fields which, in the low energy limit, look like point particles. A consistent relativistic quantum theory of closed strings has, in it, a massless spin-2 state whose interaction at low energies is governed by general relativity. Similarly, open strings gives rise to gauge fields as it's end points can carry charges. However they do not carry spin-2 massless field in their spectrum. Consistency requires the strings to have supersymmetry and to live in 10 space-time dimensions. Consistency also requires five different types of superstrings, namely type IIA, type IIB, type I, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic. However, via various perturbative and non-perturbative dualities, all of them are found to be connected [12].

In addition to strings, superstring theory also contains solitonic configurations of various dimensionality. They are known as Dirichlet branes (D-branes)[13, 14]. A Dp-brane is a $(p+1)$ dimensional hypersurface in $9+1$ dimensional space-time. Both open and closed string can interact with the D-branes and the branes can be defined as objects where open string end points live, obeying Neumann boundary condition along $p+1$ space time direction and Dirichlet boundary conditions in $(9-p)$ spatial directions. Their origin can be understood as follows. In the spectra of closed string, one has left and right moving fermions. Depending on whether we implement periodic or anti periodic boundary conditions, we can have four sectors $(R-R)$, $(R-NS)$, $(NS-R)$, $(NS-NS)$, where R stands for Ramond and NS stands for Neveu-Schwarz. The $(R-R)$, $(NS-NS)$ sectors are space-time bosons whereas $(R-NS)$, $(NS-R)$ are space-time fermions. While the $(NS-NS)$ sector contains the graviton $g_{\mu\nu}$, a two form field $B_{\mu\nu}$, dilaton ϕ, the $(R-R)$ sector contains $p+1$ form field A_{p+1}, in the massless sector. Depending on whether p is even or odd, we have type IIA or type IIB theory. The Dp-branes are charged

[2]For an excellent elementary introduction to string theory, see [6]. For more advanced discussions, see[7, 8, 9, 10, 11])

under this $p+1$ form field. The minimal coupling of Dp-branes with form fields can be written as

$$\mu_p \int A_{p+1}, \tag{1.5}$$

where

$$\mu_p = \int *F_{p+2}. \tag{1.6}$$

with $F_{p+2} = dA_{p+1}$. Being solitonic in nature, Dp-branes are heavy and its mass per unit volume, the tension T_{Dp}, can be written as[3]

$$T_{Dp} = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{1.7}$$

Since Dp-branes are BPS configurations, vanishing force between them allows us to put N number of branes stacked on top of each other. If N is large, then this stack is necessarily very heavy, and consequently, it curves the space-time. Since, in addition, Dp-branes are charged under R-R $p+1$ form potential, Dp-branes have description in terms of some classical metric and R-R form potential. This is what is known as the closed string description of D-branes. On the other hand, D-branes also have a description in terms of open strings. One can think of open strings as excitations of D-brane since open string spectrum can be identified with the fluctuations of the D-brane. The massless spectrum of the open strings, living on N number of Dp-brane, is that of a maximally supersymmetric $U(N)$ gauge theory with fermions and $9-p$ massless scalar fields which together with the gauge field provide an unique supersymmetric completion. Thus, we have two very different descriptions of a stack of D-branes: one in terms of a gauge theory and the other in terms of classical R-R charged p-brane gravity background. Exploration of this idea led to the discovery of gauge/gravity duality also known as AdS/CFT correspondence[15, 16, 17], originally proposed by Maldacena. The next two subsections serve as a brief introduction to this correspondence.

1.2.1 D-branes and gauge theories

If we consider N number of coincident D3-branes in a flat space-time, the massless spectrum of open string consists of a gauge field A_μ, six real scalar field X^i and four Weyl fermion λ^a_α in the adjoint representation[4] of $U(N)$ with R-symmetry (as explained below) index $a = 1, ..., 4$ and Weyl index $\alpha = 1, 2$. At two derivative

[3]Let us note that, Dp branes are solitonic objects in string theory, and their mass is related inversely to the string coupling , which is different from usual solitonic objects found in the gauge theory where mass goes as $\frac{1}{g^2_{YM}}$. The factor $\frac{1}{l_s^{p+1}}$ that comes in Eq.(1.7) comes from dimensional grounds and $(2\pi)^p$ is introduced as a normalization factor.

[4]Let us note that, this theory has no fields such as quarks which transform in the fundamental representation.

level, the low energy[5] effective action for massless modes turns out to be $\mathcal{N} = 4$ super Yang-Mills[6] (SYM) with gauge group $U(N)$ in $3+1$ dimensions. One can think of $SU(N) \subset U(N)$ as relative motion of branes where as $U(1) \subset U(N)$ as rigid motion of the branes. Because of the overall translation invariance[18], this mode decouples from $SU(N)$, giving us $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$. Let us note that $\mathcal{N} = 4$ has a global symmetry, the $SU(4)_R$ symmetry under which A_μ transforms as singlet, λ^a_α as 4 and scalars X^i are rank 2 anti-symmetric tensors in representation 6. One can understand the origin of $SU(4)_R$ global symmetry as follows. The directions transverse to the D-branes is isotropic and these directions correspond to scalars X^i. Isotropy therefore means that there is a global $SO(6) \sim SU(4)$ symmetry for X^i. The Lagrangian for $\mathcal{N} = 4$ super Yang-Mills theory is unique and is given by [19]

$$\begin{aligned}\mathcal{L} = \quad & Tr\Big(-\frac{1}{2g^2_{SYM}} F^{\mu\nu}F_{\mu\nu} + \frac{\theta}{8\pi^2} F^{\mu\nu}\tilde{F}_{\mu\nu} - \sum_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \\ & + \sum_{a,b,i} g_{SYM} C^{ab}_i \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g_{SYM} \bar{C}^{ab}_i \bar{\lambda}_a [X^i, \bar{\lambda}_b] + \frac{g^2_{SYM}}{2} \sum_{i,j} [X^i, X^j]^2 \Big) \end{aligned} \tag{1.8}$$

where g_{SYM} is the gauge coupling and θ is instanton angle. The constants C^{ab}_i and C_{iab} are the $Clebsh-Gordon$ coefficients needed to make a singlet out of fermions and scalars. The overall trace is taken over the $SU(N)$ indices. The gauge coupling is determined in terms of string coupling by the relation

$$g^2_{SYM} = g_s. \tag{1.9}$$

The gauge field and scalars have mass dimension 1 and fermions have mass dimension $\frac{3}{2}$. So all the terms in the action have mass dimension 4. This implies that the theory is classically scale invariant. This scaling symmetry combines with Poincare symmetry $SO(1,3)$, resulting in a conformal symmetry $SO(2,4)$. This group is generated by translations P_μ, Lorentz transformations $L_{\mu\nu}$, dilations D and conformal transformations K_μ. It turns out that, even at the quantum level, this theory remains conformally invariant. This together with supersymmetry and R-symmetry lead to the supergroup[7] $SU(2,2|4)$ as the symmetry group of

[5]By low energy we mean $E \ll \frac{1}{\sqrt{\alpha'}}$, so that massive states of the open strings on the $D-$branes are not accessible.

[6]D-branes preserve $\frac{1}{2}$ of the 32 supersymmetries in the bulk. Thus the four dimensional world volume of D3-branes has 16 supercharges which implies $\mathcal{N} = 4$ in four dimensions.

[7]Let us note that, the supergroup $SU(2,2|4)$ can be written as

$$\begin{pmatrix} SU(2,2) \simeq SO(2,4) & Q,\ \bar{S} \\ \bar{Q},\ S & SU(4)_R \end{pmatrix},$$

where Q and S are Poincare supersymmetry generators and conformal supersymmetry generators respectively.

$\mathcal{N}=4$ SYM. We shall see that this symmetry group is again appearing in the next subsection where we consider D-branes in a different perspective.

We also note that, the Lagrangian in Eq.(1.8) receives higher derivative corrections which are suppressed by terms of order $\alpha^{'}E^2$, at an energy E. It also receives corrections due to its interactions with the closed string sector. The interactions of the closed string modes with themselves and with the open strings modes are controlled by dimensionless coupling constant $\alpha^{'}\,{}^4E^8$. Hence, in the low energy limit, the Lagrangian that describes the dynamics is given in Eq.(1.8) plus the decoupled closed string modes. So we conclude that the low energy effective description for $D3$ branes can be given by $\mathcal{N}=4 \quad SU(N)$ SYM theory and decoupled closed strings or supergravity in the ten dimensional Minkowski space-time.

1.2.2 D-brane space-time geometry

As we have noted, the D-branes are massive solitonic objects and they are the sources of various (R-R) fields. One can obtain the corresponding geometry by solving the equations of motion that follow from the effective low energy type II supergravity. Let us consider the case of $D3$-brane in particular. The $D3$ brane is a solution in type IIB string theory which, like generic Dp-branes, preserves half of the space-time supersymmetry. In the low energy limit, massless fields include, among the bosonic fields, metric g^{MN}, dilaton ϕ, axion C, and a (R-R) five form self-dual field strength F_{MNPQR}. The truncated action in the Einstein frame can be written as

$$\begin{aligned} I = \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{|g|}\Big(R-\frac{1}{2}\partial_M\phi\partial^M\phi &- \frac{1}{2}e^{2\phi}\partial_M C\partial^M C \\ &- \frac{1}{2.5!}F_{MNPQR}F^{MNPQR}\Big). \end{aligned} \tag{1.10}$$

The ten dimensional Newton's constant is given by[8]

$$G_{10}=8\pi^6 g_s^2 l_s^8. \tag{1.11}$$

The D3-brane solution following from the equations of motion, after imposing self-duality $*F_5=F_5$ is,

$$ds^2 = H^{-\frac{1}{2}}(-dt^2+\sum_{i=1}^{3}(dx^i)^2)+H^{\frac{1}{2}}(dr^2+r^2d\Omega_5^2) \tag{1.12}$$

and

$$\begin{aligned} &F_5=(1+*)dt\wedge dx_1\wedge dx_2\wedge dx_3\wedge dH^{-1}, \quad g_s=e^{\phi}, \\ &C=\text{Constant}, \quad \phi=\text{Constant}, \end{aligned} \tag{1.13}$$

[8]In Eq.(1.11), l_s^8 comes from the fact that G_{10} has a dimension of *length*^{8}. For fixed l_s, we expect gravitational effect should increase with increasing g_s. The exact dependence of g_s^2 follows from computation of string scattering amplitude. The factor $8\pi^2$ is again a convention.

with

$$H(r) = 1 + \frac{L^4}{r^4}, \qquad L^4 = 4\pi g_s N l_s^4, \tag{1.14}$$

where (t, x_1, x_2, x_3) are the D3-brane world-volume coordinates and $r^2 = \sum_{i=1}^{6} y_i^2$, with y_i's are orthogonal to brane directions. This solution is referred to as an extremal D3-brane solution. The non-extremal generalization, discussed later, introduces temperature and consequently breaks space-time supersymmetry. The factor L can be thought of as characteristic length of gravitational effect of N D3-branes. Its exact dependence on l_s and g_s is explained below. The above solution is known as supergravity solution since we have neglected all possible corrections that might come from massive string modes. To be precise, in the limit $g_s N \ll 1$, the length L is much less than l_s and thus supergravity approximation is not expected to be a reliable approximation of the full string solution. On the other hand, in the limit $g_s N \gg 1$, the radius L is much greater than l_s and thus supergravity approximation is expected to be a reliable approximation to the full string solution. To have a better understanding of the geometry, we consider the following two limits. In the region $r \gg L$, the metric looks like

$$ds^2 = (1 + \mathcal{O}(\frac{L^4}{r^4}))(\eta_{MN} dx^M dx^N), \tag{1.15}$$

which is ten dimensional Minkowski space with small correction of the order of $\frac{L^4}{r^4}$. The appearance of correction terms can be understood as follows. The mass of N number of D3-branes is $M \propto N T_{D3}$. Since D3-branes extend along three spatial directions, their gravitational effect is similar to that of a point particle with mass M in the six transverse directions. So at $r \gg L$, we expect a correction of the form $\frac{G_{10} M}{r^4}$. Now using Eq.(1.11) and Eq.(1.7), we get

$$\frac{G_{10} M}{r^4} \sim \frac{g_s N l_s^4}{r^4}. \tag{1.16}$$

This explains various factors that appears in L in Eq.(1.14) except 4π which is a convention.

Now we consider the opposite limit, namely $r \ll L$. The metric in Eq.(1.12) approximates to

$$ds^2 = ds^2_{\mathrm{AdS}_5} + L^2 d\Omega_5^2, \tag{1.17}$$

where

$$ds^2_{\mathrm{AdS}_5} = \frac{r^2}{L^2}(-dt^2 + \sum_{i=1}^{3}(dx^i)^2) + \frac{L^2}{r^2} dr^2. \tag{1.18}$$

So to conclude, far away from the branes the space time is flat, ten-dimensional Minkowski space, whereas close to the branes a *throat* geometry of the form $AdS_5 \times S^5$ develops.

Let us now concentrate on two distinct sets of modes, one propagating in the Minkowski region and other propagating in the throat region. The low energy limit consists of focusing on excitations that have arbitrarily low energy with respect to an observer in the asymptotically flat Minkowski region. While in the Minkowski region, only massless ten-dimensional graviton super multiplet survives, the whole tower of string excitations contribute in the throat region. One can understand this in the following way. The energy of an object measured by an observer at constant r (say E_r) and energy E measured by an observer at infinity are related by a redshift factor

$$E = \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} E_r. \tag{1.19}$$

So a closed string of arbitrarily high proper energy in the throat region may have an arbitrarily low energy as seen by an observer at asymptotic infinity. To understand how these two modes interact, one can study the absorption cross section of mass less particles (say graviton) from the branes sent from asymptotic infinity. In the low energy limit, they decouple as the low energy absorption cross section goes to zero at energy E, as $L^8 E^3$ [20, 21]. Similarly the excitations that live deep down the throat, faces a infinite gravitational potential barrier so they can not escape to the asymptotic region. So we conclude that we get two region where, in the first region we get supergravity in Minkowski space and, in the second region, we get the full string theory on $AdS_5 \times S^5$.

Another instructive way to see this decoupling is as follows. We start with the D3-brane metric given in Eq.(1.12). Defining a new coordinate

$$z = \frac{L^2}{r}, \tag{1.20}$$

we can rewrite the metric as

$$\begin{aligned} ds^2 &= \left(1 + \frac{L^4}{z^4}\right)^{-\frac{1}{2}} \frac{L^2}{z^2} \eta_{ij} dx^i dx^j + L^2 \left(1 + \frac{L^4}{z^4}\right)^{\frac{1}{2}} \left(\frac{dz^2}{z^2} + d\Omega_5^2\right) \\ &\equiv L^2 \tilde{g}_{MN} dx^M dx^N. \end{aligned} \tag{1.21}$$

Here i, j run over the world-volume coordinates of the brane and η_{ij} is the flat metric. In the last line we introduced $g_{MN} = L^2 \tilde{g}_{MN}$ for the complete metric in ten dimensions with $M, N = 0, 1, ..., 9$. Let us now consider a closed string moving in this geometry. The world-sheet action of which is

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma} \gamma^{\alpha\beta} g_{MN} \partial_\alpha X^M \partial_\beta X^N + ... \\ &= \frac{L^2}{4\pi\alpha'} \int d^2\xi \sqrt{\gamma} \gamma^{\alpha\beta} \tilde{g}_{MN} \partial_\alpha X^M \partial_\beta X^N + ..., \end{aligned} \tag{1.22}$$

where the dots represent possible other terms which will not be relevant for the discussion to follow. In Eq.(1.22), $\gamma_{\alpha\beta}$ is the world-sheet metric with α, β running over the world-sheet coordinates. Using the relation $L^4 = 4\pi g_s N l_s^4$ and $\alpha' = l_s^2$, we re-write Eq.(1.22) as

$$S = \sqrt{\frac{\lambda}{4\pi}} \int d^2\xi \sqrt{\gamma}\gamma^{\alpha\beta}\tilde{g}_{MN}\partial_\alpha X^M \partial_\beta X^N + ..., \tag{1.23}$$

where we have used $\lambda = g_s N$. In view of Eq.(1.9), λ is the 't Hooft's coupling in the gauge theory. The Eq.(1.23) also implies that the world-sheet higher derivative corrections (the correction due to massive string modes) are now controlled by dimensionless coupling

$$\alpha'_{effective} = \sqrt{\frac{1}{4\pi\lambda}}. \tag{1.24}$$

- **Maldacena's limit:** Maldacena's limit is defined as $\alpha' \to 0$, keeping λ fixed. This is equivalent to taking $L \to 0$ since form (1.22) and (1.23)

$$\frac{L^2}{4\pi\alpha'} = \sqrt{\frac{\lambda}{4\pi}}. \tag{1.25}$$

  Interestingly, in this limit string action is well defined and the rescaled metric $\tilde{g}_{MN}$ in Eq.(1.23) reduces to

$$\tilde{g}_{MN}dx^M dx^N = \frac{1}{z^2}\eta_{ij}dx^i dx^j + \frac{dz^2}{z^2} + d\Omega_5^2. \tag{1.26}$$

  This is $AdS_5 \times S^5$ metric with unit radius of curvature, written in z coordinate. To summarize, we see that in the Maldacena limit, only the $AdS_5 \times S^5$ region of the D3-brane contributes to the closed string dynamics while asymptotically flat region effectively decouples.

Before we pass over to the next section, we end this section with a brief description of AdS_5[22]. AdS_5 is a space-time with a constant negative curvature. It can be represented by a hypersurface obeying

$$X_0^2 + X_5^2 - \sum_{i=1}^{4} X_i^2 = L^2, \tag{1.27}$$

in six dimensional flat space with metric

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2. \tag{1.28}$$

In this form, it is obvious that AdS_5 metric is endowed with $SO(2,4)$ isometry. Moreover, S^5 has an isometry group $SO(6) \sim SU(4)$. We have already discussed that, being half-BPS, 16 of 32 supersymmetries are preserved by an array of N D3-branes. In addition to this, in the decoupling limit where we are left with the AdS part, we have another 16 conformal supersymmetries which were broken by full D3-brane geometry. Thus together with supersymmetry, the $SU(4)$-symmetry and the conformal symmetry $SO(2,4)$ leads to supergroup $SU(2,2|4)$. This is also the symmetry group of the $\mathcal{N}=4$ SYM, discussed in the last subsection.

1.3 The AdS/CFT correspondence

Discussion in the previous section leads to the two different descriptions of the low energy limit of N D3-branes.

- **Open string description:** $\mathcal{N}=4$, $SU(N)$ SYM in four dimensions with gauge coupling g_{SYM} + free supergravity in flat space-time.
- **Closed string description:** Type IIB string theory on $AdS_5 \times S^5$ with parameters string coupling g_s and string length l_s + free supergravity in flat space-time.

Both the descriptions have decoupled free supergravity in flat space-time and Maldacena proposed to drop this and identify the rest. This leads to the following correspondence.

$$\mathcal{N} = 4,\ SU(N)\text{ SYM} \equiv \text{Type IIB string theory on AdS}_5 \times \text{S}^5, \tag{1.29}$$

with the parameters of both side related to each other by

$$g^2_{SYM} = g_s, \tag{1.30}$$

$$g^2_{SYM} N = \lambda = \frac{L^4}{4\pi l_s^4} \tag{1.31}$$

and the axion expectation value is given by SYM instanton angle $\langle C \rangle = \theta$.

Unfortunately, quantization of strings on $AdS_5 \times S^5$ background suffers from inadequate understanding. We have noted that above space is supported by the R-R five form flux. While the NS-R approach turns out to be difficult in presence of R-R fields, the Green-Schwarz approach is more suitable. However, finding covariant Green-Schwarz action on curved R-R background is again a complicated matter (see [23]). In these circumstances, the conjecture is mostly exploited only in a particular region of the coupling space. This region can be isolated as follows.

We note from Eq.(1.24) that the world-sheet derivative corrections are controlled by $\alpha'_{effective}$. Therefore massive string modes decouple in the limit $\alpha'_{effective} \to 0$. This, in view of Eq.(1.24), means that we must be in the strongly coupled region of the gauge theory with λ being large. We can also suppress string loop corrections by taking $g_s \to 0$. But since $\lambda = g_s N$, to keep it fixed but large, we need to take $N \to \infty$. So gauge theory in question is actually $\mathcal{N} = 4$ $SU(N)$ SYM with very large number of colours. Since we have gotten rid of massive stringy modes and also have g_s small, we have a classical type IIB supergravity on $AdS_5 \times S^5$. This is a very well understood subject. Consequently, most of the explorations are carried out in this region of AdS/CFT correspondence. In this thesis, we mostly exploit this weaker form of the conjecture.

1.3.1 The matching of spectrum

We have seen that the symmetry group of both side of the duality is given by the supergroup $SU(2,2|4)$. The AdS/CFT duality implies that the representations of the same supergroup $SU(2,2|4)$ should also match on both the sides. Stated differently, there should be a one to one correspondence between gauge invariant local operators in the gauge theory with the local fields in the gravity. In the following, we briefly discuss the spectrum of both the sides and then their mapping (see [22] for details).

The $SU(N), \mathcal{N} = 4$ SYM contains all the gauge invariant quantities that can be built out of gauge field A_μ, scalars X^i and Weyl fermions λ^a_α. Since all of the fields are in the adjoint representation, the gauge invariant operators must be product of traces of products of those fields. These can be classified into single trace and multi trace operators. We only need to consider single trace operators, since multi trace operators appear in the operator product expansions of single trace operators. Out of single trace operators, only superconformal primary operators are important since all others can be built out of them by applying Poincare supersymmetry generator Q and translation P_μ. These primaries can further be divided into chiral primary and non-chiral primary operators. Chiral primaries are those, which are annihilated by half of the supersymmetry generators. Since the supercharges have helicities $\pm\frac{1}{2}$, the other primaries in that representation will have range of helicities between $\lambda - 2$ to $\lambda + 2$ where λ is the helicity of lowest dimensional operator. This is known as short multiplet. For example,

$$\mathcal{O}^{I_1 I_2 \ldots I_n} = \Big[str(X^{I_1} X^{I_2} \ldots X^{I_n})\Big], \quad \text{with} \quad \mathrm{n} = 2, 3, \ldots \mathrm{N}, \tag{1.32}$$

where str means symmetrized trace over gauge algebra which implies that the above operator is totally symmetric under $SU(4)_R$, I indices and therefore transform in $(0, n, 0)$ representation of $SU(4)$. Further, the third bracket in the right hand side of above equation implies that one needs to take only the traceless part in the $SU(4)_R$ indices. The scaling dimension of these primaries are n, and the highest dimension primaries in this multiplet have a dimension $n + 4$, which is of the form $Q^4 \bar{Q}^4 \mathcal{O}$. The cases with $n \geq N$, are multi trace operators where N is the number of colours.

Let us consider the $n = 2$ case, which is called supergraviton representation. Since chiral primary of lowest dimension is built out of scalar ($\lambda = 0$), this representation will have the range of helicities between -2 to $+2$, and the highest dimension primaries in this multiplet have a dimension 4, instead of 6 as primaries with $\Delta > 4$ vanishes. This multiplet includes among others, a vector, the $SU(4)_R$ symmetry current J^μ of dimension 3, a symmetric tensor field, the energy momentum tensor $T^{\mu\nu}$ of gauge theory of dimension 4.

In the gravity side, the short multiplet arises as follows. As we have already described, it is not known how to compute the full type IIB string spectrum on $AdS_5 \times S^5$. Only the states that arises from the dimensional reduction [24] of the ten dimensional type IIB supergravity multiplet, are known. They all have helicity range (-2) to 2. Hence we get short multiplet and these fields are built on a lowest dimensional field which is scalar in $(0, n, 0)$ representation of $SO(6)$ with $n = 2$ [25]. This lowest dimensional scalar field arises from linear combination of metric h^a_a along S^5 and four form field A_{abcd}, where a, b, c, d are indices along S^5. For the case of $n = 2$, one has in it massless graviton field $g^{\mu\nu}$, the massless $SU(4)_R$ gauge field A_μ. It then immediately follows that the massless graviton field $g^{\mu\nu}$ corresponds to energy momentum tensor $T^{\mu\nu}$ and the massless $SU(4)_R$ gauge fieldA_μ corresponds to the $SU(4)_R$ symmetry current J^μ of the gauge theory.

1.3.2 Computing correlation function from AdS/CFT

One of the powerful aspects of duality is that, it maps the problem of finding quantum correlation function in the field theory to a classical problem in the gravity. Suppose we are interested in computing correlation function of a local gauge invariant operator operator θ in the gauge theory. For that we need to deform the theory by

$$S \to S + \int d^4x\phi(x)_0\theta(x), \tag{1.33}$$

where $\phi_0(x)$ is source conjugate to θ. According to AdS/CFT, this source can be identified with the boundary value of some bulk fields Φ, up to appropriate factors (as explained below) such that[16, 17, 27]

$$-\mathrm{log}\langle \mathrm{e}^{\int \mathrm{d^4x}\phi_0(\mathrm{x})\theta(\mathrm{x})}\rangle_{\mathrm{CFT}} \simeq \mathrm{onshell\ S}[\phi_0(\mathrm{x})]_{\mathrm{sugra}}, \tag{1.34}$$

where by on-shell we mean we solve equations of motion in the bulk subject to Dirichlet boundary condition on the boundary with the specified boundary value, and evaluate the action on the solution. Now in order to compute the n-point correlation function, all we need to do is to take derivative of this on-shell action with respect to ϕ_0, n times. More precisely,

$$\langle T[\theta(t_1, x_1)...\theta(t_n, x_n)]\rangle = \frac{\partial^n S_{Sugra}}{\partial\phi_0(x_1, t_1)....\partial\phi_0(x_n, t_n)}. \tag{1.35}$$

We illustrate this with an example here.

Let us consider a massive bulk scalar field Φ of mass m in AdS_5. This can be thought of as arising from Kaluza-Klein compactification along S^5, in which case the mass is given by $m \sim \frac{1}{L}$ where L is the radius of S^5 which is same as AdS_5 radius. For the time being, we shall take m to be a generic value. To be more precise, let us work with AdS_5 in the coordinate system z with $z = \frac{L^2}{r}$, where metric takes the form

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + \sum_{i=1}^{3}(dx^i)^2 + dz^2). \tag{1.36}$$

The action is given by

$$\begin{aligned} S &= \frac{1}{2}\int d^5x\sqrt{g}[g^{MN}\partial_M\Phi\partial_N\phi + m^2\Phi^2] \\ &= \frac{1}{2}\int_0^\infty dzd^4x\frac{L^3}{z^3}[(\partial_z\Phi)^2 + (\partial_\mu\Phi)^2 + \frac{m^2L^2}{z^2}\Phi^2], \end{aligned} \tag{1.37}$$

where M, N indices takes value along all the bulk directions where as μ, ν indices takes value along field theory directions. In the momentum space

$$\Phi(x^\mu, z) = \int d^4k e^{ik.x} f_k(z). \tag{1.38}$$

The equation of motion is given by

$$f_k^{''} - \frac{3}{z}f_k^{'} - (k^2 + \frac{m^2L^2}{z^2})f_k = 0, \tag{1.39}$$

with $k^2 = g^{\mu\nu}k_\mu k_\nu$ and prime ($'$) denotes derivative with respect to z. Solution to equation of motion Eq.(1.39) is a linear superposition of $z^2 I_{\Delta-2}(kz)$ and $z^2 K_{\Delta-2}(kz)$. In the interior of AdS space ($z \to \infty$), the Bessel functions behave as

$$I_{\Delta-2}(kz) \sim e^{kz}, \quad K_{\Delta-2}(kz) \sim e^{-kz}. \tag{1.40}$$

So by imposing regularity at $z \to \infty$ (interior of AdS), we can set the coefficient of $I_{\Delta-2}(kz)$ to zero. In the above, we have used the notation that,

$$\Delta = 2 + \sqrt{4 + m^2L^2}. \tag{1.41}$$

To have real exponents in Eq.(**??**), we require $m^2L^2 \geq -4$ which is referred as Breitenlohner-Freedman (BF) bound and is required for stability[28, 29, 30]. To understand the role of Δ in the boundary theory, let us study the near boundary behavior of the field $\Phi(x)$ in Eq.(1.38). Near the boundary ($z \to \epsilon$), it behaves as $\Phi \sim z^\Delta$. We set the boundary condition near the boundary to be

$$\Phi(x, z)|_{z=\epsilon} = \phi_0(x)\epsilon^{4-\Delta}. \tag{1.42}$$

Using this, we fix the normalization of f to be

$$f_k(z=\epsilon) = \phi_0(k)\epsilon^{4-\Delta}, \tag{1.43}$$

so that we get

$$f_k(z) = \phi_0(k) z^2 \epsilon^{2-\Delta} \frac{K_{\Delta-2}(k,z)}{K_{\Delta-2}(k,\epsilon)}. \tag{1.44}$$

In the position space, if we write Φ near the boundary, we get

$$\Phi(z,x) \to \epsilon^{\Delta}[A(x) + \mathcal{O}(\epsilon^2)] + \epsilon^{4-\Delta}[\phi_0(x) + \mathcal{O}(\epsilon^2)], \tag{1.45}$$

with

$$A(x) = \pi^{-2}\frac{\Gamma(\Delta)}{\Gamma(\Delta-2)} \int d^4x^{'} \phi_0(x^{'})|x - x^{'}|^{-2\Delta}. \tag{1.46}$$

As mentioned earlier, the term ϕ_0 dominates near the boundary whereas the other factor always goes to zero, since by definition $\Delta \rangle 0$. So the factor $\phi_0(x)$ will act as source for operator θ. Under the scaling $x \to \lambda x$, $z \to \lambda z$, the field Φ does not get scaled, but due to presence of factor $\epsilon^{4-\Delta}$ in Eq.(1.45), the factor ϕ_0, scales as $\phi_0(x\lambda) \to \lambda^{\Delta-4}\phi_0(x)$, and hence by Eq.(1.33), the scaling dimension of operator θ is Δ. So we see that the mass of the dual bulk field determines the scaling dimension of the boundary operator. In the context of the boundary theory, the BF bound arises from requirement of unitarity. Now we turn our attention to the computation of correlation functions of operators θ.

Let us first evaluate the on-shell action. By doing integration by parts in the action Eq.(1.37) and using equation of motion Eq.(1.39) we get on-shell action as a boundary term and is given by

$$S_{\text{onshell}} = \frac{1}{2}\int\int \frac{d^4k d^4k^{'}}{(2\pi)^8}\delta^4(k+k^{'})\phi_0(k_\mu)\phi_0(k^{'}_\mu)\epsilon^{4-2\Delta}\frac{z\partial_z f_{k,z}}{f(k,\epsilon)}\bigg|_{z=\epsilon}, \tag{1.47}$$

with $\epsilon \to 0$. Inserting the solution given by Eq.(1.44) in Eq.(1.47) we get

$$\begin{aligned} S_{\text{onshell}} &= \frac{1}{2}\int\int \frac{d^4k d^4k^{'}}{(2\pi)^8}\delta^4(k+k^{'})\phi_0(k_\mu)\phi_0(k^{'}_\mu)\Big(\frac{1}{\epsilon^{4-2\Delta}}\text{Polynomial}[\text{k}^2\epsilon^2] \\ &\quad -2^{1-2(\Delta-2)}(\Delta-2)k^{2(\Delta-2)}\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)} + ...\Big) \end{aligned} \tag{1.48}$$

where ... represents terms which are zero as $\epsilon \to 0$. Let us note that we have, in Eq.(1.48) some divergent pieces as $\epsilon \to 0$. If we Fourier transform back in to position space, we see that these are the contact terms. From the dual gauge theory point of view one can think of these as UV divergencies with UV cut-off[9] ϵ. These can be subtracted off by adding suitable counter terms. By taking two derivatives of the on-shell action Eq.(1.48) with respect to ϕ_0, we get

$$< \theta(k)\theta(k^{'}) >= -2^{1-2(\Delta-2)}(\Delta-2)k^{2(\Delta-2)}\frac{\Gamma(2-\Delta)}{\Gamma(\Delta-2)}, \tag{1.50}$$

which in the position space gives

$$< \theta(x)\theta(x^{'}) >= 2\pi^{-2}\frac{\Gamma(\Delta)}{\Gamma(\Delta-1)}\frac{1}{|x-x^{\prime}|^{2\Delta}}. \tag{1.51}$$

We again observe that, the scaling dimension of θ is Δ. Although we have discussed the case of scalar field, one can similarly find out Greens functions for other operators in the boundary theory by identifying dual field and calculating the on-shell action.

Till now we have discussed how to compute correlation functions in the Eucledian signature. The AdS/CFT techniques can very well be used to compute the same in the Lorentzian signature. The differences between these two are, in the Euclidean signature we are interested in the time ordered correlators where as in the Lorentzian signature there are several correlators of interest (time-ordered, advanced, retarded). We shall return to them in later sections.

1.4 Compactification along S^5

In this section, we describe briefly the S^5 compactification of the type IIB theory. Results of this section, and its generalization, will be used repeatedly in the later part of the thesis.

After dimensional reduction on S^5, the type IIB supergravity action can be written as

$$S = \frac{1}{16\pi G_5}\int d^5x[\mathcal{L}_{\rm grav}+\mathcal{L}_{\rm matt}], \tag{1.52}$$

[9]Let us note that, ϵ acts as UV cut-off for gauge theory which is an IR cut-off in AdS space. This is generally goes by the UV/IR relation in AdS/CFT. To illustrate this, let us write AdS_5 metric in the coordinate $r=\frac{L^2}{z}$, where it takes the form

$$ds^2 = (\frac{r}{L})^2\eta_{\mu\nu}dx^\mu dx^\nu + (\frac{dr}{r})^2L^2. \tag{1.49}$$

Scaling symmetry of AdS_5 implies, under scaling $x^\mu \to \lambda x^\mu$ of the gauge theory coordinates, the radial coordinate scales as energy scale that is $r \to \frac{r}{\lambda}$.Let us note that, as we approach IR of the boundary theory by doing a scaling by $x^\mu \to \lambda x^\mu$, with $\lambda > 1$, we are going deep inside the AdS. In other words, r large in the boundary theory corresponds to UV physics of the gauge theory whereas r small corresponds to IR physics. Hence, radial direction in the gravity side, can be identified as the energy scale in the dual gauge theory.

where five-dimensional Newton's constant G_5 is related to ten dimensional Newtons constant through $G_5 = \frac{G_{10}}{\pi^3 L^5}$. By using Eq.(1.11), Eq.(1.30) and Eq.(1.31) we get

$$\frac{G_5}{L^3} = \frac{\pi}{2N^2}. \tag{1.53}$$

The part $\mathcal{L}_{\rm matt}$ in Eq.(1.52) is the Lagrangian for matter fields which gets contribution from infinite tower of fields that we get after compactification along S^5. In the cases where $\mathcal{L}_{\rm matt} = 0$, the ten dimensional IIB supergravity reduces to Einstein action ($\int \mathcal{L}_{\rm grav}$) in the presence of negative cosmological constant. Details of the compactification goes as follows. We start with the metric,

$$ds^2 = g^5_{\mu\nu} dx^\mu dx^\nu + L^2 d\Omega_5^2. \tag{1.54}$$

Here $g^5_{\mu\nu}$ is the five dimensional part of the metric and $d\Omega_5^2$ is the metric on S^5, represented by five angular coordinates $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$. Since the metric is diagonal, ten dimensional Ricci scalar is totally decoupled in two independent components, one coming from the $g^5_{\mu\nu}$ part and another from the S^5 part. We denote them by $\mathcal{R}^{(5)}$ and $\mathcal{R}^{(S)}$ respectively. Since we are interested to get five dimensional action, we keep first component as it is and evaluate the second one from S^5 metric. Then the value of $\mathcal{R}^{(S)}$ is $\frac{20}{L^2}$. Similarly the five form field strength $F^{(10)}$ has non vanishing components $F^{(10)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = F^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ and $F^{(10)}_{\theta_1\theta_2\theta_3\theta_4\theta_5} = F^5_1 \epsilon_{\theta_1\theta_2\theta_3\theta_4\theta_5}$, where $F^{(5)}_1$ is a zero-form field strength on the S^5. To write down both the components of the form field in terms of zero-form field in the action, we use the Hodge dual transformation for the first component which is $F^{(5)}_{\mu_1\mu_2\mu_3\mu_4\mu_5} = \frac{1}{L^2} F^{(5)}_2 \epsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}$. Here $F^{(5)}_2$ is also a zero-form field strength on the space given by metric $g^5_{\mu\nu}$. After rearranging the all fields and integrating over the S_5, the ten dimensional action in Eq.(1.10), reduces to the five dimensional form as

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{|g^{(5)}|} \left[R^{(5)} + \frac{20}{L^2} - \frac{1}{2L^5} (F^{(5)2}_1 + F^{(5)2}_2) \right]. \tag{1.55}$$

Let us note that there is no contribution from dilaton ϕ or axion C since they are constants. The value of the last term of the above integral can easily be calculated using equations of motion of the five form field (see [24, 26] for details). This comes out to be $\frac{8}{L^2}$. Therefore, the final form of the five dimensional action is

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{|g^{(5)}|} \left[R^{(5)} + \frac{12}{L^2} \right]. \tag{1.56}$$

Because of the presence of cosmological constant $\Lambda = -\frac{12}{L^2}$, the action admits AdS_5 as a solution.

In general the above action receives other contributions if we allow rotations or other excitations on S^5. In the case of rotation, for example, the additional terms come in the form of scalars and vectors. We will come to these contribution in later sections.

1.5 Some applications of AdS/CFT: At equilibrium

So far we have considered SYM at zero temperature. In this thesis, we will primarily be interested in gauge theories at finite temperature as well as at finite chemical potentials. This section serves as an attempt to address some of the general features of $\mathcal{N} = 4$, $SU(N)$ SYM at non-zero temperature and chemical potentials.

At non-zero temperature

In the light of gauge/gravity duality, there are two ways to introduce temperature in the gauge theory. First is to compactify the Euclidean time direction of AdS_5. The periodicity then determines the temperature of the gauge theory. This is known as the thermal AdS space. Second way is to incorporate a black hole into the AdS geometry. The Hawking temperature and the entropy of the black hole then determine the temperature and the entropy of the dual. Moreover, according to AdS/CFT, the free energy of the gauge theory is determined by the temperature times the on-shell Euclidean supergravity action. This was computed in [17]. For $\mathcal{N} = 4$, $SU(N)$ SYM, the free energy (density) and the entropy (density), at large N, were found to have a N^2 dependence resulting from the contributions due to all the degrees of freedom of $SU(N)$. We call this phase as the deconfined phase. Further, the same computation on thermal AdS produces a N^0 dependence in the corresponding thermodynamic quantities. Naturally, this space is then identified as the gravity dual of the confined phase. It was further shown in [17] that for SYM on S^3, the transition from one phase to another takes place at a finite non-zero temperature and can be identified as the Hawking-Page transition from thermal AdS to the black hole space-time[31]. However, for gauge theories on R^3 with dual as the black hole with flat horizon, the deconfined phase was found to be stable at all non-zero temperature. In what follows, we shall concentrate on the black holes with the flat horizon. An excellent discussion on gauge theories on S^3, in this context can be found in [17].

At non-zero temperature and chemical potential

As we have noted previously, $\mathcal{N} = 4$, $SU(N)$ SYM has a global R-symmetry given by the group $SU(4)$. Consequently, there can be three independent R-charges coming from three independent $U(1)$ Cartans of the group. Conjugate of this charges are the chemical potentials. Therefore, one can study this SYM in the presence of three non-zero chemical potentials and hence at finite density of charges conjugate to this chemical potentials. The gauge/gravity duality says that the *global* symmetries of the gauge theory appear as a *local* symmetries on its dual [32]. It is easy to see as to where from the gauge fields could appear in the geometry. The sphere S^5 has a $SU(4)$ symmetry with precisely three independent

$U(1)'$s. Rotating the sphere along the three independent directions would therefore produce three gauge fields on AdS after compactification. So the gravity duals are the five dimensional AdS black holes with these gauge charges. Following the literature we call these general class of black holes as R-charged black holes. In the last sub-section of this section, we present a brief discussion on these holes and their gravity duals.

1.5.1 Finite temperature

The AdS Schwarzschild black holes are the solutions of Eq.(1.56), with

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + \sum_{i=1}^{3}(dx^i)^2 + \frac{1}{f(z)}dz^2), \tag{1.57}$$

where

$$f(z) = 1 - (\frac{z}{z_H})^4. \tag{1.58}$$

In the above equation z_H is a constant. Horizon is given by the solution of $f(z) = 0$. This happens at $z = z_H$. The horizon of the black hole is flat, and are called black branes. The Hawking temperature of the black hole can be computed in the following way. Close to horizon, we define $z = z_H + \frac{\kappa\ z_H^2}{2}\rho^2$ with $\kappa = \frac{f'(z_H)}{2}$, called the surface gravity. In the Euclidean space, where $t \to i\tau$, the metric reads

$$ds^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 + \frac{L^2}{z_H^2}\sum_{i=1}^{3} dx_i^2. \tag{1.59}$$

Comparing the first two terms of the right hand side of the above equation with $d\rho^2 + \rho^2 d\phi^2$ (where $\phi = \phi + 2\pi$), we see that to avoid conical singularity, we need to identify

$$\begin{aligned} \kappa\tau &\sim \kappa\tau + 2\pi \\ \Rightarrow \tau &\sim \tau + \frac{2\pi}{\kappa}. \end{aligned} \tag{1.60}$$

The Hawking temperature is simply the inverse of this periodicity and is given by

$$T = \frac{\kappa}{2\pi} = \frac{1}{\pi z_H}. \tag{1.61}$$

From the gauge theory point of view, this can be interpreted as the temperature of the SYM. Using the relation between the entropy of the black hole and the area of the horizon, we can write entropy density to be[10]

$$s = \frac{A}{4VG_5} = \frac{L^3}{4G_5 z_H^3}. \tag{1.62}$$

[10] Since the horizon has an infinite volume, one needs to put a cut-off in order to define thermodynamic quantities. Thermodynamic densities are then defined by dividing respective quantities by the volume V.

This is interpreted as the entropy density of the dual gauge theory. Using the expression for the Hawking temperature we can re-write the entropy as

$$s = \frac{1}{4G_5}(\pi L)^3 T^3. \tag{1.63}$$

Further, one can compute the free energy of the gauge theory by using the relation

$$Z_{CFT} \equiv e^{\frac{F}{T}} = e^{-S_g[g]}, \tag{1.64}$$

where g is the Euclidean saddle point metric which extremizes the action in Eq.(1.56). However, it turns out that, on-shell action evaluated on the solution given in Eq.(1.57) is infinite. Therefore one needs to have a regularization scheme. There are two different way of doing this. First is to subtract the AdS background keeping the geometries of the AdS background and black hole in the asymptotic region same[17]. The other way is to introduce counter terms (see for example [32]). Though we shall use the counter term method to calculate on-shell action, both the ways give the same result.

It is well known that, in order to have well defined variational principle and on-shell finite action, one needs to add counter terms to the action Eq.(1.56). The modified form of the action is given by [32],

$$\begin{aligned} S_g &= S_{E.H.} + S_{G.H.} + S_{ct} \\ &= -\frac{1}{16\pi G_5}\int d^5x\sqrt{g}\ (\mathcal{R} + \frac{12}{L^2}) \\ &\quad +\frac{1}{8\pi G_5}\int_{z\to 0} d^4x\sqrt{\gamma}\ (\mathcal{K} - \frac{3}{L}), \end{aligned} \tag{1.65}$$

where γ is the induced metric at the boundary of the space time and $\mathcal{K}$ is the trace of extrinsic curvature. $S_{G.H.}$ is Gibbons-Hawking term that is required to have a well defined variational principle. However for asymptotically AdS space Gibbons-Hawking boundary term gives a vanishing contribution to the on-shell action. S_{ct} is required to render the on-shell action finite. Now, in order to evaluate free energy as in Eq.(1.64), we use the solution as given in Eq.(1.57). After plugging this in the right hand side of Eq.(1.65) and using the definition in Eq.(1.64), we get

$$\frac{F}{V} = -\frac{1}{16G_5}(\pi L)^3 T^4. \tag{1.66}$$

As mentioned earlier, instead of introducing counter terms, above expression could have been obtained by subtracting the AdS background, keeping the geometries of the AdS background and black hole in the asymptotic region same. Since, free energy in Eq.(1.66) is always negative[11], it is the black hole phase and not the AdS that minimizes the free energy. We, therefore, conclude that at any non-zero temperature, black hole is the stable phase of the gravity system. Now turning our attention to the gauge theory side, we notice that the free energy in Eq.(1.66) should be identified as the free energy of the gauge theory at the same temperature. On natural ground, we expect for SYM at temperature T, the free energy density is given by

$$\frac{F}{V} = -c^{'}T^4, \tag{1.67}$$

where $c^{'}$ is a measure of number of degree of freedom of the CFT. Upon comparing this with Eq.(1.66) and using Eq.(1.53), we get

$$c^{'} = \frac{1}{16G_5}(\pi L)^3 = \frac{\pi^2 N^2}{8}. \tag{1.68}$$

Since free energy density has a leading N^2 dependence, we conclude that the gauge theory is in the deconfined phase. Let us note that, in order to define thermodynamics properly for gauge theory, we also need to introduce a IR cut-off. The volume of the space is V, which appears in Eq.(1.66) and in Eq.(1.67) .

Let us end this subsection with the following comment. At a much higher energy compared to the scale set by the temperature, we expect SYM to have negligible effect of temperature. In this sense, the temperature modifies the IR physics. In the gravity dual, the temperature modifies the geometry by putting a horizon into the deep interior of the AdS. However asymptotically far away, it preserves the AdS structure. Hence we expect that the near horizon physics of the black hole captures the IR physics of the gauge theory where as the asymptotic region dictates the UV physics of the theory.

1.5.2 Finite temperature and chemical potential

The S^5 reduction of type IIB supergravity gives rise to $\mathcal{N} = 8$, $D = 5$ gauged supergravity with $SO(6)$ Yang-Mills gauge group. The complete details of this reduction is quite complex (see for example [33]). However, truncation of this

[11]A more interesting situation arises when we consider the gauge theory on $S^1 \times S^3$. In this case the dual gravity background is a black hole with spherical horizon. Here one finds that below a critical temperature, the thermal AdS space has lesser free energy than the black hole phase and hence there is a phase transition from black hole to thermal AdS as we lower the temperature. This, in the gauge theory, is interpreted as deconfinement to confinement transition, where thermal AdS space represents the confined phase of the gauge theory. Let us note that, the transition temperature is inversely proportional to the radius of the space S^3 where field theory lives. Hence, in the limit where radius of the sphere goes to a very large value, the transition temperature tends to zero. This is what we got from the study of thermodynamics of black hole with flat horizon.

five dimensional theory to $\mathcal{N}=2$ gauged supergravity with gauge group $U(1)\times U(1)\times U(1)$ which is the Cartan subgroup of $SO(6)$ is known. In the bosonic sector, it contains three gauge bosons, the metric and two scalars. However, it is more convenient to parametrize these two scalars in terms of three real scalar fields with a constraint. We give a brief description of the black holes and their various thermodynamic properties. We refer them as R-charged black holes. More details of R-charged black holes can be found in [34, 35, 33, 36].

The truncated action is given by

$$\begin{aligned} S_5 &= \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big(\mathcal{R}+\frac{2}{L^2}V-\frac{1}{4}G_{ij}F^i_{\mu\nu}F^{\mu\nu\,j}-G_{ij}\partial_\mu X^i\partial^\mu X^j \\ &+ \frac{1}{24\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}\epsilon_{ijk}F^i_{\mu\nu}F^j_{\rho\sigma}A^k_\lambda\Big), \end{aligned} \tag{1.69}$$

where

$$G_{ij}=\frac{L^2}{2}\mathrm{diag}\left[(X^1)^{-2},\,(X^2)^{-2},\,(X^3)^{-2}\right]\,, \tag{1.70}$$

and the scalar potential is given by

$$V=2\sum_{i=1}^{3}\frac{1}{X^i}. \tag{1.71}$$

The $F^i_{\mu\nu}$ with $i=1..3$ are the field strength of the three $U(1)$ gauge fields and X^i with $i=1..3$ are three real scalars subject to constraint $X_1X_2X_3=1$. The Newtons constant and gauge theory variables are related by $\frac{1}{16\pi G_5}=\frac{N^2}{8\pi^2L^3}$, as before. The solution of the equations of motions that follow from the action in Eq.(1.69) are summarized below. The metric is given by

$$ds_5^2=-\mathcal{H}^{-2/3}\frac{(\pi T_0L)^2}{u}\,f\,dt^2+\mathcal{H}^{1/3}\frac{(\pi T_0L)^2}{u}\,\left(dx^2+dy^2+dz^2\right)+\mathcal{H}^{1/3}\frac{L^2}{4fu^2}du^2\,, \tag{1.72}$$

where

$$f(u)=\mathcal{H}(u)-u^2\prod_{i=1}^{3}(1+\kappa_i)\,,\qquad H_i=1+\kappa_iu\,,\qquad \mathcal{H}=\prod_{i=1}^{3}H_i, \tag{1.73}$$

and

$$X^i=\frac{\mathcal{H}^{1/3}}{H_i(u)}\,, \tag{1.74}$$

where as the gauge field is given by

$$A^i_t=\frac{\pi T_0\sqrt{2k_i(1+k_1)(1+k_2)(1+k_3)u}}{H_i(u)}. \tag{1.75}$$

For convenience, we have used the coordinate system u in which $u = 1$ is the horizon[12] and $u = 0$ is the boundary which is AdS_5. Let us note that, if we set all the chemical potential to zero $\kappa_i = 0$, then we see that the u coordinate is related to z coordinate of the previous section by the relation $u = \frac{z^2}{z_H^2}$ and hence T_0 is identified as the temperature of the black hole at zero chemical potential. Now we summarize various thermodynamic quantities. The Hawking temperature and entropy density can be computed as done in the last subsection and are given by

$$T_H = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1\kappa_2\kappa_3}{2\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}} T_0 \,, \qquad s = \frac{\pi^2 N^2 T_0^3}{2} \prod_{i=1}^{3} (1+\kappa_i)^{1/2} \,. \tag{1.76}$$

As discussed previously, in order compute free energy, we need to add appropriate counter terms. Including counter terms, the full action takes the form

$$S = S_5 + \frac{1}{8\pi G_5} \int_{\text{boundary}} d^4x \sqrt{-h}\, \mathcal{K} + \frac{1}{8\pi G_5} \int_{\text{boundary}} d^4x \sqrt{-h} W(X), \tag{1.77}$$

where

$$W = -\frac{1}{L} \sum_{i=1}^{3} X^i, \tag{1.78}$$

and was derived originally in [37]. Let us note that in four or higher dimension, we do not require any boundary term for the Maxwell fields. Upon evaluating on-shell action, we get free energy of the dual gauge theory. The pressure (P) of dual gauge theory, which is related to free energy by $P = -\frac{F}{V}$ is given by

$$P = \frac{\pi^2 N^2 T_0^4}{8} \prod_{i=1}^{3} (1+\kappa_i) \,. \tag{1.79}$$

The energy density of the gauge theory is related to ADM mass of the black hole and is given by[38]

$$\varepsilon = \frac{3\pi^2 N^2 T_0^4}{8} \prod_{i=1}^{3} (1+\kappa_i) \,, \tag{1.80}$$

and hence we see $\epsilon = 3P$. The densities of physical charges and conjugate chemical potentials are

$$\rho_i = \frac{\pi N^2 T_0^3}{8} \sqrt{2\kappa_i} \prod_{l=1}^{3} (1+\kappa_l)^{1/2} \,, \qquad \mu_i = A_t^i(u) \Big|_{u=1} = \frac{\pi T_0 \sqrt{2\kappa_i}}{(1+\kappa_i)} \prod_{l=1}^{3} (1+\kappa_l)^{1/2} \,, \tag{1.81}$$

[12] Let us note that $f = 0$, has three roots. The largest root corresponds to black hole horizon.

One can now easily check that the relation

$$\epsilon + P = sT_H + \sum_{i=1}^{3} \rho_i \mu_i \tag{1.82}$$

holds. As is the case with temperature, introduction of chemical potential effects IR physics of the gauge theory. This is evident from the fact that, solution Eq.(1.72) asymptotically (or near the boundary) approaches AdS_5.

It is well known that, unless the charges satisfy certain constraints, these black holes undergo a local instability [39, 38, 40]. While at high temperature, black holes remain stable, once we reduce the temperature down to a critical value, the specific heat and susceptibility diverge. In order to see this, let us compute those quantities. The specific heat associated with the black holes has the following form

$$\begin{aligned} C = \left(T\frac{\partial s}{\partial T}\right)_{\mu_1,\mu_2,\mu_3} &= (\pi T_0 L)^3(2+\kappa_1+\kappa_2+\kappa_3-\kappa_1\kappa_2\kappa_3)\times \\ \frac{3-(\kappa_1+\kappa_2+\kappa_3)-(\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1)+3\kappa_1\kappa_2\kappa_3}{4\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}(2-(\kappa_1+\kappa_2+\kappa_3)+\kappa_1\kappa_2\kappa_3)}&. \end{aligned} \tag{1.83}$$

The expressions for susceptibility can be found in [41]. What we note from above expressions that the specific heat diverges over the critical hypersurface

$$2-(\kappa_1+\kappa_2+\kappa_3)+\kappa_1\kappa_2\kappa_3 = 0. \tag{1.84}$$

Same is true for susceptibility as well. Hence the black hole background is thermodynamically stable provided the κ_i's satisfies the constraint

$$2-(\kappa_1+\kappa_2+\kappa_3)+\kappa_1\kappa_2\kappa_3 > 0. \tag{1.85}$$

It turns out that the Lagrangian in Eq.(1.69) can further be truncated down to a smaller one. For example, one can truncate it to a theory with diagonal $U(1)$ of the group $U(1)^3$. In this case the fields $X_i = 1$ for $i = 1...3$. Action can be written as[33]

$$S_5 = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\Big(\mathcal{R} + \frac{12}{L^2} - \frac{1}{4}F_{(2)}^2 + \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}F^{\rho\sigma}A_\lambda\Big). \tag{1.86}$$

The solution of the equations of motions that follows from the above action is asymptotically AdS Reissner-Nordstrom black hole in five dimensions. The embedding of this truncated Lagrangian in $D = 10$ dimensions can be found in [42]. In the light of AdS/CFT, thermodynamics and instabilities of these black holes have also been discussed in [42].

1.6 Some applications of AdS/CFT: Dissipation near the equilibrium

Till now, we have seen how time independent homogeneous gravity backgrounds can be used to study equilibrium properties of dual gauge theories. We now consider the response of the gauge theory to small space and time dependent external perturbations about its equilibrium. This has been developed in [43, 44, 1] and has been extremely useful to study the transport properties of strongly coupled gauge theories.

The basic quantity that we want to compute is the retarded Green's function. It encodes the causal response of system to external perturbation. Let us consider a perturbation of the field theory of the form

$$\Delta S_{QFT} = \int d^4x \Theta_a(x)\phi_a(x), \tag{1.87}$$

where ϕ_a is source and Θ_a is an operator in the field theory. When the source of the perturbation $\phi_a(t,x)$ is small then in the linear response regime, we can write

$$\delta\langle\Theta_a(x)\rangle = -\int_y G^R_{ab}(x-y)\phi_b(y)\Big|_{\phi\to 0}, \tag{1.88}$$

where by $\delta\langle\Theta_a(x)\rangle$ we mean deviation from the average value of operator at equilibrium. In Eq.(1.88) $G^R_{ab}(x-y)$ is the retarded [13] Greens function and can be written as

$$G^R(x-y) = -i\theta(x^0-y^0)\langle[\Theta_a(x),\Theta_b(y)]\rangle. \tag{1.89}$$

Taking a Fourier transform of Eq.(1.89) we get

$$\delta\langle\Theta_a\rangle = G^R_{ab}(\omega,k)\phi_b(\omega,k), \tag{1.90}$$

where we have assumed space-time translation invariance. Similarly taking a Fourier transform of Eq.(1.89) and using Eq.(1.90) we get

$$G^R_{ab}(\omega,\overrightarrow{k}) = -i\int d^3x dt e^{-i\omega t - ik.x}\langle[\Theta_a(x),\Theta_b(0)]\rangle. \tag{1.91}$$

In the long wavelength and low frequency limit, where the field theory at finite temperature is defined by hydrodynamics, one can use the Kubo's formula (elaborated later)

$$\delta\langle\Theta_a\rangle = i\omega\chi_{ab}\phi_b\Big|_{\omega,k\to 0}, \tag{1.92}$$

[13] In the Lorentzian signature, we have several choices for correlator, namely time-ordered, advanced, retarded. The choice of retarded Greens function here, over others follows from the causality.

where χ_{ab} is some response function (transport coefficient) which characterizes the hydrodynamic regime. The Eq.(1.92) together with Eq.(1.90) implies,

$$\chi_{ab} = -\lim_{\omega,k\to 0} \frac{1}{i\omega} G^R_{ab}(\omega, k). \tag{1.93}$$

If we consider $\Theta = T^x_y$ or $\Theta = J^i$ then $\chi = \eta$, the shear viscosity or $\chi = \sigma$, the conductivity of the dual gauge theory respectively.

In the next few subsections, we provide a brief review of these developments. In the later chapters of this thesis, we will discuss universal nature of some of the transport coefficients of strongly coupled theories using gauge/gravity duality.

1.6.1 Hydrodynamics

Let us consider an interacting QFT, in global thermal equilibrium at temperature (T) and chemical potentials (μ) dual to various conserved charges. There is a characteristic length scale in QFT, namely the mean free path (l_{mfp}). Now if we perturb the system out of equilibrium with fluctuations whose wave length is large compared to scale set by the mean free path, one describes the system in terms of an effective theory called hydrodynamics, which is formulated in-terms of equations of motion. Perturbation away from the equilibrium, in this limit, can be thought of as if we are allowing the thermodynamic variables of the system to fluctuate at a scale sufficiently large compared to scale set by temperature or energy density in equilibrium. Then its natural to expect, around any given point, a region where local temperature is roughly constant and one can use basic thermodynamic variables to describe the physical properties of the region. The role of hydrodynamics is to describe how these different regions exchange thermodynamic quantities among themselves. The dynamics in this regime is captured by conservation of energy momentum tensor and other conserved global charges. The dynamical equations are

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu_I = 0, \tag{1.94}$$

where $T^{\mu\nu}$ is stress tensor and J^μ_I is the charged currents and I specifies number of conserved charges required to specify the system. Now all that we have to do is to solve Eq.(1.94) for energy momentum tensor and current. By the virtue of local thermal equilibrium, we should be able to express $T^{\mu\nu}$ and J^μ in terms of thermodynamic variables. Since we would like to understand how thermodynamic variables flow from one region of local thermal equilibrium to the other, we associate a velocity field $u^\mu(x)$ to each region. It turns out that, local thermodynamic variables together with velocity field completely describes the system in the hydrodynamic regime. We therefore need to know as to how the stress tensor and the currents can be expressed in terms of variables like temperature T, energy density ϵ, pressure P, chemical potentials μ and fluid velocity u^μ. We do this for the ideal fluid and then for the dissipative fluids. For further details, we refer reader to[46, 36, 47, 45].

Ideal fluid

For an ideal fluid, there is no dissipation. One can go to a local rest frame where velocity field is aligned in the direction of energy flow. In this case one can write

$$T^{\mu\nu}_{\text{ideal}} = \epsilon u^{\mu}u^{\nu} + PP^{\mu\nu}, \quad J^{\mu}_{I,\text{ideal}} = \rho_I u^{\mu}, \tag{1.95}$$

with $u_{\mu}u^{\mu} = -1$, and $P^{\mu\nu} = g^{\mu\nu} + u^{\mu}u^{\nu}$ which can be thought of it as projecting orthogonal to velocity. In the local rest frame, $P^{\mu\nu}$ is used to decompose energy momentum tensor into temporal and spatial components. In Eq.(1.95), ϵ, P and ρ_I are the energy density, pressure and conserved charged of the system. Since there is no dissipation one expects zero entropy production. This can be understood by defining entropy current

$$J^{\mu}\Big|_{S} = su^{\mu}, \tag{1.96}$$

which keeps track of how local entropy density varies in the fluid. In the above equation s is the entropy density of the fluid. For ideal case we have $\partial_{\mu}J^{\mu}\Big|_{S} = 0$, a statement of no entropy production.

Dissipative fluid

The fluid perturbed away from equilibrium, tries to equilibriate through dissipation (see [46, 36, 47, 45] for details). Microscopically dissipation arises because of interaction term in QFT. In this case we expect flow of fluid to create entropy consistent with second law of thermodynamics. To model dissipation, one might simply adds extra terms in the energy momentum tensor and current as

$$T^{\mu\nu}_{\text{Dissipation}} = T^{\mu\nu}_{\text{Ideal}} + \Pi^{\mu\nu}, \quad J^{\mu}_{I,\text{ Dissipative}} = J^{\mu}_{I,\text{ ideal}} + Y^{\mu}_{I}. \tag{1.97}$$

So we now need to determine $\Pi^{\mu\nu}$ and Y^{μ}_{I}. One way of doing this is to demand positivity of entropy current and determine set of allowed most general terms in $\Pi^{\mu\nu}, Y^{\mu}_{I}$ consistent with symmetries. Here one allows terms that are gradient in velocity and thermodynamic variables. In addition we need to choose the velocity field. In the Landau frame [46],

$$u_{\mu}\Pi^{\mu\nu} = u_{\mu}Y^{\mu}_{I} = 0. \tag{1.98}$$

In other words we find $T^{\mu\nu}u_{\nu} = -\epsilon u^{\mu}$. So u^{μ} can be thought of as eigenvector with eigenvalue ϵ. So u^{μ} determines how energy-momentum is transported in the system. Before writing down dissipative parts in terms of gradient expansion of thermodynamic variables and velocity field, let us look at the following . We know that

$$\nabla_\mu T^{\mu\nu} = 0. \tag{1.99}$$

Contracting it with velocity and using expression for ideal part of energy momentum tensor, we get

$$\begin{aligned} u_\nu \nabla_\mu T^{\mu\nu}_{\text{ideal}} &= 0 \\ \Rightarrow (\epsilon + p) \nabla_\mu u^\mu + u^\mu \nabla_\mu \epsilon &= 0. \end{aligned} \tag{1.100}$$

Projecting orthogonal to velocity field we get

$$\begin{aligned} P_{\nu\alpha} \nabla_\mu T^{\mu\nu}_{\text{Ideal}} &= 0 \\ \Rightarrow P^\mu_\alpha \nabla_\mu P + (\epsilon + p) P_{\nu\alpha} u^\mu \nabla_\mu u^\nu &= 0. \end{aligned} \tag{1.101}$$

We observe that there is a relation between gradient of thermodynamic variables such as energy density, pressure to gradient of velocity. So we conclude that energy momentum tensor can only be expressed in terms of derivative of velocity field.

The velocity gradient can be decomposed along and orthogonal to velocity field. The orthogonal part can further be decomposed into trace part (θ), traceless symmetric ($\sigma^{\mu\nu}$) and antisymmetric parts ($\omega^{\mu\nu}$). For a four dimensional system we can write,

$$\nabla^\mu u^\nu = -a^\nu u^\mu + \sigma^{\mu\nu} + \omega^{\mu\nu} + \frac{1}{3}\theta P^{\mu\nu}, \tag{1.102}$$

where

$$\begin{aligned} \theta &= \nabla_\mu u^\mu && : \text{ The divergence part} \\ a^\mu &= u^\nu \nabla_\nu u^\mu && : \text{ The accelaration} \\ \sigma^{\mu\nu} &= \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) + \frac{1}{2}(u^\mu a^\nu + u^\nu a^\mu) - \frac{1}{3}\theta P^{\mu\nu} \\ \omega^{\mu\nu} &= \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu) + \frac{1}{2}(u^\mu a^\nu - u^\nu a^\mu). \end{aligned} \tag{1.103}$$

It follows from the definition that,

$$u_\mu a^\mu = \sigma^{\mu\nu} u_\mu = \omega^{\mu\nu} u_\mu = 0. \tag{1.104}$$

We are now ready to write down the most general form of the dissipative part of the energy momentum tensor ($\Pi^{\mu\nu}$) that appears in Eq.(1.97). In order to do so, we should keep in mind that, the energy momentum tensor should be symmetric and it should obey Landau frame condition stated in Eq.(1.98). With these constraint in mind, the dissipative part of energy momentum tensor can be expressed as

$$\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}, \tag{1.105}$$

where we have introduced two new parameters, the shear viscosity η and the bulk viscosity ζ. Further, if the system is conformally invariant, the bulk viscosity ζ vanishes. Before concentrating on how to compute shear viscosity η, we shall discuss the dissipative part Y_I^μ that appears in the Eq.(1.97).

Keeping in mind the Landau frame condition stated in Eq.(1.98), one can express Y_I^μ in terms of acceleration a^μ, and derivatives of thermodynamic variables. However, using Eq.(1.101), we see that a^μ can be written in terms of gradient of thermodynamic variables. For convenience, Y_I^μ is expressed in terms of gradient of intensive variables such as chemical potentials μ_I or temperature T, instead of expressing it in terms of gradient of energy density, charge densities. The most general form that is consistent with Eq.(1.98) is given by

$$Y_I^\mu = -\varkappa_{IJ} P^{\mu\nu} \bigtriangledown_\nu \frac{\mu_J}{T} - \gamma_I P^{\mu\nu} \bigtriangledown_\nu T. \tag{1.106}$$

If we are interested in the case of a conformal system such as $\mathcal{N} = 4$ SYM, then the only contribution that should come from chemical potential and temperature is in the scale-free combination $\frac{\mu_I}{T}$, and hence $\gamma_I = 0$. The negative signs are chosen to make the divergence of the entropy current positive. This is required by second law of thermodynamics since we have dissipation. To simplify matters a little more, we consider the field theory to live in flat space so that covariant derivatives can be replaced by ordinary derivatives. The coefficient $\varkappa_{IJ}$ can be related to the thermal conductivity of the field theory in the following way. The thermal conductivity[46, 36], is defined as response to temperature gradient (which induces a heat flow and hence energy flow $T^{t\ i} \neq 0$), in the absence of any charge current i.e. $J_I^i = 0$.[14] For small u^α, the vanishing of charge current, upon using Eq.(1.97) gives

$$\rho_I u^i = \sum_{J=1}^{m} \varkappa_{IJ} \partial^i \frac{\mu^J}{T},$$

From which one obtains

$$\sum_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_J u^i = \sum_{I=1}^{m} \rho_I \partial^i \frac{\mu^I}{T}\,, \tag{1.107}$$

hence

$$u^i = \frac{1}{\sum\limits_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_J} \sum_{l=1}^{m} \rho_l \partial^i \frac{\mu^l}{T}\,. \tag{1.108}$$

Using thermodynamic relations

$$\epsilon + P = Ts + \sum_{I=1}^{m} \mu^I \rho_I, \qquad dP = sdT + \sum_{I=1}^{m} \rho_I d\mu^I, \tag{1.109}$$

[14] In our notation $i, \mu, \nu..$ are the field theory space-time indices where as I, J are the charge indices.

we get

$$\sum_{I=1}^{m} \rho_I \partial^i \frac{\mu^I}{T} = -\frac{\epsilon + P}{T^2}\partial^i T + \frac{\partial^i P}{T}. \tag{1.110}$$

After substitution this in Eq.(1.108), we get

$$u^i = -\frac{1}{\sum\limits_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_I}(\frac{\epsilon + P}{T^2})(\partial^i T - \frac{T}{\epsilon + P}\partial^i P)\,. \tag{1.111}$$

Therefore

$$T^{t\ i} = (\epsilon + P)u^i = -\frac{1}{\sum\limits_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_J}(\frac{\epsilon + P}{T})^2(\partial^i T - \frac{T}{\epsilon + P}\partial^i P). \tag{1.112}$$

In the non-relativistic case, heat flow is proportional to temperature gradient, where as in the relativistic case, in addition we have pressure gradient. The proportionality coefficient is known as thermal conductivity hence [48]

$$\kappa_T = \left(\frac{\epsilon + P}{T}\right)^2 \frac{1}{\sum\limits_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_J}. \tag{1.113}$$

Let us further note that for systems with a single conserved current[36], $\frac{1}{\rho_I \varkappa_{IJ}^{-1} \rho_J} = \frac{\varkappa}{\rho^2}$. Therefore one gets[36]

$$\kappa_T = \left(\frac{\epsilon + P}{\rho T}\right)^2 \varkappa = \left(\frac{\epsilon + P}{\rho}\right)^2 \frac{\sigma}{T}. \tag{1.114}$$

1.6.2 Kubo formula for various transport coefficients

The set of transport coefficients η, the shear viscosity, k_T, the thermal conductivity, which characterizes the hydrodynamic regime and encodes dissipation, can be related to Greens function by using Kubo formula. This is discussed below.

Shear viscosity

Let us consider field theory in flat space-time and, on it, a spatially homogeneous time dependent metric perturbation of the form [43, 49]

$$\begin{aligned} g_{ij}(t,x) &= \delta_{ij} + h_{ij}(t), \quad h_{ij} \ll 1. \\ g_{00}(t,x) &= -1, g_{0i}(t,x) = 0. \end{aligned} \tag{1.115}$$

In the rest frame, where $u^\mu = (1,0,0,0)$, the dissipative part of the energy momentum tensor Eq.(1.105) is given by

$$\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}, \tag{1.116}$$

and up to the linearized order it takes the form (using Eq.(1.104))

$$\Pi_{xy} = -\eta\partial_0 h_{xy}(t), \tag{1.117}$$

giving

$$\Rightarrow T_{xy} = \quad -\eta\partial_0 h_{xy}(t). \tag{1.118}$$

Now by going to Fourier space and comparing with Eq.(1.88) and Eq.(1.92), in the low frequency limit and at zero spatial momentum, we get

$$\begin{aligned} G^R_{xy,xy}(\omega,0) &= -i\int dtd^3xe^{i\omega t}\theta(t)\langle[T_{xy},T_{xy}]\rangle \\ &= -i\eta\omega, \end{aligned} \tag{1.119}$$

implying

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\Im G^R_{xy,xy}(\omega). \tag{1.120}$$

Thermal conductivity

Consider putting the system in a slowly varying background gauge fields (A^μ_I) which couple to conserved currents. This field will induce a current, proportional to electric field as

$$\begin{aligned} J^i_I &= \sigma_{IJ}E^i_J \\ &= \sigma_{IJ}(\partial^t A^i_J - \partial^i A^t_J), \end{aligned} \tag{1.121}$$

where the coefficients σ_{IJ} represent the electrical conductivity of the system. The field A^t_I can be identified with the chemical potential μ_I. Now comparison between Eq.(1.121) and Eq.(1.106) suggests $\sigma_{IJ} = \frac{\varkappa_{IJ}}{T}$ and hence

$$J^i_I = \frac{\varkappa_{IJ}}{T}(\partial^t A^i_J - \partial^i A^t_J). \tag{1.122}$$

In the Fourier space, at zero spatial momentum and low frequency limit, with (spatially homogeneous) time dependent background field, above equation simplifies to

$$J^i_I = i\frac{\varkappa_{IJ}}{T}\omega A^i_J. \tag{1.123}$$

Comparing with the relation $J^i_I = -G^R_{IJ} A^i_J$, that follows from linear response theory, we get

$$\begin{aligned} G_{x,x,\ IJ}(\omega,0) &= \int dtdxe^{i\omega t}\theta(t)\langle[J_{x,I},J_{y,J}]\rangle \\ &= -i\frac{\varkappa_{IJ}}{T}\omega, \end{aligned} \tag{1.124}$$

which implies[36]

$$\frac{\varkappa_{IJ}}{T} = \sigma_{IJ} = -\lim_{\omega\to 0}\frac{1}{\omega}\Im G^R_{x,x,IJ}(\omega). \tag{1.125}$$

Now the thermal conductivity can be related to above using Eq.(1.113).

Given a weakly coupled field theory, in principle we should be able to compute using perturbation technique, the transport coefficients such as $\eta, \varkappa$. However it turns out to be a difficult exercise [50, 51]. Since we are interested in the transport coefficients of strongly coupled theories, the known techniques fails to provide any meaningful results. However for certain classes of strongly coupled gauge theories such as $\mathcal{N} = 4$ SYM, we can use their gravity duals to compute transport coefficients. This is what we discuss in the next sections.

1.7 Computation of real time correlators from gauge/gravity duality

Gauge/gravity duality allows us to compute gauge theory correlators using classical supergravity computations which are other wise hard to compute. We have discussed in subsection (1.3.2) how AdS/CFT can be used to compute the Euclidean correlator. However, for many purposes such as computation of transport coefficients, we need real time correlators. One might argue that by doing analytic continuation of two point Euclidean correlators, one can find retarded Green's function. The relation between the retarded and the Euclidean two-point functions in momentum space is given by

$$G_R(\omega, \overrightarrow{k}) = G_E(-i(\omega + i\epsilon), \overrightarrow{k}). \tag{1.126}$$

However in most cases, the Euclidean correlation functions can only be found numerically. Consequently analytic continuation to Lorentzian signature becomes difficult. In particular, the problem that one faces in order to extract the hydrodynamic limit ($\omega,\ k \to 0$) of real time correlators from Euclidean ones is that one needs to perform analytic continuation from a discrete set of frequencies (Matsubara frequencies) having lowest value $\omega = 2\pi T$ to real and small frequencies such that $\omega \ll 2\pi T$. Thus, it is important to be able to compute real-time correlation functions directly. A working prescription for the computation of real time correlator was given in [44, 1] and later on, in [52], it was established rigorously. An alternate way to compute the real time correlator was given in [4]. Here we summarize both the ways of computation.

1.7.1 Son, Starinets prescription for computing real time correlators

In this subsection we shall briefly describe the recipe for computing the real time correlator prescribed first by Son and Starinets in [44, 1]. Suppose we are interested in computing the retarded two point correlator

$$G(x-y) = -i\langle T\ \Theta(x)\ \Theta(y)\rangle \tag{1.127}$$

where Θ is some scalar operator in the gauge theory side, which is dual to some massless scalar field (ϕ) in the gravity side. The boundary value of ϕ acts as a source and we have

$$S \to S + \int \phi_0 \Theta. \tag{1.128}$$

For the time being we shall consider a generic black hole background given by

$$ds^2 = g_{tt}(z)dt^2 + g_{zz}(z)dz^2 + g_{xx}(z)\sum_{i=1}^{d-1}(dx^i)^2, \tag{1.129}$$

where z is the radial coordinate. The action for this scalar field in this background is given by

$$S = -\frac{1}{2}\int d^{d-1}x dt \int_{z=0}^{z_H} dz\sqrt{-g}[g^{zz}(\partial_z\phi)^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi], \tag{1.130}$$

where z_H is the location of horizon. The equation for scalar field which follow from this action is

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z\phi) + g^{\mu\nu}\partial_\mu\partial_\nu\phi = 0, \tag{1.131}$$

where μ, ν runs in the field theory directions. The above equation needs to be solved with the boundary condition

$$\lim_{z\to 0}\phi(z) \to \phi_0. \tag{1.132}$$

In momentum space we can write,

$$\phi(z,x) = \int \frac{d^dk}{(2\pi)^d} e^{ik.x} f_k(z)\phi_0(k), \tag{1.133}$$

with $f_k(z \to 0) = 1$, which upon using Eq.(1.131) reduces to

$$\frac{1}{\sqrt{-g}}\partial_z(\sqrt{-g}g^{zz}\partial_z f_k) + g^{\mu\nu}\partial_\mu\partial_\nu f_k = 0. \tag{1.134}$$

In order to get the retarded correlator, we also need to put incoming wave boundary condition at the horizon (this is natural, since classically we do not expect things to come out of the horizon). The on-shell action therefore reduces to

$$S = -\frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\phi_0(k)F(k,z)\phi_0(k)\Big|_{z\to 0}^{z_H}, \tag{1.135}$$

where

$$F(k,z) = \sqrt{-g}g^{zz}f_{-k}(z)\partial_z f_k(z). \tag{1.136}$$

Now if we differentiate the above action with respect to boundary value ϕ_0, we get

$$G(k) = -\frac{1}{2}F(k,z)\Big|_{z\to 0}^{z_H} - \frac{1}{2}F(-k,z)\Big|_{z\to 0}^{z_H}. \tag{1.137}$$

Now using Eq.(1.134) and the fact that, $f_k^* = f_{-k}$, we get

$$\partial_z \Im(F(k,z)) = 0, \tag{1.138}$$

so we can evaluate imaginary part of F at any radius. Consequently the imaginary part of Greens function in Eq.(1.137) vanishes. To circumvent this problem, in [44, 1], the following proposal was put forward

$$\begin{aligned} G^R(k) &= -F(k, z\to 0) \\ &= -\sqrt{-g}g^{zz}f_{-k}(z)\partial_z f_k(z). \end{aligned} \tag{1.139}$$

In order to verify that the prescription works, in[44], retarded Greens function was computed in theories where it is known from other methods. Further the Eq.(1.139) was established rigorously in [52], using connection between closed time path formulation of real time QFT with dynamics of whole Penrose diagram of black hole. Although we have shown here the computations for a scalar field, above prescription can be followed for other fields as well. We summarize this in the following.

Suppose we are interested in computing retarded correlator of some operator $\mathcal{O}$ whose dual field in the gravity side is Ψ.

1. Extract out the coefficient of kinetic term $A(z)$ from the classical action of field Ψ written in the gravity side. $A(z)$ is defined as

$$S_{cl} = \frac{1}{2}\int dz d^d x A(z)(\partial_z \Psi)^2 + ... \tag{1.140}$$

2. Find solution to the equation of motion with in-going boundary condition at the horizon and a constant value $\Psi(z,k) \to \Psi_0(k)$ at the boundary $z \to 0$. Let us assume the solution in the Fourier space has the form

$$\Psi(z,k) = f_k(z)\Psi_0(k), \tag{1.141}$$

where $f_k(z=0) = 1$.

3. The prescription then tells that the retarded Greens function is

$$G^R(k) = A(z) f_{-k} \partial_z f_k(z)\Big|_{z\to 0}. \tag{1.142}$$

We end this discussion with the computation of a few transport coefficients using prescription mentioned above.

<u>Example : The shear viscosity</u>

First we compute the shear viscosity of $\mathcal{N} = 4$ SYM at finite temperature T. The metric Eq.(1.57) in the coordinate system $u = (\frac{z}{z_H})^2$ with $z_H = \frac{1}{\pi T}$, can be written as

$$ds^2 = \frac{(\pi T L)^2}{u}(-f(u)dt^2 + dx^2 + dy^2 + dz^2) + \frac{L^2}{4u^2 f(u)}du^2, \tag{1.143}$$

where $f(u) = 1 - u^2$, with $u = 1$ being horizon and $u = 0$ is the boundary. The entropy for this case is given by

$$s = \frac{\pi^2}{2} N^2 T^3. \tag{1.144}$$

To compute shear viscosity, we need to take the background perturbation of the form

$$g^x_y \to g^x_y + \phi, \tag{1.145}$$

where $\phi = h^x_y$. The action and the equation of motion for ϕ is that of a massless scalar field in the background Eq.(1.143). With appropriate normalization, the action is given by

$$S = -\frac{1}{32\pi G_5}\int d^3x dt \int du \sqrt{-g}[g^{uu}(\partial_u\phi)^2 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi]. \tag{1.146}$$

In the Fourier space we write

$$\phi(t, x^i, u) = \int \frac{d^4k}{(2\pi)^4} e^{ik.x}\phi_k(u)\phi_0(k). \tag{1.147}$$

The object that we want to compute is T_{xy} correlator which is related to shear viscosity as discussed in the previous section. In order to find the Greens function, we now need to solve $\phi_k(u)$. The equation of motion takes the form

$$\phi_k^{''} - \frac{1+u^2}{uf(u)}\phi_k^{'} + \frac{\omega^2 - k^2 f}{(2\pi T)^2 uf^2}\phi_k = 0, \tag{1.148}$$

where prime denotes derivative with respect to radial coordinate u. This equation can not be solved for all values of $\omega,\ q$. However in the limit $\frac{\omega}{T}, \frac{q}{T} \ll 1$, we can write a series solution in $\frac{\omega}{T}, \frac{q}{T}$. There are two solution which are complex conjugate to each other, which represents incoming and out going solutions at the horizon ($u = 1$). The incoming solution at $u = 1$ can be written as

$$\phi_k = (1-u)^{-i\frac{\omega}{4\pi T}}\Big(1 - i\frac{\omega}{4\pi T}\ln\frac{1+\mathrm{u}}{2} + \mathrm{O}(\omega^2, \mathrm{q}^2)\Big). \tag{1.149}$$

Now using the prescription as summarized in Eq.(1.142) we get

$$G^R_{xy,xy}(\omega) = -\frac{\pi N^2 T^3}{8} i\omega, \tag{1.150}$$

where we have used the relation $G_5 = \frac{\pi L^3}{2N^2}$ as given in Eq.(1.53). Now the Kubo's formula for η, immediately gives

$$\eta = \frac{\pi}{8} N^2 T^3. \tag{1.151}$$

So we see that

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{1.152}$$

Example : electrical conductivity

As we have already discussed, if we are interested in computing current-current correlator in $\mathcal{N} = 4$ SYM, we then need to analyze linearized perturbation of $U(1)$ gauge field A_μ on the dual gravity back ground. The five dimensional Maxwell action in this background can be written as [44, 1]

$$S = -\frac{N^2}{16\pi^2 L}\int d^5x \frac{1}{4}\sqrt{-g} F_{\mu\nu}F^{\mu\nu}, \tag{1.153}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The gauge fields A_μ obey Maxwell equation of the form

$$\partial_\nu[\sqrt{-g} g^{\nu\alpha} g^{\beta\gamma} F_{\alpha\gamma}] = 0. \tag{1.154}$$

We choose the gauge where radial component of the gauge field is zero ($A_u = 0$). As before, we work in the Fourier space where,

$$A_\mu(t, x, u) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + i\vec{q}.\vec{x}} A_\mu(\omega, q, u). \tag{1.155}$$

For our purpose we choose perturbation to be spatially homogeneous so that we can set $q = 0$. Suppose we are interested in computing $\langle J^x J^x \rangle$ correlator, then we should focus on A_x component of the gauge field in the bulk. The spatial component of the gauge field A_x obeys the equation

$$A_x^{''} + \frac{f^{'}}{f} A_x^{'} + \frac{1}{uf^2} \frac{\omega^2}{4\pi^2 T^2} A_x = 0, \tag{1.156}$$

where prime denotes derivative with respect to radial coordinate u. Upto linear order in ω, the solution to Eq.(1.156), takes the form

$$A_x(\omega, u) = A_x^0 (1-u)^{-i\frac{\omega}{4\pi T}} \Big(1 + i\frac{\omega}{4\pi T} \ln\frac{1+\mathrm{u}}{2} + \mathrm{O}(\omega^2, \mathrm{q}^2)\Big). \tag{1.157}$$

Finally, following the same procedure as for shear viscosity along with the use of appropriate Kubo's formula, we conclude that the response function is given by

$$\sigma = \frac{N^2 T}{16\pi}. \tag{1.158}$$

Actually to define the above response function as the electrical conductivity of the SYM, we need to first gauge the global $U(1)$ symmetry of SYM with small electromagnetic gauge coupling (say e). This implies, the current operators are multiplied with a factor of e that is $J_\mu \to eJ_\mu$ and hence there will be a factor of e^2 in the two point current correlator [53, 3]. However we shall drop that extra factor of e^2 from our discussion. Since e is small, to the leading order in e, the effect of gauging can be neglected and response can be computed from original theory. For details see[53, 18].

1.7.2 Iqbal-Liu prescription for computing real time correlator

It turns out that the Son-Starinets prescription can be reformulated in terms of boundary values of the canonical momenta of the bulk field by treating the AdS radial direction as time. This reformulation has various advantages. For example, many of the boundary transport coefficients can be expressed in terms of quantities evaluated at the horizon. Universality of transport coefficients can therefore be understood via certain universal behavior of the black hole horizon. According to AdS/CFT, the one point function is defined as

$$\langle\Theta\rangle\Big|_{\phi_0} = \frac{\delta}{\delta\phi_0}S_{cl}[\phi_0], \tag{1.159}$$

where ϕ_0 is the boundary value of the massless scalar field dual to operator Θ. Let us note that, in the classical mechanics derivative of an on-shell action with respect to boundary value of a field is simply equal to the canonical momentum conjugate to the field evaluated at the boundary. For example

$$S = \int_{X(t_0)}^{X(t_f)} dt\ \mathcal{L}, \tag{1.160}$$

and

$$\frac{\delta}{\delta X(t_f)}S_{cl} = P(t_f), \tag{1.161}$$

where $P(t)$ is the momentum conjugate to $X(t)$. For the case of scalar field ϕ, we can write

$$\begin{aligned} < \Theta(x) >_\phi &= \frac{\delta}{\delta\phi_0}S_{cl}[\phi_0] \\ &= \Pi(z,x)\Big|_{z\to 0}, \end{aligned} \tag{1.162}$$

where $\Pi(z,x)$ is the canonical momentum conjugate to ϕ with respect to radial z foliation. Equivalently in the Fourier domain, we can write

$$< \Theta(k) >= \Pi(z,k)\Big|_{z\to 0}. \tag{1.163}$$

In the domain of linear response, we have

$$< \Theta >= -G^{\mathrm{ret}}(\omega,k)\lim_{z\to 0}\phi(k,z), \tag{1.164}$$

This implies

$$G^{\mathrm{ret}} = -\frac{\Pi(z,k)}{\phi(z,k)}\Big|_{z\to 0}, \tag{1.165}$$

which upon using Eq.(1.93) gives,

$$\chi = \lim_{\omega,k\to 0}\frac{\Pi(z,k)}{i\omega\phi(z,k)}\Big|_{z\to 0}. \tag{1.166}$$

For illustration, let us consider the case of massless scalar field propagating in the background

$$ds^2 = g_{tt}(z)dt^2 + g_{zz}(z)dz^2 + g_{xx}(z)\sum_{i=1}^{d-1}(dx^i)^2, \tag{1.167}$$

where z is the radial coordinate. We have assumed full rotational symmetry in x^i directions so that $g_{ij} = g_{xx}\delta_{ij}$, where i, j run over all the indices except z, t. We also assume that metric components depend only on radial coordinate. We assume that the metric has an event horizon, where g_{tt} has a first order zero and g_{zz} has a first order pole. We also require that all the other metric components are finite as well as non vanishing at the horizon. The action for the scalar field is same as given in Eq.(1.130). The canonical momenta for this case is,

$$\Pi(z,k) = -\sqrt{-g}g^{zz}\partial_z\phi. \tag{1.168}$$

Using Eq.(1.131), we can compute

$$\partial_z\Pi(z,k) = \sqrt{-g}k^2\phi(z,k). \tag{1.169}$$

In the limit $k \to 0$, we have both

$$\lim_{k\to 0}\partial_z\Pi(z,k) = 0, \qquad \lim_{k\to 0}\partial_z(\omega\phi(z,k)) = 0. \tag{1.170}$$

So in the limit $k \to 0$, both $\omega\phi(z)$ and $\Pi(z,k)$ is independent of z, which implies

$$\chi(z \to 0) = \chi(z \to z_H). \tag{1.171}$$

In other words, the radial evolution of response function χ, which we refer as flow, is trivial. This can be used to show, in particular that response function of gauge theory dual to some gravity theory, can be expressed in terms of geometrical quantities evaluated at the horizon. Let us note that, had we considered the massive scalar field of mass m in the bulk, then

$$\lim_{k\to 0}\partial_z\Pi(z,k) \sim m^2\phi(k,z) \neq 0, \tag{1.172}$$

which implies that there is a non-trivial flow of the transport coefficient if we go from horizon to boundary. Hence evaluating response function at the horizon will not give same result as boundary response function. One such example is, the computation of electrical conductivity at finite chemical potential. We shall discuss this issue in later chapters. Following [4] and stretching the previous discussion a bit more, one can define response function at any radial position z, through

$$\chi(z) = \lim_{\omega,k\to 0}\frac{\Pi(z,k)}{i\omega\phi(z,k)} \tag{1.173}$$

which in the limit $z \to 0$ gives AdS/CFT results. It is possible to compute Eq.(1.173) at the horizon and then, by solving flow equation, we can relate it with AdS/CFT result which is evaluated at the boundary. This leads to a connection between dual gauge theory with the fictitious fluid living on the horizon. This goes by the name *membrane paradigm* (for a brief discussion on *membrane paradigm*, see Appendix). Following the Iqbal-Liu proposal, we end this section with the computations of two transport coefficients. Both these two, however, have trivial flow from the horizon to the boundary AdS.

Example : The shear viscosity

In order to find out the shear viscosity, we need to look at fluctuation $\phi = h^x_y$ of the metric field g^x_y, where x, y are the field theory directions. As previously discussed, the shear fluctuation mode decouples from rest of the fluctuations and behaves as massless scalar field with the action Eq.(1.146). As before we shall work with spatially homogeneous fluctuations, so that we can set $\overrightarrow{q}$, the spatial part of k zero and we shall also work in the limit where $\omega \to 0$. Evaluating the canonical momentum, we get

$$\begin{aligned} \Pi(z \to 0, \omega \to 0, q = 0) &= \Pi(z \to z_H, \omega \to 0, q = 0) \\ &= \frac{\sqrt{-g}}{16\pi G} \frac{1}{\sqrt{-g_{zz} g_{tt}}}\Big|_{z_H} i\omega\phi(z_H, \omega \to 0, q = 0), \end{aligned} \tag{1.174}$$

where G is the Newtons constant and in the second line of above equation we have used in-going boundary condition at the horizon, which states

$$\lim_{z \to z_H} \frac{d}{dz}\phi(z) = -i\omega \lim_{z \to z_H} \sqrt{\frac{g_{zz}}{-g_{tt}}}\phi(z) + \mathcal{O}(\omega^2). \tag{1.175}$$

Now using definition of response function we get shear viscosity η to be

$$\eta = \Big[\frac{\sqrt{-g}}{16\pi G} \frac{1}{\sqrt{-g_{zz} g_{tt}}}\Big]_{z_H}. \tag{1.176}$$

Entropy density of the black hole is area of the horizon divided by $4G$, which gives

$$s = \frac{1}{4G} \frac{\sqrt{-g}}{\sqrt{-g_{zz} g_{tt}}}\Big|_{z_H}. \tag{1.177}$$

Now the shear viscosity to entropy density ratio is given by

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{1.178}$$

This result coincides with Eq.(1.152), which was computed for particular back ground dual to $\mathcal{N} = 4$ SYM. So we already see, for large class of gauge theories with gravity dual having metric of the form Eq.(1.167) subject to certain constraint, the shear viscosity to entropy density ratio is $\frac{1}{4\pi}$ and is universal.

The origin of Eq.(1.175), perhaps require some elaboration. Near the horizon, the metric can be written as

$$g_{tt} = -a(z_H - z), \quad g_{zz} = \frac{b}{z_H - z}. \tag{1.179}$$

In this region, at vanishing spatial momentum $\overrightarrow{q} = 0$, the Eq.(1.134) takes the form

$$\sqrt{\frac{a}{b}}(z_H - z)\partial_z(\sqrt{\frac{a}{b}}(z_H - z)\partial_z\phi) + \omega^2\phi = 0, \tag{1.180}$$

which has solutions of the form

$$\phi \propto e^{-i\omega(t\pm x)}, \quad dx = \sqrt{\frac{g_{zz}}{-g_{tt}}}dz. \tag{1.181}$$

The in falling boundary condition on the horizon picks up the positive sign in the exponent. This implies, the solution near the horizon takes the form

$$\phi \propto e^{-i\omega v}, \quad dv = dt + \sqrt{\frac{g_{zz}}{-g_{tt}}}dz. \tag{1.182}$$

So solution can only depend on the non singular combination v. This gives, near the horizon

$$(\partial_z - \sqrt{\frac{g_{zz}}{-g_{tt}}}\partial_t)\phi = 0. \tag{1.183}$$

This, in turn, means

$$\lim_{z\to z_H}\frac{d}{dz}\phi(z) = -i\omega \lim_{z\to z_H}\sqrt{\frac{g_{zz}}{-g_{tt}}}\phi(z) + \mathcal{O}(\omega^2). \tag{1.184}$$

<u>Example : electrical conductivity</u>

Let us consider a Maxwell field propagating in the uncharged black brane background. The Maxwell field action is

$$S = -\int d^{d+1}x \ \sqrt{-g}\frac{1}{4g^2_{d+1}(z)}F_{MN}F^{MN}, \tag{1.185}$$

where $g^2_{d+1}(z)$ in general is a z dependent gauge coupling, where background value of gauge field is zero and we take only nonzero component to be A_x. Here again one can show that equation for A_x is same as that for massless field with a substitution

$$\sqrt{-g} \to \frac{1}{g^2_{d+1}(z)}\sqrt{-g}g^{xx}. \tag{1.186}$$

Now using

$$\langle J_x(k)\rangle = \sigma(k)\lim_{z\to 0} E_x(z,k), \quad E_x = -i\omega A_x, \tag{1.187}$$

we get

$$\sigma = \frac{J(z_H)}{-i\omega A_x(z_H)} = \Big[\frac{1}{g^2_{d+1}(z)}\frac{\sqrt{-g}}{\sqrt{-g_{zz}g_{tt}}}g^{xx}\Big]_{z_H}. \tag{1.188}$$

Applying this for $\mathcal{N}=4$ SYM (dual gravity background is the AdS_5 Schwarzschild black hole) we get

$$\sigma = \frac{1}{g_5^2}(\pi LT) \tag{1.189}$$

which is same as expression in Eq.(1.158) provided we make the identification $\frac{1}{g_5^2}=\frac{N^2}{16\pi^2 L}$.

Let us note that, in the above computations, we have not assumed any particular gravity background. Rather, we have only imposed few generic constraints on the gravity background. So, above results are applicable to the gravity dual of $\mathcal{N}=4$ SYM at finite temperature as well as any other gauge theory at finite temperature with a gravity dual and a few generic constraints. In fact we shall push these ideas further and present computations of transport coefficients for the backgrounds where Iqbal-Liu prescription might not be readily applicable. Our main focus will again be on finding features which are independent of details of these particular models. Though in some part of the thesis, we shall work with gravity backgrounds for which dual gauge theory might not always be well defined, we hope our results might be applicable to situations where it is well defined. With this brief introduction, in the next section we discuss the plan of the thesis.

1.8 Plan of the thesis

The plan of the thesis is as follows. In the next chapter, we compute electrical conductivity in the presence of one and more chemical potentials for several models [48, 54]. What we observe is that, in the presence of multiple chemical potentials, there is a nontrivial mixing between current operators which, from the bulk point of view, can be understood to be arising because of interactions through graviton. We find that the boundary electrical conductivity takes a universal form in the presence of chemical potential for a large class of black branes which include R-charged black branes in various dimensions in asymptotically AdS spaces as well as charged Dp branes in various dimensions. We also observe that the boundary conductivity is related to horizon conductivity by thermodynamic quantities. We further note for Lifshitz like black branes, the form of conductivity is different than one observed for other examples. Subsequently, we focus on understanding relation between the conductivity of the fluid described by membrane paradigm[15]. In order

[15] In the appendix A, we give a brief account of the membrane paradigm.

to do that, we compute conductivity at arbitrary cut-off out side the horizon for gauge theory dual to charged asymptotically AdS black hole and show that there is a smooth interpolation between conductivity at the horizon and at the boundary.

In the third chapter, we provide a proof that under general assumptions in the gravity side together with precise condition on the bulk stress tensor, the electrical conductivity is the same as one we observed in the second chapter[55]. This immediately explains as to why the Lifshitz like theories does not have the form of conductivity as proposed since the bulk stress tensor does not satisfy the constraint. In this chapter we also give a general form of conductivity matrix encoding the mixing between current operators, in the presence of multiple chemical chemical potential.

In the fourth chapter, we compute thermal conductivities for various field theories with gravity duals and observe that the thermal conductivity to the shear viscosity ratio is independent of number of chemical potentials. This observation together with observation that at zero chemical potential the above ratio remains unchanged, lead us to conjecture that it is universal. Further, for CFT's with a gravity dual, using thermodynamic relations, one can express the above ratio in terms of central charges of the dual conformal field theories [56]. We also observe that the thermal conductivity to the viscosity ratio is again universal for non conformal theories. All these observations give us a way to express the conductivity solely in terms boundary thermodynamic variables.

We then turn our attention to study of transport coefficients of gauge theories at zero temperature which corresponds to extremal black hole in the bulk, in chapter five. We find that, for several examples, the form of conductivity at zero temperature is same. Under the general assumption that extremal black brane has double pole structure at the horizon together with requirement that bulk stress tensor satisfies same constraint as non extremal cases, we show that form of electrical conductivity is universal. We also provide a simple proof that shear viscosity to entropy density ratio is $\frac{1}{4\pi}$ even at zero temperature.

In chapter six, we give a brief summary of the results presented in this thesis. In the appendix A, we give a brief account of membrane paradigm, we also provide details of R-charged black holes in various dimensions in appendix B.

2

Electrical conductivity at finite chemical potential

2.1 Introduction

Based on several examples, we find out a general expression for electrical conductivity for gauge theories in the presence of chemical potentials having a gravity dual. It turns out that the electrical conductivity can be determined in terms of geometrical quantities evaluated at the horizon and thermodynamic quantities.

At finite temperature and at large length scales, an interacting QFT is described by hydrodynamics. In the gravity side, finite temperature amounts to having a black hole and the long wave length physics of the field theory is governed by the near horizon physics of the black hole. This idea was employed in [4] to show that, in the low frequency limit, the linear response of the boundary theory is captured completely by the near horizon physics. In [4], the authors studied the transport coefficients which correspond to the massless modes in the bulk, resulting in trivial flow from horizon to boundary. This in turn, gave an equality between the boundary and the horizon transport coefficients. So when there is a nontrivial flow from the horizon to the boundary (like massive bulk modes), horizon physics will no longer be able to capture the whole low frequency AdS/CFT response. Calculation of electrical conductivity in the presence of non-zero chemical potential is one such example where corresponding mode in the bulk shows a non trivial flow from horizon to boundary. These flows are in general governed by complicated differential equations (if more than one charge is present they are coupled as well) and, a priori, there is no reason that electrical conductivity for different theories will show some universal features. In spite of this, as we shall find, electrical conductivity does show some universal features.

This chapter is structured as follows. In section 2 we discuss the effective action approach in the gravity side to compute electrical conductivity following [48, 57]. We set it up in way that allows us to study different gravity backgrounds in a unified way. In section 3, we take up several examples such as R-charge black hole in $4, 5$ and 7 dimensions. We compute electrical conductivity for single charge case as well as multiple charge cases. For multiple charge case, we observe non-trivial mixing between current operators. In section 4 we demonstrate the relation between horizon and boundary conductivity, based on these examples. In section 5 and 6, we check that the relation continues to hold for Reissner-Nordstrom AdS black hole in arbitrary dimension and for black Dp-branes, which in general corresponds to non-conformal gauge theory. However in section 7, we check that Lifshitz like black holes do not satisfy the same relation. In section 8, we study radial evolution of electrical conductivity. We end this chapter with a discussion of our results.

2.2 Holographic computation of electrical conductivity at finite chemical potential: The perturbation equation

We start with a gauge theory at finite temperature with multiple chemical potentials with a gravity dual. In the gravity side, this gauge theory correspond to black branes charged under multiple $U(1)$ gauge fields. In the boundary theory one has current operator dual to every $U(1)$ bulk gauge field. At equilibrium, there are no mixing between the different current operators. When perturbed away from equilibrium, in general there might be nontrivial mixing between them. This mixing arises naturally in the context of gauge/gravity duality due to the presence of graviton in the bulk which induces interaction between different gauge fields modes, hence nontrivial mixing. To understand that, we consider the bulk action of the form

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}(R - \frac{1}{4}G_{IJ}F^I_{\mu\nu}F^{\mu\nu\, J} + ...), \tag{2.1}$$

where dots contains other bulk fields such as neutral scalar fields. The metric that we take is of the form

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)\sum_{i=1}^{d-1}(dx^i)^2, \tag{2.2}$$

where r is the radial coordinate. We have assumed full rotational symmetry in x^i directions so that[16] $g_{ij} = g_{xx}\delta_{ij}$, where i, j run over all the indices except r, t. We also assume that metric components depend on radial coordinate only. We shall work with the metric which has an event horizon[17], where g_{tt} has a first order zero and g_{rr} has a first order pole[18]. We also assume that all the other metric

[16]Let us note that, we are using the notation where $g_{\mu\nu}(r) \equiv g_{\mu\nu}$.
[17]For charged black holes, there exists inner horizons also.
[18]Therefore it excludes extremal black holes

components are finite as well as non vanishing at the horizon. The boundary of the space time is at $r = \infty$. The gauge coupling G_{IJ} may be constant or in general can be a function of r. The constant κ is related to Newtons constant. Maxwell equation can be written as

$$\partial_\mu \left(\sqrt{-g} G_{IJ} F_J^{\nu\mu} \right) = 0. \tag{2.3}$$

If we consider G_{IJ} to be diagonal and only $A_t(r)$ component to be non zero, we can define charge density to be,

$$\rho_I = \frac{1}{2\kappa^2} \sqrt{-g} G_{II} g^{rr} g^{tt} F_{rt}^I. \tag{2.4}$$

Since our aim is to compute the electrical conductivity using Kubo formula, it is sufficient to consider perturbations in the tensor (metric) and the vector (gauge fields) modes around the black hole solution and keep other fields such as scalars unperturbed. Therefore we consider perturbation of the form

$$g_{\mu\nu} = \mathbf{g}_{\mu\nu}^{(0)} + h_{\mu\nu} \ , \qquad A_\mu^I = \mathbf{A}_\mu^{I(0)} + \mathcal{A}_\mu^I \ . \tag{2.5}$$

where $\mathbf{g}_{\mu\nu}^{(0)}$ and $\mathbf{A}_\mu^{I(0)}$ are background metric and gauge fields. In order to determine electrical conductivity it is enough to consider perturbations in (tx^1) and (x^1x^2) component of the metric tensor and x^1 component of the gauge fields. Moreover one can choose the perturbations to depend on radial coordinate r, time t and one of the spatial coordinates say x^2. A convenient ansatz, with the above restrictions in mind, is

$$\begin{aligned} h_{tx^1} = \mathbf{g}_{0x^1x^1}\ T(r)\ e^{-i\omega t + iqx^2}, \qquad & h_{x^2x^1} = \mathbf{g}_{0xx}\ Z(r)\ e^{-i\omega t+iqx^2}, \\ \mathcal{A}_{x^1}^I \ = \ \phi_I(r)\ \ e^{-i\omega t+iqx^2}. & \end{aligned} \tag{2.6}$$

Here ω and q represent the frequency and the momentum in x^2 direction respectively and we set perturbations in the other components to be equal to zero. Next step is to find linearized equations which follow from the equations of motion. It turns out that at the level of linearized equation and at zero momentum limit, metric perturbation $Z(r)$ decouple from the rest [57, 58]. The linearized equation that we get are of the form

$$\frac{d}{dr}(N_I \frac{d}{dr}\phi_I(r)) - \omega^2 N_I\ g_{rr} g^{tt} \phi_I(r) + N_I g_{xx} g^{tt} \frac{d}{dr}(g^{xx} h_{x^1t}) = 0. \tag{2.7}$$

with

$$N_I = \sqrt{-g} G_{II} g^{xx} g^{rr}, \tag{2.8}$$

and

$$\frac{d}{dr}(g^{xx} h_{xt}) = \sum_{J=1}^{m} G_{JJ} F_{rt}^J \phi_J. \tag{2.9}$$

We observe that we can use Eq.(2.9) in Eq.(2.7) to get an equation only in terms of gauge field fluctuations. Upon substitution we get

$$\frac{d}{dr}(N_I \frac{d}{dr}\phi_I(r)) - \omega^2 N_I \; g_{rr} g^{tt} \phi_I(r) + \sum_{J=1}^{m} M_{IJ}\phi_J(r) = 0. \tag{2.10}$$

with

$$M_{IJ} = F^I_{rt}\sqrt{-g} G_{II} g^{xx} g^{rr} g^{tt} G_{JJ} F^J_{rt}. \tag{2.11}$$

Let us note that $M_{IJ} = M_{JI}$.

Following [48], we now write down the effective action which reproduces the Eq.(2.10) and extract out the expression for electrical conductivity using Kubo formula.

2.2.1 Effective action and expression for conductivity

The electrical conductivity is usually computed from current-current correlator[19]

$$\begin{aligned} \lambda &= -\lim_{\omega\to 0} \frac{G_{xx}(\omega, q=0)}{i\omega} \\ &= \lim_{\omega\to 0} \frac{1}{2\omega} \int_{-\infty}^{\infty} dt \;\; e^{-i\omega t} \int d\vec{x} \langle [J_x(t,\vec{x}), J_x(0,\vec{0})] \rangle. \end{aligned} \tag{2.13}$$

The current-current correlator can be computed by taking second derivative of effective action which reproduces the Eq.(2.10) with respect to boundary fields [44, 1]. The expression for electrical conductivity can formally be written as $\lambda = i\sigma_0 + \sigma$.

[19] If there is more than one conserved current then one can define conductivity matrix using

$$\begin{aligned} \lambda_{ij} &= -\lim_{\omega\to 0} \frac{G^{ij}_{xx}(\omega, q=0)}{i\omega} \\ &= \lim_{\omega\to 0} \frac{1}{2\omega} \int_{-\infty}^{\infty} dt \;\; e^{-i\omega t} \int d\vec{x} \langle [J^i_x(t,\vec{x}), J^j_x(0,\vec{0})] \rangle, \end{aligned} \tag{2.12}$$

where indices, i, j are for different gauge fields, for which the currents are defined.

- $\sigma(=\Re(\lambda))$**:** In order to determine the real part of the conductivity (σ), we follow [48]. Effective action [20] can be written as

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int\frac{d^dq}{(2\pi)^d}dr\Big[\frac{1}{2}\sum_{I=1}^{m}N_I(r)\frac{d}{dr}\phi_I(r,\omega,q)\frac{d}{dr}\phi_I(r,-\omega,-q) \\ &+ \frac{1}{2}\sum_{I,J=1}^{m}M_{IJ}(r)\phi_I(r,\omega,q)\phi_J(r,-\omega,-q)\Big]. \end{aligned} \tag{2.14}$$

Boundary action is given by

$$\begin{aligned} S_\epsilon &= \lim_{r\to\infty}\frac{1}{2\kappa^2}\int\frac{d^dq}{(2\pi)^d}\left(\frac{1}{2}\sum_{I=1}^{m}N_I(r)\frac{d}{dr}\phi_I(r,\omega,q)\phi_I(r,-\omega,-q)\right) \\ &= \int\frac{d^dq}{(2\pi)^d}\sum_{I\geq K,I,K=1}^{m}\phi_I^0(\omega,q)\mathcal{F}_{IK}(\omega,q)\,\phi_K^0(-\omega,-q). \end{aligned} \tag{2.15}$$

where the boundary value of the field $\phi_I(r)$ is $\phi_I^0(\omega,q)$. Next, the retarded correlators are given by

$$G^R = \begin{cases} -2\mathcal{F}_{JK}(\omega,q)\,, & J=K, \\ -\mathcal{F}_{JK}(\omega,q)\,, & J\neq K. \end{cases} \tag{2.16}$$

The expression for diagonal and off diagonal parts of the conductivity can be written as

$$\begin{aligned} \sigma_{II} &= -\lim_{\omega\to0}\frac{\Im\Big(G^R(\omega,q=0)\Big)}{\omega} \\ &= \frac{2\,\Im\Big(\mathcal{F}_{II}(\omega\to0,q=0)\Big)}{\omega}, \end{aligned} \tag{2.17}$$

[20] One can obtain this effective action, starting from the action written in Eq.(2.1) and evaluating it to the quardratic order in fluctuations ϕ, T and using Eq.(2.9). Let us note that, in general Eq.(2.1) includes other parts such as contributions coming from matter fields other than the gauge fields, which are denoted by dots. However, they does not play any role in evaluating effective action.

and

$$\sigma_{IJ} = -\lim_{\omega\to 0}\frac{\Im\Big(G^R(\omega, q=0)\Big)}{\omega} = \frac{\Im\Big(\mathcal{F}_{IJ}(\omega\to 0, q=0)\Big)}{\omega}, \tag{2.18}$$

respectively.

In order to find out $\Im\Big(\mathcal{F}\Big)$, we need to compute,

$$\Im\Big[\lim_{r\to\infty}\frac{1}{2\kappa^2}\int\frac{d^dq}{(2\pi)^d}\Bigg(\frac{1}{2}\sum_{I=1}^{m}N_I(r)\frac{d}{dr}\phi_I(r,\omega,q)\phi_I(r,-\omega,-q)\Bigg)\Big]. \tag{2.19}$$

Now

$$\begin{aligned}&\frac{d}{dr}\Im\Big(\sum_{I=1}^{m}N_I(r)\frac{d}{dr}\phi_I(r,\omega,q)\phi_I(r,-\omega,-q)\Big)\\ &=\Im\Big[\sum_{I=1}^{m}\frac{d}{dr}(N_I(r)\frac{d}{dr}\phi_I(r,\omega,q))\phi_I(r,-\omega,-q)\\ &+\sum_{I=1}^{m}N_I(r)\frac{d}{dr}\phi_I(r,\omega,q)\frac{d}{dr}\phi_I(r,-\omega,-q)\Big].\end{aligned} \tag{2.20}$$

Using (2.10), right hand side of above equation reduces to

$$\Im\Big[-\sum_{I,J=1}^{m}M_{IJ}(r)\phi_I(r,\omega,q)\phi^J(r,-\omega,-q)+\sum_{I=1}^{m}N_I(u)\frac{d}{dr}\phi_I(r,\omega,q)\frac{d}{dr}\phi_I(r,-\omega,-q)\Big], \tag{2.21}$$

which is equal to zero since the quantity in the bracket is real. Then (2.18) can as well be calculated at the horizon i.e. at $r = r_h$. This simplifies calculations significantly. Regularity at the horizon implies

$$\lim_{r\to r_h}\frac{d}{dr}\phi_I(r) = -i\omega\lim_{r\to r_h}\sqrt{-\frac{g_{rr}}{g_{tt}}}\phi_I(r)+\mathcal{O}(\omega^2). \tag{2.22}$$

Hence (2.18) reduces to

$$\Im\Big[-i\omega\lim_{r\to r_h}\frac{1}{2\kappa^2}\int\frac{d^dq}{(2\pi)^d}\sqrt{-\frac{g_{rr}}{g_{tt}}}\Bigg(\frac{1}{2}\sum_{I=1}^{m}N_I(r)\phi_I(r,\omega,q)\phi_I(r,-\omega,-q)\Bigg)\Big]. \tag{2.23}$$

Let us note that, if we take the solutions of the form

$$\phi_I(r,\omega,q) = \sum_{A=1}^{m} \psi_A^I(r,\omega,q)\phi_A^0, \tag{2.24}$$

where

$$\lim_{r\to\infty} \phi_I(r,\omega,q) = \phi_I^0, \tag{2.25}$$

then we get

$$\Im\Big(\mathcal{F}_{II}\Big) = \omega\frac{1}{2\kappa^2}\sqrt{-\frac{g_{rr}}{g_{tt}}}\sum_{A=1}^{m}\frac{1}{2}N_A\psi_A^I(r)\psi_A^I(r)\Bigg|_{r_h}, \tag{2.26}$$

and

$$\Im\Big(\mathcal{F}_{IJ}\Big) = \omega\frac{1}{2\kappa^2}\sqrt{-\frac{g_{rr}}{g_{tt}}}\sum_{A=1}^{m}N_A\psi_A^I(r)\psi_A^J(r)\Bigg|_{r_h}. \tag{2.27}$$

- **Single charge case:** For single charge case, consider $\phi(r) = \psi(r)\phi_0$, then we get

$$\Im\Big(\mathcal{F}\Big) = \omega\frac{1}{2\kappa^2}\sqrt{-\frac{g_{rr}}{g_{tt}}}\frac{1}{2}N_1\psi(r)\psi(r)\Bigg|_{r_h}. \tag{2.28}$$

Using Eq.(2.16), we get

$$\begin{aligned}\sigma &= \frac{1}{2\kappa^2}\sqrt{-\frac{g_{rr}}{g_{tt}}}N_1\psi(r)\psi(r)\Bigg|_{r_h}\\ &= \frac{1}{2\kappa^2}G_{11}(r)\ g_{xx}^{\frac{d-3}{2}}\ \psi^2(r)\Big|_{r=r_h}\\ &= \sigma_H\ \ \psi^2(r)\Big|_{r=r_h},\end{aligned} \tag{2.29}$$

where

$$\sigma_H = \frac{1}{2\kappa^2}G_{11}(r)\ g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}. \tag{2.30}$$

We can also compute conductivity at any arbitrary radius say at r_c. This is given by

$$\sigma(r_c) = \frac{1}{2\kappa^2}G_{11}(r)\ g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\Big[\frac{\phi(r=r_h)}{\phi(r_c)}\Big]^2. \tag{2.31}$$

Let us note that, at the horizon the expression for conductivity (which we shall call as horizon conductivity) reduces to σ_H, where as at the boundary conductivity is related to the horizon conductivity by Eq.(2.28).

- **Imaginary part of conductivity** $\sigma_0(=\Im(\lambda))$**:** The imaginary part of the conductivity is

$$\Im(\lambda)=\frac{1}{\omega\phi^0}\lim_{r\to\infty}\frac{1}{2\kappa^2}N(r)\frac{d}{dr}\phi(r). \tag{2.32}$$

Using Eq.(2.10) and Eq.(2.11), we can write

$$N(r)\frac{d}{dr}\phi(r)|_{\infty}^{r_h}=-(2\kappa^2)^2\rho^2\int_{\infty}^{r_h}dr\frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}\phi(r), \tag{2.33}$$

which implies [21]

$$\lim_{r\to\infty}N(r)\frac{d}{dr}\phi(r)=-(2\kappa^2)^2\rho^2\int_{\infty}^{r_h}dr\ \ \frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}\phi(r). \tag{2.34}$$

Defining $\phi(r)=\psi(r)\phi_0$, we find

$$\begin{aligned}\Im(\lambda) &= \frac{1}{\omega\phi^0}\lim_{r\to\infty}\frac{1}{2\kappa^2}N(r)\frac{d}{dr}\phi(r)\\ &= -2\kappa^2\rho^2\int_{\infty}^{r_h}dr\ \ \frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}\psi(r).\end{aligned} \tag{2.35}$$

In order to compute imaginary part of conductivity, we can use both Eq.(2.31) as well as Eq.(2.34).

2.3 Electrical conductivity for R-charged black hole in 4,5,7 dimensions in asymptotically AdS space

In this section we compute electrical conductivity for gauge theories dual to 4, 5, 7 dimensional R-charge black branes[22]. We observe that the behavior of conductivity with temperature is $\sigma\sim T^{d-3}$ for d dimensional dual gauge theory which also follows from dimensional analysis. For multi-charge black hole we get conductivity matrix whose off diagonal parts comes solely due to effective interaction between gauge fields. We first compute conductivity with single chemical potential and then turn to multiple chemical potential cases. In the following, we shall use radial coordinate to be u for convenience and we shall use notation $\kappa^2=8\pi G_{d+1}$ in $d+1$ dimensions. In this coordinate system, $u=1$ and $u=0$ are respectively position of horizon and boundary. The details of these background in coordinate system u can be found in [34, 35, 33, 36] and are summarized in the Appendix.

[21] Let us note that, $N(r=r_h)=0$, since $N(r)=\sqrt{-g}G(r)g^{xx}g^{rr}$ and $g^{rr}(r=r_h)=0$, and at the boundary if $N(r\to\infty)\sim r^{1-n}$ then $\phi(r\to\infty)\sim\phi^0+\phi^1r^n$.

[22] Black branes in 4, 7 dimensions arise from rotating M2, M5 brane solutions in a similar as 5 dimensional R-charged black brane arises which is discussed in chapter 1 section 1.5.2.

2.3.1 Single charge black hole in various dimension

For single charge black hole one finds

$$(\phi_1)'' + \left(\frac{f'}{f} + \frac{H_1'}{H_1} - \frac{c}{u}\right)(\phi_1)' - \frac{au^b(1+k_1)}{fH_1^2}\Big(k_1\phi_1\Big) = 0. \tag{2.36}$$

The expression for conductivity reduces to

$$\sigma = \frac{1}{8\pi G_{d+1}}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\frac{N(u)\phi(u,\omega,q)\phi(u,-\omega,-q)}{(\phi)^0(\phi)^0}\right]_{u\to1,q\to0}. \tag{2.37}$$

In this case we have $\sigma_H = \left[\frac{1}{8\pi G_{d+1}}\sqrt{-\frac{g_{uu}}{g_{tt}}}N(u)\right]_{u\to1}$.

- **D=4:** In this case one gets $c=0, a=1, b=2$. Relevant parts are

$$\sigma_H = \frac{N^{\frac{3}{2}}}{24\sqrt{2}\pi}(1+k)^{\frac{3}{2}}, \tag{2.38}$$

$$\phi(u) = \phi^0\frac{1+\frac{2ku}{3}}{1+ku} \tag{2.39}$$

which implies

$$\sigma = \frac{(3+2k)^2N^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k)}}. \tag{2.40}$$

We see that for three dimensional gauge theory, conductivity is independent of temperature. Now we can compare this result with the the result for $\mu=0$, case.

$$\frac{\sigma_\mu}{\sigma_{\mu=0}} = \frac{(1+\frac{2k}{3})^2}{\sqrt{1+k}} \geq 1. \tag{2.41}$$

Since there exist a critical line $k=\frac{3}{2}$ [58], one can not make conductivity arbitrarily large. This discussion also holds true for rest of the cases with only difference in location of critical line.

- **D=5:** Here $c=0, a=1, b=1$. Summary of the results are

$$\sigma_H = \frac{N^2T_0(1+k)^{\frac{3}{2}}}{16\pi}, \tag{2.42}$$

$$\phi(u) = \phi_0\frac{1+\frac{ku}{2}}{1+ku}. \tag{2.43}$$

So one gets conductivity

$$\sigma = \frac{(2+k)^2N^2T_0}{64\pi\sqrt{(1+k)}} = \frac{N^2T_H(2+k)}{32\pi}. \tag{2.44}$$

where $T_H = \frac{(2+k)T_0}{2\sqrt{1+k}}$ is the Hawking temperature of the black hole.

- **D=7:** In this case $c = -1, a = 4, b = 3$. Relevant parts are

$$\sigma_H = \frac{4N^3T_0^3(1+k)^{\frac{3}{2}}}{81}, \tag{2.45}$$

$$\phi(u) = \phi_0 \frac{1 + \frac{ku^2}{3}}{1 + ku^2}. \tag{2.46}$$

Conductivity in this case is given by

$$\sigma = \frac{4(3+k)^2N^3T_0^3}{3^6\sqrt{(1+k)}} = \frac{4N^3T_H^3(1+k)}{27(3+k)}. \tag{2.47}$$

where $T_H = \frac{(3+k)T_0}{3\sqrt{1+k}}$ is the Hawking temperature of the black hole.

2.3.2 Two charge black hole in various dimension

Now we turn to cases where two chemical potentials are turned on in the boundary gauge theory. Differential equations are[23]

$$\begin{aligned}(\phi_1)'' + &\left(\frac{f'}{f} + 2\frac{H_1'}{H_1} - \frac{\mathcal{H}'}{\mathcal{H}} - \frac{c}{u}\right)(\phi_1)' \\ &- \frac{au^b(1+k_1)(1+k_2)}{fH_1^2}\left[k_1\phi_1 + \sqrt{k_1k_2}\ \phi_2\right] = 0,\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}(\phi_2)'' + &\left(\frac{f'}{f} + 2\frac{H_2'}{H_2} - \frac{\mathcal{H}'}{\mathcal{H}} - \frac{c}{u}\right)(\phi_2)' \\ &- \frac{au^b(1+k_1)(1+k_2)}{fH_2^2}\left[k_2\phi_2 + \sqrt{k_1k_2}\ \phi_1\right] = 0,\end{aligned} \tag{2.49}$$

Note that $\sigma_{H,ii} = \left[\frac{1}{8\pi G_{d+1}}\sqrt{-\frac{g_{uu}}{g_{tt}}}N_i(u)\right]_{u\to 1}$ where $N_i = \frac{fH_i^2}{u^m\mathcal{H}}$. Now we compute case by case.

- **D=4:** Here one has $c = 0, a = 1, b = 2$. Solutions are

$$\phi_1 = \frac{(a_0 + \frac{2a_0k_1 - b_0\sqrt{k_1k_2}}{3}u)}{1+k_1u}, \qquad \phi_2 = \frac{(b_0 + \frac{2b_0k_2 - a_0\sqrt{k_1k_2}}{3}u)}{1+k_2u}, \tag{2.50}$$

and

$$\sigma_{H,ii} = \frac{N^{\frac{3}{2}}(1+k_i)^2}{24\pi\sqrt{2(1+k_1)(1+k_2)}}. \tag{2.51}$$

[23] These form of equations are different than the form written in [41] , because as mentioned earlier that we have done a rescaling.

Using these we get following form of conductivity.

$$\begin{pmatrix} \frac{N^{\frac{3}{2}}(9+(12+k_2)k_1+4k_1^2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} & -\frac{2N^{\frac{3}{2}}\sqrt{k_1k_2}(3+k_1+k_2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} \\ -\frac{2N^{\frac{3}{2}}\sqrt{k_1k_2}(3+k_1+k_2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} & \frac{N^{\frac{3}{2}}(9+(12+k_1)k_2+4k_2^2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} \end{pmatrix}.$$

- **D=5:** Here we have $c = 0, a = 1, b = 1$. In this case solutions are

$$\phi_1 = \frac{(a_0 + \frac{a_0k_1 - b_0\sqrt{k_1k_2}}{2}u)}{1+k_1u}, \qquad \phi_2 = \frac{(b_0 + \frac{b_0k_2 - a_0\sqrt{k_1k_2}}{2}u)}{1+k_2u}. \tag{2.52}$$

Where as

$$\sigma_{H,ii} = \frac{N^2T_0(1+k_i)^2}{16\pi\sqrt{(1+k_1)(1+k_2)}}. \tag{2.53}$$

So we get conductivity as

$$\begin{pmatrix} \frac{(4+k_1^2+k_1(4+k_2))N^2T_o}{64\pi\sqrt{(1+k_1)(1+k_2)}} & -\frac{(4+k_1+k_2)N^2T_o}{64\pi}\sqrt{\frac{k_1k_2}{(1+k_1)(1+k_2)}} \\ -\frac{(4+k_1+k_2)N^2T_o}{64\pi}\sqrt{\frac{k_1k_2}{(1+k_1)(1+k_2)}} & \frac{(4+k_2^2+k_2(4+k_1))N^2T_o}{64\pi\sqrt{(1+k_1)(1+k_2)}} \end{pmatrix}.$$

So σ increases linearly with T_H.

- **D=7:** In this case $c = -1, a = 4, b = 3$. Solutions are

$$\phi_1 = \frac{(a_0 + \frac{a_0k_1 - 2b_0\sqrt{k_1k_2}}{3}u^2)}{1+k_1u^2}, \qquad \phi_2 = \frac{(b_0 + \frac{b_0k_2 - 2a_0\sqrt{k_1k_2}}{3}u^2)}{1+k_2u^2} \tag{2.54}$$

Now

$$\sigma_{H,ii} = \frac{4N^3T_0^3(1+k_i)^2}{81\sqrt{(1+k_1)(1+k_2)}}. \tag{2.55}$$

Using these one finds conductivity matrix as

$$\begin{pmatrix} \frac{4(9+k_1(k_1+4k_2+6))N^3T_o^3}{3^6\sqrt{(1+k_1)(1+k_2)}} & -\frac{(6+k_1+k_2)N^3T_o^3}{3^6}\sqrt{\frac{k_1k_2}{(1+k_1)(1+k_2)}} \\ -\frac{(6+k_1+k_2)N^3T_o^3}{3^6}\sqrt{\frac{k_1k_2}{(1+k_1)(1+k_2)}} & \frac{4(9+k_2(k_2+4k_1+6))N^3T_o^3}{3^6\sqrt{(1+k_1)(1+k_2)}} \end{pmatrix}.$$

- Notice that off diagonal components of the conductivity matrix are negative but they are important, and plays crucial role.

- Observe that off diagonal components goes as

$$\sigma_{ij} \sim \Omega_i\Omega_j T^{d-3}, \tag{2.56}$$

where $\Omega_i = \frac{\mu_i}{2\pi T}$. So switching off one of the chemical potential will make it zero, where as diagonal parts of conductivity goes as $\sigma_{ii} \sim T^{d-3} f_{ii}(\Omega_1, \Omega_2)$, where $f_{ii}(0,0) \neq 0$ ($\mu = 0$, implies total charge density is zero i.e. there exist equal number of positive as well as negative charge and applying external electric field will induce flow of both in opposite direction which will contribute to electrical current). Since charged particles moves in opposite direction, there will be collisions among them and it ensures finite conductivity. As one increases μ, conductivity should increase as relative number of collisions between opposite charges are less compared to zero chemical potential case.

2.3.3 Three charge black hole in various dimension

Now we turn to three charge black hole cases. General form of differential equations are

$$(\phi_i)'' + \left(\frac{f'}{f} + 2\frac{H_i'}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}}\right)(\phi_i)' - \frac{u^b \prod_{j=1}^{3}(1+k_j)\sqrt{k_i}}{f H_i^2}\Big[\sum_{j=1}^{3} \sqrt{k_j}\phi_j\Big] = 0, \quad (2.57)$$

where i takes value up to three.

- **D=4:** For this case the one gets $b = 2$. Relevant results in this case are

$$\phi_i = \frac{\Big[3\phi_i^0 + \sqrt{k_i}\Big(3\sqrt{k_i}\phi_i^0 - \sum_{j=1}^{3}\sqrt{k_j}\phi_j^0\Big)u\Big]}{3(1+k_i u)}\ , \quad \sigma_{H,ii} = \frac{N^{\frac{3}{2}}(1+k_i)^2}{24\pi\sqrt{2\prod_{j=1}^{3}(1+k_j)}}, \quad (2.58)$$

  where ϕ_i^0 is the boundary value of ith perturbed gauge field.

  Let us introduce, $\sigma_{ij} = \frac{N^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k_1)(1+k_2)(1+k_3)}}\sigma_{ij}^0$, where σ_{ij}^0, is given by

$$\begin{pmatrix} 9+(12+k_2+k_3)k_1+4k_1^2 & -\sqrt{k_1k_2}(6+2k_1+2k_2-k_3) & -\sqrt{k_1k_3}(6+2k_1+2k_3-k_2) \\ -\sqrt{k_1k_2}(6+2k_1+2k_2-k_3) & 9+(12+k_1+k_3)k_2+4k_2^2 & -\sqrt{k_2k_3}(6+2k_2+2k_3-k_1) \\ -\sqrt{k_1k_3}(6+2k_1+2k_3-k_2) & -\sqrt{k_2k_3}(6+2k_2+2k_3-k_1) & 9+(12+k_1+k_2)k_3+4k_3^2 \end{pmatrix}.$$

- **D=5:** For this case $b = 1$. Results needed for conductivity calculation are

$$\phi_i = \frac{\Big[2\phi_i^0 + \sqrt{k_i}\Big(2\sqrt{k_i}\phi_i^0 - \sum_{j=1}^{3}\sqrt{k_j}\phi_j^0\Big)u\Big]}{2(1+k_i u)}, \quad \sigma_{H,ii} = \frac{N^2 T_0 (1+k_i)^2}{16\pi\sqrt{(1+k_1)(1+k_2)(1+k_3)}}. \quad (2.59)$$

Defining as before $\sigma_{ij} = \frac{N^2 T_0}{64\pi\sqrt{(1+k_1)(1+k_2)(1+k_3)}}\sigma^0_{ij}$, where σ^0_{ij} is given by

$$\begin{pmatrix} 4+k_1^2+k_1(4+k_2+k_3) & -\sqrt{k_1k_2}(4+k_1+k_2-k_3) & -\sqrt{k_1k_3}(4+k_1+k_3-k_2) \\ -\sqrt{k_1k_2}(4+k_1+k_2-k_3) & 4+k_2^2+k_2(4+k_1+k_3) & -\sqrt{k_2k_3}(4+k_2+k_3-k_1) \\ -\sqrt{k_1k_3}(4+k_1+k_3-k_2) & -\sqrt{k_2k_3}(4+k_2+k_3-k_1) & 4+k_3^2+k_3(4+k_1+k_2) \end{pmatrix}$$

2.3.4 Four charge black hole

Differential equations are

$$(\phi_i)'' + \left(\frac{f'}{f} + 2\frac{H_i'}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}}\right)(\phi_i)' - \frac{u^2\prod_{j=1}^{4}(1+k_j)\sqrt{k_i}}{fH_i^2}\Big[\sum_{j=1}^{4}\sqrt{k_j}\phi_j\Big] = 0, \quad (2.60)$$

and solutions are

$$\phi_i = \frac{\Big[3\phi_i^0 + \sqrt{k_i}\Big(3\sqrt{k_i}\phi_i^0 - \sum_{j=1}^{4}\sqrt{k_j}\phi_j^0\Big)u\Big]}{3(1+k_iu)}, \quad \sigma_{H,ii} = \frac{N^{\frac{3}{2}}(1+k_i)^2}{24\pi\sqrt{2\prod_{j=1}^{4}(1+k_j)}}. \quad (2.61)$$

Using these we get following form of conductivity.

$$\sigma_{ij} = \frac{N^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k_1)(1+k_2)(1+k_3)(1+k_4)}}\sigma^0_{ij}. \quad (2.62)$$

Where

$$\sigma^0_{ii} = 9 + \Big(12 + \sum_{j=1}^{4}k_j\Big)k_i + 3k_i^2 \quad \text{and} \quad \sigma^0_{\mathrm{ij}} = -\Big(6 - \sum_{\mathrm{l}=1}^{4}\mathrm{k_l} + 3\big(\mathrm{k_i}+\mathrm{k_j}\big)\Big).$$

- **Some special cases :** Using above results one can study special cases such as effect of small chemical potential or the case with equal chemical potential. Let us note that in the case when all the chemical potential are equal then there exist no second order phase transition. In this case, temperature i.e. $T \geq 0$ gives a constraint on the possible maximum value of chemical potential.

  Note that as $T \to 0$, $\sigma \to 0$ quadratically in the parameter k, irrespective of which dimension we are in[24].

[24]Determinant of conductivity matrix for general μ also vanishes in similar way once we approach extremality even for M2 brane case where conductivity is independent of temperature.

Dimension	Constraint$(T \geq 0)$	σ
5	$k \leq 2$	$\frac{3(2-k)^2 N^2 T_0}{32\pi(1+k)^{\frac{3}{2}}}$
4	$k \leq 3$	$\frac{(3-k)^2 N^{\frac{3}{2}}}{27\sqrt{2}(1+k)^2\pi}$
7	$k \leq 3$	$\frac{16(3-k)^2 N^3 T_0^3}{729(1+k)}$

Table 2.1: Conductivity at equal charges

2.4 Relating boundary and horizon electrical conductivity:

In this section we reconsider the examples of previous section (all are asymptotically AdS spaces) and show that for each case there exist a universal relation between boundary and horizon conductivity. We tabulate the results below (see [54]).

Gravity theory in $d+1$ dimension	σ_H	$\sigma_H(\frac{sT}{\epsilon+P})^2$	σ_B
R-charge black hole in $4+1$ dim.	$\frac{N^2T(1+k)^2}{16\pi(1+\frac{k}{2})}$	$\frac{N^2T(2+k)}{32\pi}$	$\frac{N^2T(2+k)}{32\pi}$
R-charge black hole in $3+1$ dim.	$\frac{N^{\frac{3}{2}}}{24\sqrt{2}\pi}(1+k)^{\frac{3}{2}}$	$\frac{(3+2k)^2 N^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k)}}$	$\frac{(3+2k)^2 N^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k)}}$
R-charge black hole in $6+1$ dim.	$\frac{4N^3T^3(1+k)^3}{81(1+\frac{k}{3})^3}$	$\frac{4N^3T^3(1+k)}{27(3+k)}$	$\frac{4N^3T^3(1+k)}{27(3+k)}$
Reissner-Nordstrom black hole in $3+1$ dim.	$\frac{1}{g^2}$	$\frac{1}{g^2}(\frac{sT}{\epsilon+P})^2$	$\frac{1}{g^2}(\frac{sT}{\epsilon+P})^2$

Table 2.2: Real part of electrical conductivity at the horizon (σ_H) and at the Boundary (σ_B) are related by $\sigma_B = \sigma_H\left(\frac{sT}{\epsilon+P}\right)^2$.

- **Single charge:** We propose based on the observation in **Table** 2.2 that for the gauge theory with single chemical potential the expression for real part of the conductivity is given by

$$\begin{aligned} \sigma_B &= \frac{1}{2\kappa^2} G_{11}\, g_{xx}^{\frac{d-3}{2}}\Big|_{r_h} \left(\frac{sT}{\epsilon+P}\right)^2 \\ &= \sigma_H\left(\frac{sT}{\epsilon+P}\right)^2, \end{aligned} \tag{2.63}$$

where s, T, P, ϵ are entropy, temperature, pressure and energy density of the boundary fluid respectively. We observe that boundary conductivity can be expressed in terms of geometrical quantities evaluated at the horizon and some combination of other thermodynamic quantities.

- **Multiple charge:** For multiple charge case (say there are m number of chemical potential present in the gauge theory side), then boundary conductivity is $m \times m$ symmetric matrix (see [48]) where as horizon conductivity is $m \times m$ diagonal matrix. One can check by explicit computation that in each case the relation

$$\frac{1}{\rho_I \sigma_{IJ}^{-1} \rho_J} = \frac{1}{\rho_I \sigma_{H,II}^{-1} \rho_I} \Big(\frac{sT}{\epsilon + P} \Big)^2, \tag{2.64}$$

holds where σ_{IJ} and $\sigma_{H,II}$ are boundary and horizon conductivity respectively. For the action of the form Eq.(2.1), the expression for horizon conductivity can be written as

$$\sigma_{H,II} = \frac{1}{2\kappa^2} G_{II} \; g_{xx}^{\frac{d-3}{2}} \Big|_{r_h}. \tag{2.65}$$

Let us note that this expression reduces to Eq.(2.62) in the case when single chemical potential is present. As discussed in section 1.5 and Appendix.B. that, one can restrict attention to the diagonal $U(1)$ case where one obtains Reissner-Nordstrom solution. In the next section, we focus our attention on general Reissner-Nordstrom solution where one can show that the form of conductivity is again given by Eq.(2.62). In other words, the conductivity obtained from setting all the charges same for R-charge black hole is consistent with that obtained from the Reissner-Nordstrom black hole.

2.5 Reissner-Nordstrom black hole in arbitrary dimension:

In this section our main focus will be on the Reissner-Nordstrom black branes in various dimensions. For computation in four dimension see [32, 59, 60]. Our main aim is to check the validity of Eq.(2.62). Action for Reissner-Nordstrom case is given by

$$S = \int d^{d+1}x \sqrt{-g} \Big[\frac{1}{2k^2} (R + \frac{d(d-1)}{L^2}) - \frac{1}{4g^2} F^2 \Big]. \tag{2.66}$$

The expression for the metric and gauge field for Reissner-Nordstrom black hole in arbitrary dimension are

$$ds^2 = \frac{L^2}{r^2} \Big(- f(r) dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d-1} dx^i dx^i \Big), \tag{2.67}$$

and

$$A_t = \mu\Big[1 - (\frac{r}{r_+})^{d-2}\Big], \tag{2.68}$$

where $f(r) = 1 - (1 + \frac{r_+^2\mu^2}{\gamma^2})(\frac{r}{r_+})^d + \frac{r_+^2\mu^2}{\gamma^2}(\frac{r}{r_+})^{2(d-1)}$ and $\gamma^2 = \frac{(d-1)g^2L^2}{(d-2)k^2}$. Let us note that boundary is at $r = 0$ and μ and r_+ are chemical potential and horizon radius respectively. Various thermodynamic quantities are given by

$$P = \frac{L^{d-1}}{2k^2 r_+^d}(1 + \frac{r_+^2\mu^2}{\gamma^2}); \qquad \rho = (d-1)\frac{L^{d-1}}{k^2 r_+^{d-2}}\frac{\mu}{\gamma^2}, \tag{2.69}$$

and

$$T = \frac{1}{4\pi r_+}\Big[d - \frac{(d-2)r_+^2\mu^2}{\gamma^2}\Big], \quad s = \frac{2\pi}{k^2}\frac{L^{d-1}}{r_+^{d-1}}. \tag{2.70}$$

In order to compute the electrical conductivity we have to solve the Eq.(2.10) for this back ground. The Eq.(2.10) takes the form (in $\omega \to 0$ limit)

$$\frac{d}{dr}(\frac{f(r)}{r^{d-3}}\frac{d}{dr}\phi(r)) + \frac{2k^2\mu^2(d-2)^2 r^{d-1}}{g^2L^2 r_+^{2(d-2)}}\phi(r) = 0. \tag{2.71}$$

The solution takes the form

$$\phi(r) = \phi_0\left(1 - r^{d-2}\frac{2(d-1)(d-2)\; k^2\mu^2 r_+^{4-d}}{d\;[g^2\; L^2\; (d-1) + (d-2)\; k^2\mu^2 r_+^2]}\right), \tag{2.72}$$

where ϕ_0 is the boundary value of the perturbed field $\phi(r)$. Now according to Eq.(2.28)

$$\begin{aligned} \sigma &= \sigma_H \left(\frac{\phi(r = r_H)}{\phi_0}\right)^2 \\ &= \sigma_H\left(\frac{(d-1)dg^2L^2 - (d-2)^2k^2\mu^2 r_+^2}{d[(d-1)g^2L^2 + (d-2)k^2\mu^2 r_+^2]}\right)^2. \end{aligned} \tag{2.73}$$

Now using the fact that $\epsilon = (d-1)P$ and the thermodynamic quantities in Eq.(2.68) and Eq.(2.69) we can express right hand side of Eq.(2.72) as

$$\sigma = \sigma_H\left(\frac{sT}{\epsilon + P}\right)^2. \tag{2.74}$$

So we have shown explicitly that for Reissner-Nordstrom black hole in any dimension, the expression for conductivity in Eq.(2.62) is valid. In the following we shall check again whether the form of conductivity in Eq.(2.62) holds if we consider non-conformal boundary theory and its dual.

2.6 Electrical conductivity for non-conformal boundary theory

All the examples that we have discussed till now are for asymptotically AdS space time which corresponds to boundary theory to be conformal. In view of relation Eq.(2.62), a natural question arises whether similar relation holds for other known gravity theories which is supposed to have gauge theory dual. Recently in [61], authors studied electrical conductivity for charged $D1$ brane. In the following we shall check that their results does obey Eq.(2.62). However, before procedding let us note that for $D1$ brane, it is not possible to have perturbation like $h_{x^1x^2}$ as defined in Eq.(**??**) since there is only one spatial dimension. However, our analysis continues to hold here since as we saw, in the limit of zero spatial monetum $h_{x^1x^2}$, does not play any role. After analysing $D1$ brane, we shall check explicitly the validity of Eq.(2.62) and Eq.(2.63) for non-conformal gauge theories dual to general charged Dp brane.

- **Electrical conductivity for charged $D1$ brane:** Let us consider the following action

$$\begin{aligned} I &= \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\Big[R(g) - \frac{8}{9}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\Psi^2 e^{-\frac{4}{3}\phi}F_{\mu\nu}F^{\mu\nu} \\ &- \frac{1}{2\Psi^2}\partial_\mu\Psi\partial^\mu\Psi + \frac{2}{3\Psi}\partial_\mu\phi\partial^\mu\Psi + \frac{12}{L^2}e^{\frac{4}{3}\phi}(1+\Psi^{-1})\Big]. \end{aligned} \tag{2.75}$$

In the following discussion, the radial coordinate is r and r_h is the position of horizon. The boundary is at $r \to \infty$. The metric, gauge field and scalar fields are given by

$$\begin{aligned} ds^2 &= \left(-c_T^2 dt^2 + c_X^2 dz^2 + c_R^2 dr^2\right), \qquad (2.76) \\ c_T^2 &= \left(\frac{r}{L}\right)^8 K, \qquad c_X^2 = \left(\frac{r}{L}\right)^8 H, \qquad c_R^2 = \frac{H}{K}\left(\frac{r}{L}\right)^2, \\ A_t &= -\frac{r_0^3 l}{L^2(r^2+l^2)}, \qquad \phi = -3\log\left(\frac{r}{L}\right), \qquad \Psi = 1 + \frac{l^2}{r^2}. \end{aligned}$$

Here H and K are defined as

$$H = 1 + \frac{l^2}{r^2}, \qquad K = 1 + \frac{l^2}{r^2} - \frac{r_0^6}{r^6}. \tag{2.77}$$

Different thermodynamic quantities are given by,

$$T = \frac{1}{2\pi L^3}\frac{r_h^5}{r_0^3}(3+2k), \qquad s = \frac{1}{4G_3}\frac{r_0^3 r_h}{L^4}, \tag{2.78}$$

where k is given by

$$k = \frac{l^2}{r_h^2}, \tag{2.79}$$

and r_h is the radius of the horizon which is given by the largest root of the equation

$$r_h^6 + r_h^4 l^2 - r_0^6 = 0. \tag{2.80}$$

The energy density (ϵ) and the pressure (p) is given by

$$\epsilon = \frac{1}{4\pi G_3}\frac{r_0^6}{L^7}, \qquad p = \frac{1}{8\pi G_3}\frac{r_0^6}{L^7} = \frac{\epsilon}{2}. \tag{2.81}$$

The charge density ρ and its conjugate the chemical potential μ are given by

$$\rho = \frac{1}{8\pi G_3}\frac{r_0^3 l}{L^5}, \qquad \mu = A_t(r)\Big|_{r\to\infty} - A_t(r)\Big|_{r_h} = \frac{l r_h^4}{L^2 r_0^3}. \tag{2.82}$$

So conductivity should be ,

$$\begin{aligned}
\sigma &= \frac{1}{16\pi G_3}\frac{1}{g_{\text{eff}}^2} g_{xx}^{-\frac{1}{2}}\Big|_{r=r_h}\left(\frac{sT}{\epsilon+P}\right)^2 \\
&= \frac{1}{16\pi G_3}\Psi^2 e^{-\frac{4}{3}\phi} g_{xx}^{-\frac{1}{2}}\Big|_{r=r_h}\left(\frac{sT}{\epsilon+P}\right)^2 \\
&= \frac{1}{16\pi G_3}\frac{(2k+3)^2}{9\sqrt{1+k}},
\end{aligned} \tag{2.83}$$

which is same as the one computed in [61]. In that paper authors also computed electrical conductivity for four equal charge case. The results follow from Eq.(2.62) in a straight forward manner.

Now we shall check Eq.(2.62) and Eq.(2.63) for general Dp branes.

2.6.1 Charged Dp brane

Let us consider the background obtained from Kaluza-Klein spherical reduction of rotating black Dp brane to d dimension (see for details [62, 63, 64]).

$$ds^2 = -(g\ r)^{\frac{n+1}{d-2}} h^{-\frac{d-3}{d-2}} f(r) dt^2 + (g\ r)^{\frac{n+1}{d-2}} h^{\frac{1}{d-2}} \sum_{i=1}^{p} dx_i^2 + (g\ r)^{1-n+\frac{n+1}{d-2}} h^{\frac{1}{d-2}} \frac{1}{f(r)} dr^2, \tag{2.84}$$

where

$$f(r) = h - \frac{2m}{r^{n-1}}, \quad h = \prod_{i=1}^{b}(1+H_i), \qquad H_i = 1 + \frac{l_i^2}{r^2}, \tag{2.85}$$

where b is the number of independent gauge fields (which is same as number of independent rotations that a higher dimensional Dp brane can have before compactification). The action is of the form

$$S = \frac{1}{16\pi G}\int d^{p+2}x\sqrt{-g}\Big[R - \frac{1}{4}\sum_{i=1}^{b}\frac{1}{X_i^2}F^i_{\mu\nu}F^{i\ \mu\nu} + \textit{all the other terms..}\Big], \quad (2.86)$$

where

$$X_i = g^{-\frac{a^2(D-2)}{4(d-2)}} r^{-\frac{a^2(D-2)}{4(d-2)}} h^{\frac{d-3}{2(d-2)}}\frac{1}{H_i}, \quad (2.87)$$

and

$$A^i_t = -\sqrt{2m}g^{\frac{n-3}{2}}\frac{1-\frac{1}{H_i}}{l_i}. \quad (2.88)$$

In the following we define all the required thermodynamic quantities. The expression for charge density is,

$$\rho_i = \frac{1}{8\pi G}\sqrt{2m}g^{\frac{n+3}{2}}l_i, \quad (2.89)$$

the chemical potentials are given by

$$\mu_i = \sqrt{2m}g^{\frac{n-3}{2}}\frac{l_i}{r_h^2 H_i(r_h)}. \quad (2.90)$$

The Hawking temperature is given by

$$T = \frac{\sqrt{m}}{\sqrt{2}\pi r_h}g^{\frac{n-1}{2}}\Big(\frac{n-1}{2} - \frac{1}{r_h^2}\sum_{j=1}^{b}\frac{l_i^2}{H_i(r_h)}\Big). \quad (2.91)$$

The expression for entropy and other required quantities are

$$s = \frac{1}{4G}g^{\frac{n+1}{2}}r_h\sqrt{2m}, \quad \epsilon + P = \frac{(n-1)m}{8\pi G}g^n. \quad (2.92)$$

The equation[25] that we have to solve in order to find out conductivity is given by

$$\frac{d}{dr}\Big(N_i\frac{d}{dr}\phi_i(r)\Big) + \sum_{j=1}^{m}M_{ij}\phi_j(r) = 0. \quad (2.93)$$

where

$$N_i = \sqrt{-g}\frac{1}{X_i^2}g^{xx}g^{rr}, \quad (2.94)$$

and

$$M_{ij} = F^i_{rt}\sqrt{-g}\frac{1}{X_i^2}g^{xx}g^{rr}g^{tt}\frac{1}{X_j^2}F^j_{rt}. \quad (2.95)$$

[25] Unless explicitly mentioned, there is no sum over repeated indices i, j.

Plugging the background values we can show

$$N_i = g^3 r^3 f(r) H_i^2 \frac{1}{h}, \quad M_{ij} = -8m\ l_i\ l_j g^3 r^{-n} \frac{1}{h}. \tag{2.96}$$

- **Single charge case:** Here one can easily check that

$$\sigma = \frac{1}{16\pi G} \frac{1}{X^2} g_{xx}^{\frac{p-2}{2}} \Big|_{r_h} \left(\frac{sT}{\epsilon + P} \right)^2. \tag{2.97}$$

- **Multi charge case:** The expression for electrical conductivity at the horizon is given by,

$$\begin{aligned} \sigma_{H,ii} &= \frac{1}{16\pi G} G_{ii}(r)\ g_{xx}^{\frac{p-2}{2}} \Big|_{r=r_h} \\ &= \frac{1}{16\pi G} \frac{1}{X_i^2}\ g_{xx}^{\frac{p-2}{2}} \Big|_{r=r_h} \\ &= \frac{g^{\frac{7-n}{2}} r_h^3 H_i^2(r_h)}{16\sqrt{2m}\ \pi G}. \end{aligned} \tag{2.98}$$

For simplicity we just give example of $D1$ brane and a general result will be presented in the next chapter.

- **D1 brane with four unequal charges:** In this case, the coupled set of equations for i^{th} field are given by

$$\frac{d}{dr}(N_i \frac{d}{dr}\phi_i(r)) + \sum_{j=1}^{4} M_{ij}\phi_j(r) = 0, \tag{2.99}$$

where index i, can take value from 1 to 4 (there is no sum over i in the above) and

$$N_i = g^3 r^3 f(r) H_i^2 \frac{1}{h}, \quad M_{ij} = -8m\ l_i\ l_j g^3 r^{-7} \frac{1}{h}, \quad h = \prod_{i=1}^{4}(1 + H_i). \tag{2.100}$$

Demanding regularity (in going boundary condition) at the horizon and at the boundary $\phi_i = \phi_i^0$, we get the solution to 4 coupled equation to be

$$\phi_i = \frac{\phi_i^0 + \frac{l_i}{6r^2}(6l_i\phi_i^0 - 2\sum_{j=1}^4 \phi_j^0 l_j)}{H_i^2}. \tag{2.101}$$

The expression for diagonal part of electrical conductivity is given by

$$\sigma_{ii} = \frac{9r_h^4 + 12r_h^2 l_i^2 + 3l_i^4 + l_i^2 \sum_{j=1}^4 l_j^2}{144\sqrt{2m}\ \pi G r_h}, \tag{2.102}$$

where as off diagonal part of the conductivity is given by

$$\sigma_{ij} = -\frac{l_i l_j}{144\sqrt{2m}\ \pi G r_h}(6r_h^2 + \sum_{k=1}^{4} l_k^2 - 3(l_i^2 + l_j^2)). \tag{2.103}$$

We can now explicitly check that, for multi charge case

$$\rho_i \sigma_{ij}^{-1} \rho_j = \rho_i \sigma_{H,ii}^{-1} \rho_i \left(\frac{\epsilon + P}{sT}\right)^2, \tag{2.104}$$

where

$$\sigma_{H,ii} \quad = \quad \frac{r_h^3 H_i^2(r_h)}{16\sqrt{2m}\ \pi G}, \tag{2.105}$$

is the electrical conductivity evaluated at the horizon and depends only on the geometrical quantities evaluated at the horizon.

To conclude, we have checked that, at and away from conformality the form of boundary conductivity is given by Eq.(2.62) and Eq.(2.63). Next we shall check whether the general form of conductivity holds for Lifshitz like black holes. First we give a brief details of Lifshitz like black holes and then we shall compute conductivity for both charged and uncharged cases.

2.7 Lifshitz like black holes:

Due to possible applications in condensed matter systems, there have been lots of work [65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76] going on to understand transport properties of gauge theories dual to both uncharged and charged Lifshitz like black holes. Motivated by this, our aim in this section is to explore electrical conductivities for these class of black holes. This section is organized as follows. First we review the geometry and thermodynamics of uncharged Lifshitz like black holes. Then we discuss transport coefficients such as electrical conductivity. For the charged Lifshitz case, after discussing geometry and thermodynamics, we focus our attention to the computation of electrical conductivity.

2.7.1 Uncharged Lifshitz black holes:

The metric for this case is given by

$$ds^2 = L^2[-\frac{r_0^{2z}}{u^{2z}}f(u)dt^2 + \frac{du^2}{u^2 f(u)} + \frac{r_0^2}{u^2}\sum_{i=1}^{d} dx_i^2], \quad f(u) = 1 - u^{z+d}. \tag{2.106}$$

The horizon is located at $u = 1$ and the boundary at $u = 0$. We take uncharged Lifshitz black brane metric in Eq.(2.105) as the background and treat the Maxwell action

$$S_F = -\frac{1}{4g_{d+2}^2}\int d^{d+2}x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \tag{2.107}$$

as perturbations. Here g_{d+2} denotes the gauge coupling constant. The Maxwell equation is

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0. \tag{2.108}$$

The electrical conductivity reads

$$\sigma_B = \frac{1}{g_{d+1}^2}G(u)\ g_{xx}^{\frac{d-3}{2}}\left[\frac{\phi(u)}{\phi_0}\right]^2\Big|_{u=1}. \tag{2.109}$$

At $\omega \to 0$, to get conductivity we need to solve

$$\frac{d}{du}(N\frac{d}{du}\phi(u)) = 0, \tag{2.110}$$

where $N(u) = \sqrt{-g}\frac{1}{g_{d+1}^2}g^{xx}g^{uu}$. Solution of Eq.(2.109) that is regular at the horizon is given by $\phi(u) = \phi_0$, where ϕ_0 is the boundary value of the perturbed field. Since at zero chemical potential, conductivity at the horizon and at the boundary are the same (as $\phi(u = 1) = \phi(u = 0)$), we get

$$\begin{aligned}\sigma_H &= \sigma_B \\ &= \frac{1}{g_{d+2}^2}g_{xx}^{\frac{d-2}{2}}\Big|_{u=1},\end{aligned} \tag{2.111}$$

which upon using metric (2.105), gives

$$\sigma = \frac{1}{g_{d+2}^2}(Lr_0)^{d-2} = (\frac{4\pi}{z+d})^{\frac{d-2}{z}}T^{\frac{d-2}{z}}. \tag{2.112}$$

Where we have used $r_0^z = \frac{4\pi}{z+d}T$. Using the definition of $\frac{\rho}{\mu}$ one finds,

$$\begin{aligned}\frac{\rho}{\mu} &= \left[\int_{r_0}^{\infty} dr\ \frac{g_{rr}g_{tt}g_{d+1}^2(r)}{\sqrt{-g}}\right]^{-1} \\ &= \frac{L^{d-2}}{g_{d+2}^2}(d-z)r_0^{d-z}.\end{aligned} \tag{2.113}$$

For these class of black hole, we have

$$\epsilon = \frac{d}{z}P, \qquad \epsilon + P = Ts. \tag{2.114}$$

Taking $P = c^{'}T^{\frac{d+z}{z}}$ we get $s = c^{'}\frac{z+d}{z}T^{\frac{d}{z}}$. Moreover $r_0^z = \frac{4\pi}{z+d}T$, so that we can write

$$\begin{aligned} \chi &= \frac{L^{d-2}}{g_{d+2}^2}(d-z)(\frac{4\pi}{z+d})^{\frac{d-z}{z}}T^{\frac{d-z}{z}} \\ &= k^{'}T^{\frac{d-z}{z}}. \end{aligned} \tag{2.115}$$

In this notation the conductivity can be expressed as

$$\begin{aligned} \sigma &= k^{'}\frac{1}{d-z}(\frac{4\pi}{z+d})^{\frac{z-2}{z}}T^{\frac{d-2}{z}} \\ &= \frac{1}{d-z}(\frac{4\pi}{z+d})^{\frac{z-2}{z}}\ \chi\ T^{\frac{z-2}{z}}. \end{aligned} \tag{2.116}$$

2.7.2 Charged Lifshitz black holes:

It was noted in [70] that the following action

$$S = \frac{1}{16\pi G_{d+2}}\int d^{d+2}x\sqrt{-g}(R - 2\Lambda - \frac{1}{4}F^2 - \frac{1}{2}m^2A^2) \tag{2.117}$$

admits $(d+2)-$dimensional Lifshitz space-time with arbitrary z

$$ds^2 = L^2(-r^{2z}dt^2 + \frac{1}{r^2}dr^2 + r^2\sum_{i=1}^{d}dx_i^2) \tag{2.118}$$

as a solution. If one adds a second Maxwell field (F_1) i.e.

$$S = \frac{1}{16\pi G_{d+2}}\int d^{d+2}x\sqrt{-g}(R - 2\Lambda - \frac{1}{4}F^2 - \frac{1}{2}m^2A^2 - \frac{1}{4}F_1^2), \tag{2.119}$$

then the metric of the black hole turns out to be ,

$$ds^2 = L^2(-r^{2z}dt^2 + \frac{1}{r^2}dr^2 + r^2\sum_{i=1}^{d}dx_i^2), \quad f(r) = 1 - \frac{q^2}{2d^2r^z}. \tag{2.120}$$

The mass parameter and the cosmological constant are given by

$$m^2 = \frac{zd}{L^2}, \quad \Lambda = -\frac{1}{2L^2}[z^2 + z(d-1) + d^2], \tag{2.121}$$

while the massive vector field and the second Maxwell field strength are given by

$$A_t = \sqrt{\frac{2(z-1)}{z}}Lr^zf(r), \quad F_{1\ rt} = qLr^{z-d-1}. \tag{2.122}$$

Let us note that in the above $z = 2d$ and $r_0^z \equiv q^2/2d^2$.

When $z = 1$, the above ansatz leads to

$$ds^2 = L^2[-r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^{d} dx_i^2], \quad f(r) = 1 - \frac{m}{r^{d+1}} + \frac{q^2}{2d(d-1)r^{2d}}, \tag{2.123}$$

and

$$A_t = 0, \quad F_{rt} = 0, \tag{2.124}$$

which is nothing but asymptotically AdS black brane. The second Maxwell field and the cosmological constant are given by

$$F_{1\ rt} = \frac{qL}{r^d}, \quad \Lambda = -\frac{d(d+1)}{2L^2}. \tag{2.125}$$

In order to complete our discussion on charged Lifshitz black hole, we now discuss thermodynamics of these solutions. The temperature and entropy are given by

$$T = \frac{z}{4\pi} r_0^z, \quad S_{\text{BH}} = \frac{L^d V_d}{4G_{d+2}} r_0^d, \tag{2.126}$$

where $r_0^z \equiv q^2/2d^2$ and V_d denotes the volume of the $d-$dimensional spatial part. Let us note that

$$\begin{aligned}\chi &= \frac{\rho}{\mu} \\ &= \frac{1}{16\pi G_{d+2}}(z-d)L^{d-2} r_0^{d-z} \\ &= \frac{1}{16\pi G_{d+2}}(z-d)L^{d-2}(\frac{4\pi}{z})^{\frac{d-z}{z}} T^{\frac{d-z}{z}},\end{aligned} \tag{2.127}$$

and

$$\rho\mu = \frac{1}{16\pi G_{d+2}} \frac{q^2 L^d}{z-d} r_0^{z-d}. \tag{2.128}$$

Now assuming that the first law of thermodynamics is satisfied we get,

$$\epsilon + P = \frac{1}{8\pi G_{d+2}} z L^d r_0^{3d}. \tag{2.129}$$

2.7.3 Electrical conductivity

For convenience we shall use the coordinates $u = (\frac{r_0}{r})^{\frac{z}{2}}$. In this coordinate, $f(u) = 1 - u^2$. Using the differential Eq.(2.10) we reach at,

$$\frac{d^2}{du^2}\phi(u) + (\frac{1}{f(u)}\frac{df(u)}{du} + \frac{4-2z}{zu})\phi(u) - \frac{2}{f(u)}\phi(u) = 0. \tag{2.130}$$

The solution that satisfies regularity at the horizon takes the form

$$\begin{aligned}\phi(u) = -\phi_0 &\frac{u^{\frac{-4+3z}{z}}\Gamma\left[-\frac{1}{2}+\frac{2}{z}\right]\Gamma\left[\frac{5}{4}-\frac{1}{z}-\frac{\sqrt{16-8z-7z^2}}{4z}\right]\Gamma\left[\frac{5}{4}-\frac{1}{z}+\frac{\sqrt{16-8z-7z^2}}{4z}\right]}{\Gamma\left[\frac{5}{2}-\frac{2}{z}\right]\Gamma\left[-\frac{1}{4}+\frac{1}{z}-\frac{\sqrt{16-8z-7z^2}}{4z}\right]\Gamma\left[-\frac{1}{4}+\frac{1}{z}+\frac{\sqrt{16-8z-7z^2}}{4z}\right]} \\ &{}_2F_1\left[\frac{5}{4}-\frac{1}{z}-\frac{\sqrt{16-8z-7z^2}}{4z},\frac{5}{4}-\frac{1}{z}+\frac{\sqrt{16-8z-7z^2}}{4z},\frac{5}{2}-\frac{2}{z},u^2\right] \\ &+\phi_0\, {}_2F_1\left[-\frac{1}{4}+\frac{1}{z}-\frac{\sqrt{16-8z-7z^2}}{4z},-\frac{1}{4}+\frac{1}{z}+\frac{\sqrt{16-8z-7z^2}}{4z},-\frac{1}{2}+\frac{2}{z},u^2\right],\end{aligned} \tag{2.131}$$

where ϕ_0 is the boundary value of $\phi(u)$. The boundary conductivity is given by

$$\begin{aligned}\sigma_B &= \sigma_H\left(\frac{\phi(u=1)}{\phi(u=0)}\right)^2 \\ &= \frac{1}{16\pi G_{d+2}}(Lr_0)^{d-2}\left(\frac{\phi(u=1)}{\phi(u=0)}\right)^2.\end{aligned} \tag{2.132}$$

To compute conductivity we need to calculate $(\frac{\phi(u=1)}{\phi(u=0)})^2$.

- $z=4, d=2$**:** In this case $(\frac{\phi(u=1)}{\phi(u=0)})^2 \approx 0.24$, so that conductivity is given by

$$\begin{aligned}\sigma_B &= 0.24\ \sigma_H \\ &= \frac{0.24}{16\pi G_{d+2}}.\end{aligned} \tag{2.133}$$

- $z=6, d=3$**:** Here $(\frac{\phi(u=1)}{\phi(u=0)})^2 \approx 0.27$, which gives

$$\begin{aligned}\sigma_B &= 0.27\ \sigma_H \\ &= \frac{0.27}{16\pi G_{d+2}}(Lr_0) \\ &= \frac{0.27}{16\pi G_{d+2}}L\left(\frac{2\pi}{3}\right)^{\frac{1}{6}}T^{\frac{1}{6}}.\end{aligned} \tag{2.134}$$

In general the conductivity can be written as

$$\sigma_B = C\ \chi\ \left(\frac{4\pi}{z}\right)^{\frac{z-2}{z}}T^{\frac{z-2}{z}}, \tag{2.135}$$

where $C = (\frac{\phi(u=1)}{\phi(u=0)})^2$. It is important to note that above expressions only depends on temperature (no dependence in chemical potential), since charge and temperature are related by $T = \frac{q^2}{2\pi z}$.

Let us note that using Eq.(2.125) and Eq.(2.128), Eq.(2.62) gives

$$\begin{aligned}\sigma_{Boundary} &= \frac{(sT)^2}{(\epsilon+P)^2}\sigma_{Horizon} \\ &= 0.25 \ \ \sigma_H, \end{aligned} \tag{2.136}$$

which is independent of z, d. What we observe is that, electrical conductivity of charged Lifshitz like black holes given in Eq.(2.132) and in Eq.(2.133) differs slightly from Eq.(2.135).

2.8 Radial evolution of electrical conductivity

The aim of this section is to study electrical conductivity at any radial position r. To make life simple, we shall consider single charged asymptotically AdS black hole to find out the form of conductivity. As we shall see, at any radial position r, the conductivity is given by a simple expression which interpolates smoothly between the one computed at the horizon and at the boundary.

2.8.1 Relation between universal conductivity of stretched horizon and boundary conductivity

Consider the Maxwell part of the action of the form

$$S = -\int d^{d+1}x \ \ \sqrt{-g}\frac{1}{4g^2_{d+1}(r)}F_{MN}F^{MN}, \tag{2.137}$$

where $g^2_{d+1}(r)$ in general is a r dependent coupling[26]. The electrical conductivity at any radius is given by (see Eq.(2.30) and for further details see [57, 48])

$$\sigma(r_c) = \left(\frac{1}{4g^2_{d+1}(r)} \ g_{xx}^{\frac{d-3}{2}}\right)_{r=r_h} \left[\frac{\phi(r=r_h)}{\phi(r_c)}\right]^2. \tag{2.138}$$

Let us note that at the horizon conductivity is

$$\sigma(r_c = r_h) = \left(\frac{1}{4g^2_{d+1}(r)} \ g_{xx}^{\frac{d-3}{2}}\right)_{r=r_h}, \tag{2.139}$$

which is entirely given by geometrical quantities evaluated at the horizon. In order to understand radial evolution of conductivity we consider the cases with vanishing and non-vanishing chemical potential separately.

[26] Let us note that, in the notation used in Eq.(1.69), $\frac{1}{4g^2_{d+1}(r)} \equiv G(r)$

2.8.2 Radial evolution of conductivity at zero chemical potential

Let us note that at vanishing chemical potential, the term $M(r) = 0$ in Eq.(2.10). If we impose in going boundary condition at the horizon and impose $\phi(r \to \infty) = \phi_0$, at the boundary , then one can show that solution to Eq.(2.10) is given by $\phi(r) = \phi_0$, at any radius i.e. ϕ is a constant. Now using Eq.(2.137) we get,

$$\begin{aligned} \sigma_{\mu=0}(r_c) &= \left(\frac{1}{4g_{d+1}^2(r)} \, g_{xx}^{\frac{d-3}{2}} \right)_{r=r_h} \\ &= \sigma_{\mu=0}(r_h). \end{aligned} \tag{2.140}$$

So we conclude that at vanishing chemical potential boundary and horizon conductivity is the same.

- **Relation with universal conductivity of the stretched horizon:** The universal conductivity of the stretched horizon is given by (see [4]) $\sigma_{mb} = \frac{1}{g_{d+1}^2(r_h)}$. Now we see
$$\sigma_{CFT,\mu=0} = \sigma_{mb} \, g_{xx}^{\frac{d-3}{2}}(r_h), \tag{2.141}$$
where factor $g_{xx}^{\frac{d-3}{2}}(r_h)$ converts the length scale in CFT to proper length scale at horizon [4, 57] (let us note that in d dimension, conductivity has a mass dimension $d-3$).

2.8.3 Cutoff dependence of conductivity at finite chemical potential:

At finite chemical potential, boundary and horizon conductivity are no longer same. In this section we study how conductivity evolves radially in this case. In [77], it was demonstrated that there is a simple relation even interpolation between the fluid at the horizon defined by membrane paradigm and fluid at the boundary defined by gauge/gravity duality. The authors in that paper introduced fluid at any arbitrary radius r outside horizon which reduces to gauge/gravity fluid as $r \to \infty$. For convenience we take the metric and gauge fields as taken in [77]. The asymptotically AdS black charged p-brane solution are of the form

$$\begin{aligned} ds_{p+2}^2 &= -h(r)dt^2 + \frac{dr^2}{h(r)} + e^{2t(r)} dx^i dx_i, \\ A_t &= \frac{Qr_h}{p-1}\left(1 - \frac{r_h^{p-1}}{r^{p-1}}\right), \end{aligned} \tag{2.142}$$

where

$$\begin{aligned} h(r) &= \frac{r^2}{L^2}\left(1-(1+\alpha Q^2)\frac{r_h^{p+1}}{r^{p+1}}+\alpha Q^2\frac{r_h^{2p}}{r^{2p}}\right), \\ e^t &= \frac{r}{r_h}. \end{aligned} \tag{2.143}$$

What we observe is that these are Reissner-Nordstrom black hole in $p+2$ dimension with gauge coupling set to one and $\alpha = \frac{L^2\kappa^2}{p(p-1)}$. Let us consider a cutoff at radius $r = r_c$ outside the horizon. One can define thermodynamic quantities there. If the hawking temperature is T_H, the local temperature at the cutoff radius can expressed as

$$T_c \equiv T(r_c) = \frac{T_H}{\sqrt{h(r_c)}}, \qquad T_H = \frac{h^{'}(r_h)}{4\pi}. \tag{2.144}$$

The entropy density of the fluid at r_c is given by $s = \frac{2\pi}{\kappa^2}e^{-pt(r)}$, which reduces to $s = \frac{2\pi}{\kappa^2}$ as $r_c \to r_h$. One can find out local Brown-York stress tensor[27]to define

$$\epsilon + P = \frac{\sqrt{h}}{\kappa^2}\left(\frac{h^{'}}{2h} - t^{'}\right), \tag{2.147}$$

where ϵ and P are energy density and pressure of the fluid at r_c. Let us note that for $r_c \to r_h$

$$\epsilon + P = sT_c. \tag{2.148}$$

The chemical potential at r_c is

$$\mu = \frac{A_t}{\sqrt{h}}, \tag{2.149}$$

which vanishes at the horizon. So that the thermodynamic relation

$$\epsilon + P = sT_c + \rho\mu, \tag{2.150}$$

holds at any arbitrary radius. In order to find out electrical conductivity we need to solve Eq.(2.10) for this back ground and then use Eq.(2.30) to find out conductivity

[27]For a hypersurface Σ with unit normal n, Brown-York stress tensor is defined as

$$t_{ab} = \frac{1}{\kappa^2}(\gamma_{ab}\mathcal{K} - \mathcal{K}_{ab} + C\gamma_{ab}), \tag{2.145}$$

with $\gamma_{ab} = g_{ab} - n_a n_b$, where g_{ab} is the space-time metric. Let us note that t_{ab} is ambiguous upto a constant multiple of induced metric γ_{ab} on the hypersurface. However this dependence does not appear in the combination $\epsilon + P$. The extrinsic curvature $\mathcal{K}_{ab}$ is defined as

$$\mathcal{K}_{ab} = \frac{1}{2}\mathcal{L}_n\gamma_{ab}, \tag{2.146}$$

where $\mathcal{L}_n$ is the Lie derivative along n.

at radius r_c. The solution can be obtained easily and conductivity can be written down at any radius r_c. But here we follow a slightly different route which might be helpful to generalize the results in more general background. We propose that the form of conductivity at any radius r_c is given by

$$\sigma_c = \left(\frac{sT}{\epsilon + P}\right)^2\Bigg|_{r_c} \sigma_H \quad , \tag{2.151}$$

where $\sigma_c \equiv \sigma(r_c)$, and $\sigma_H \equiv \sigma(r_h)$. The expression for σ_H is same as given in Eq.(2.29). Let us note that, at the boundary Eq.(2.150) reproduces the desired result where as at the horizon, because of Eq.(2.147), σ_c reduces to σ_H which it should. Comparing Eq.(2.150) with Eq.(2.30), we get

$$\begin{aligned} \frac{\phi(r_c)}{\phi(r_h)} &= \frac{\epsilon + P}{sT}\Bigg|_{r_c} \\ &= \frac{sT + \rho\mu}{sT}\Bigg|_{r_c} \\ &= 1 + \frac{\rho\mu}{sT}\Bigg|_{r_c} , \end{aligned} \tag{2.152}$$

where ρ and s, the charge and entropy densities are related to total charge Q and entropy S by a multiplicative factor of volume respectively. So we get $\frac{\rho}{s} = \frac{Q}{S}$. It was also noted in [77], that S, Q are independent of cutoff radius r_c. Using Eq.(2.143) and Eq.(2.148) we get[28]

$$\begin{aligned} \frac{\phi(r_c)}{\phi(r_h)} &= 1 + \frac{\rho}{sT_H} A_t(r_c) \\ &= 1 + \frac{Q}{ST_H} A_t(r_c). \end{aligned} \tag{2.153}$$

Now only work that is remaining is to find whether the solution of the form given in Eq.(2.152) solves Eq.(2.10) for the particular background that we are interested in. One can very easily check that this is indeed the case (more general cases will be discussed in the next chapter). So to summarize, the solution to Eq.(2.10) for this particular background is given by

$$\begin{aligned} \phi(r) &= \frac{\epsilon + P}{sT}\Bigg|_{r} \phi(r_h) \\ &= \left(1 + \frac{\rho}{sT} A_t(r)\right)_r \left(\frac{sT}{\epsilon + P}\right)_{r\to\infty} \phi_0, \end{aligned} \tag{2.154}$$

[28] For the cases where $A_t(r_h) \neq 0$, the solution takes the the form $\frac{\phi(r_c)}{\phi(r_h)} = 1 + \frac{\rho}{sT_H}[A_t(r_c) - A_t(r_h)]$.

where $r \to \infty$ is the boundary and ϕ_0 is the boundary value of ϕ. The electrical conductivity for the fluid at any radius r_c is given by

$$\sigma_c = \left(\frac{sT}{\epsilon + P}\right)^2\Bigg|_{r_c} \sigma_H. \tag{2.155}$$

- **Relation with universal conductivity of the stretched horizon at finite chemical potential:** Again the universal conductivity of the stretched horizon is given by $\sigma_{mb} = \frac{1}{g_{d+1}^2(r_h)}$. Once again we observe that,

$$\sigma_{\mu\neq 0}(r_h) = \sigma_{mb}\ g_{xx}^{\frac{d-3}{2}}(r_h), \tag{2.156}$$

and

$$\sigma_{CFT,\mu\neq 0}(r\to\infty) = \sigma_{mb}\ g_{xx}^{\frac{d-3}{2}}(r_h)\left(\frac{sT_H}{\epsilon + P}\right)^2. \tag{2.157}$$

Let us note that at $\mu = 0$, Eq.(2.156) reduces to Eq.(2.140).

2.8.4 Imaginary part of conductivity $\sigma_0 = \Im(\lambda)$:

In order to gain full knowledge of current-current correlator we need to determine the imaginary part of the electrical conductivity. As we will see, this part of the conductivity also behave in a universal way. Using Eq.(2.153) and Eq.(2.151) we can write,

$$\begin{aligned} \frac{d}{dr}\phi(r) &= \left(\frac{sT}{\epsilon+P}\right)_{r\to\infty}\left(\frac{\rho}{sT}\right)_{r\to\infty} A_t^{'}(r)\ \ \phi_0 \\ &= \left(\frac{\rho}{\epsilon+P}\right)_{r\to\infty} A_t^{'}(r)\ \ \phi_0, \end{aligned} \tag{2.158}$$

where primes denote derivative with respect to r. At the boundary, imaginary part of the conductivity is given by

$$\Im(\lambda) = \frac{1}{\omega\phi^0}\lim_{r\to\infty}\frac{1}{2\kappa^2}N(r)\frac{d}{dr}\phi(r). \tag{2.159}$$

Using Eq.(2.8) and $\rho = -\frac{1}{2\kappa^2}\sqrt{-g}G_{11}g^{tt}g^{rr}A_t^{'}(r)$, we get

$$\Im(\lambda) = -\frac{1}{\omega}\frac{\rho^2}{\epsilon+P}. \tag{2.160}$$

It is interesting to compare Eq.(2.159) with Eq.(2.34). Up on comparison we find,

$$\frac{1}{\epsilon+P} = 2\kappa^2\int_0^1 dr\ \ \frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}\frac{\phi(r)}{\phi^0}. \tag{2.161}$$

Let us note that, in the case when $\mu = 0$, $\frac{\phi(r)}{\phi_0} = 1$. So we get

$$\frac{1}{\epsilon + P} = 2\kappa^2 \int_0^1 du \quad \frac{g_{uu} g_{tt}}{\sqrt{-g} g_{xx}}, \tag{2.162}$$

which is the result reported in [4].

Again one can study the cutoff dependence of imaginary part of the conductivity. Rather than providing details, here we write the result

$$\Im(\lambda)_{r_c} = -\frac{1}{\omega} \left(\frac{g_{tt}}{g_{xx}} \right)_{r_c} \left(\frac{\rho^2}{\epsilon + P} \right)_{r \to \infty}. \tag{2.163}$$

So at the horizon, imaginary part of the conductivity vanishes (since $g_{tt}(r_h) = 0$). To summarize, we see that there emerges a nice and simple picture. The boundary conductivity can expressed in terms of geometrical quantities evaluated at the horizon and thermodynamic quantities. At any radial position r_c outside the horizon the expression for cutoff dependent electrical conductivity ($\sigma(r_c)$), which interpolates smoothly between horizon conductivity $\sigma_H(r_c \to r_h)$ and boundary conductivity $\sigma_B(r_c \to \infty)$.

2.9 Discussion

We conclude that the boundary electrical conductivity takes a universal form in the presence of chemical potential for a large class of black branes which include $R-$charged black branes in various dimensions in asymptotically AdS spaces as well as charged Dp branes in various dimensions. As discussed already, presence of chemical potential brings limitations on the use of Iqbal, Liu results[4]. In fact, we have explicitly seen, boundary and horizon results are no longer the same. In fact, we have seen that there is a smooth interpolation between them. The imaginary part of conductivity can be written as

$$\Im(\lambda) = -\frac{\rho^2}{\epsilon + P} \frac{1}{\omega}, \tag{2.164}$$

where ρ, ϵ and P are the charge density, energy density and pressure of the fluid respectively. Let us mention here that the imaginary part of the conductivity has a pole at $\omega \to 0$ limit because of the translational invariance of the system. The appearance of pole will further be discussed in the next chapter, where we shall also show that the Eq.(2.163) is in fact valid for a wide class of gauge theory with gravity dual. We have also seen that the Lifshitz like black brane does not satisfy the universal form. The question therefore arises: what is the most general background for which the form of boundary conductivity as in Eq.(2.62) and Eq.(2.63) are satisfied? In the next chapter we look for an answer to this question.

3

Universality in electrical conductivity

3.1 Introduction

The fluid/gravity correspondence provides us with two distinct fluids dual to a given black hole geometry: first, the fluid given by membrane paradigm (discussed in the appendix A), which is described by quantities at the black hole horizon and second, the fluid at the boundary of the space time known from gauge/gravity duality and is described by quantities at the boundary. By exploiting the fact that changing radial position in the bulk corresponds to RG flow in the boundary fluid, authors of [4, 77] proposed a number of relations and even interpolation between them. For example, radial independence of certain quantities is used to show that, the shear viscosity (η) to entropy density (s) ratio ($\frac{\eta}{s}$) for both the fluids on the membrane and at the boundary are the same. It can also be shown that the low frequency limit of electrical conductivities of these two fluids computed at zero chemical potential, are related[4]. However, the situation changes significantly at finite chemical potential in the boundary theory (which corresponds to charged black hole in the bulk), where radial independence, exploited earlier in relating electrical conductivities of these two fluids, gets completely destroyed. One needs to solve flow equations in order to relate conductivities of these two fluids. In the last chapter we have seen, for several examples, the electrical conductivity is universal and that there exists a simple relation between the conductivities of the fluids at horizon and at boundary. It was also discussed in the previous chapter that at any radial position r, the conductivity is given by a simple expression which interpolates smoothly between the one computed at the horizon and at the boundary. However, for gauge theories dual to charged Lifshitz like gravity backgrounds, the above mentioned universality does not hold. The purpose of this chapter is to find out the most general background for which the form of boundary conductivity as in Eq.(2.62) and Eq.(2.63) are satisfied.

This chapter is structured as follows. Section 2 is a review of the earlier chapter. This section also serves us to figure out, what we should show in order to prove the universality of electrical conductivity. In section 3, we find the condition on the gravity side energy momentum tensor under which the dual gauge theory will show the universality. This section also discusses several examples, which include theories at and away from conformality. This section also explains as to why the Lifshitz like theories do not show the universality. In section 4, we work with gauge theories at multiple chemical potentials and give general form of the electrical conductivity matrix. In appendix 5, we elaborate upon the condition that we get on energy momentum tensor. Finally we summarize our results of this chapter in section 6.

3.2 What to prove?

Consider action of the form

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x \ \ \sqrt{-g}(R - \frac{1}{4g_{\rm eff}^2(u)}F_{\mu\nu}F^{\mu\nu} + \text{Other terms}), \tag{3.1}$$

and the metric of the form

$$ds^2 = g_{tt}(u)dt^2 + g_{uu}(u)du^2 + g_{xx}(u)\sum_{i=1}^{d-1}(dx^i)^2, \tag{3.2}$$

The perturbed gauge field satisfies

$$\frac{d}{dr}(N(r)\frac{d}{dr}\phi(r)) - \omega^2 N(r)\ g_{rr}g^{tt}\phi(r) + M(r)\phi(r) = 0, \tag{3.3}$$

with

$$N(r) = \sqrt{-g}\frac{1}{g_{\rm eff}^2}g^{xx}g^{rr}, \tag{3.4}$$

and

$$M(r) = \Big(\frac{1}{g_{\rm eff}^2}\Big)^2\sqrt{-g}g^{xx}g^{rr}g^{tt}F_{rt}F_{rt}. \tag{3.5}$$

We can rewrite $M(r)$ in a better way as

$$M(r) = (2\kappa^2)^2\rho^2\frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}. \tag{3.6}$$

where,

$$\rho = \frac{1}{2\kappa^2 g_{\rm eff}^2}\sqrt{-g}g^{rr}g^{tt}F_{rt}. \tag{3.7}$$

Let us note that the Maxwell equations can be written as,

$$\partial_\mu\Big(\frac{1}{g_{\rm eff}^2}\sqrt{-g}F^{\nu\mu}\Big) = 0, \tag{3.8}$$

and we choose the gauge where only $A_t(r)$ component of the background gauge field is non zero (we work with electrically charged black hole).

For evaluating the conductivity in the low frequency limit and for non-extremal backgrounds, we only need to solve equations up to zeroth order in ω. To that order one finds,

$$\frac{d}{dr}(N(r)\frac{d}{dr}\phi(r)) + M(r)\phi(r) = 0. \tag{3.9}$$

The expression for electrical conductivity is given by (see [54, 48, 57] for details),

$$\begin{aligned}\sigma &= \frac{1}{2\kappa^2}\left(\sqrt{\frac{g_{rr}}{g_{tt}}}N(r)\right)_{r=r_h}\left(\frac{\phi(r_h)}{\phi(r\to\infty)}\right)^2 \\ &= \frac{1}{2\kappa^2}\left(\frac{1}{g^2_{\text{eff}}}\ g_{xx}^{\frac{d-3}{2}}\right)_{r=r_h}\left(\frac{\phi(r_h)}{\phi(r\to\infty)}\right)^2 \\ &= \sigma_H\ \left(\frac{\phi(r_h)}{\phi(r\to\infty)}\right)^2,\end{aligned} \tag{3.10}$$

where σ_H is the conductivity evaluated at the horizon and its expression is given by,

$$\sigma_H = \frac{1}{2\kappa^2 g^2_{\text{eff}}}\ g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}. \tag{3.11}$$

we have discussed in the previous chapter, that boundary conductivity is given by

$$\begin{aligned}\sigma &= \sigma_H\ \left(\frac{\phi(r_h)}{\phi(r\to\infty)}\right)^2 \\ &= \sigma_H\left(\frac{sT}{\epsilon+P}\right)^2.\end{aligned} \tag{3.12}$$

Suppose we take the solution of Eq.(3.9) to be

$$\frac{\phi(r)}{\phi(r_h)} = 1 + \frac{\rho}{sT}(A_t(r) - A_t(r_h)), \tag{3.13}$$

where Eq.(3.13) at the boundary reduces to

$$\begin{aligned}\frac{\phi(r\to\infty)}{\phi(r_h)} &= 1 + \frac{\rho}{sT}\mu \\ &= \frac{\epsilon+P}{sT}.\end{aligned} \tag{3.14}$$

So we conclude that proposed form of solution in Eq.(3.13) reproduces exact form of conductivity both at horizon and at the boundary. So in order to show Eq.(3.12) we need to prove Eq.(3.13). In the next section we show that Eq.(3.13) indeed, is the solution to Eq.(3.9).

3.3 Proof for Singly charged black brane

The way we shall proceed is, first we shall assume that the solution to Eq.(3.9) is given by Eq.(3.13). Then we shall use Einstein equation to find out the constraint that our assumption leads to. Then we show how this constraints can be expressed in a compact form in terms of the stress energy momentum tensor of the matter content of the system. We shall also discuss possible meaning of this constraint in the boundary gauge theory.

We start by plugging Eq.(3.13) in Eq.(3.9). This gives

$$\frac{d}{dr}\left(\sqrt{-g}\frac{1}{g_{\rm eff}^2}g^{xx}g^{rr}\frac{\rho}{sT}\frac{d}{dr}A_t(r)\right)+(2\kappa^2)^2\rho^2\frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}\left(1+\frac{\rho}{sT}\big(A_t(r)-A_t(r_h)\big)\right)=0.$$

Using $F_{rt}=\frac{d}{dr}A_t$ and definition of charge density as in Eq.(3.7) we obtain

$$\begin{aligned} 2\kappa^2\frac{\rho^2}{sT}\frac{d}{dr}(g^{xx}g_{tt})+(2\kappa^2)^2\rho^2\frac{g_{rr}g_{tt}}{\sqrt{-g}g_{xx}}\left(1+\frac{\rho}{sT}\big(A_t(r)-A_t(r_h)\big)\right)&=0,\\ \text{or,}\qquad \frac{1}{2\kappa^2}\frac{\sqrt{-g}g_{xx}}{g_{rr}g_{tt}}\frac{d}{dr}(g^{xx}g_{tt})=-sT\left(1+\frac{\rho}{sT}\big(A_t(r)-A_t(r_h)\big)\right). \end{aligned} \tag{3.15}$$

Evaluating Eq.(3.15) at $r=r_h$,we get

$$\frac{1}{2\kappa^2}\frac{\sqrt{-g}g_{xx}}{g_{rr}g_{tt}}\frac{d}{dr}(g^{xx}g_{tt})\bigg|_{r_h}=-sT. \tag{3.16}$$

Subtracting Eq.(3.15) from Eq.(3.16) we get

$$\begin{aligned} \frac{\sqrt{-g}g_{xx}}{g_{rr}g_{tt}}\frac{d}{dr}(g^{xx}g_{tt})\bigg|_{r_h}^{r}&=-2\kappa^2\rho\big(A_t(r)-A_t(r_h)\big)\\ \Rightarrow\quad \left[\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right]_{r_h}^{r}&=-2\kappa^2\rho A_t\bigg|_{r_h}^{r}. \end{aligned} \tag{3.17}$$

Now we use Einstein equations to find out conditions under which Eq.(3.17) is valid. Let us consider the background of the form given in Eq.(A.2). The Einstein equation is given by

$$\begin{aligned} R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R&=T^{E.M.}_{\mu\nu}+T^{Matter}_{\mu\nu}\\ &=\frac{1}{2g_{\rm eff}^2}\left(F_{\mu\lambda}F^{\lambda}_{\nu}-\frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)+T^{Matter}_{\mu\nu}, \end{aligned} \tag{3.18}$$

where $T_{\mu\nu}^{Matter}(r)$, will include all the other stuffs which may come from scalar fields, cosmological constant or any other fields present in the theory. Since only $A_t(r)$ is non-zero, we have $F_{rt} \neq 0$. Using Eq.(3.18), we can write

$$R_t^t - \frac{1}{2} g_t^t R = \frac{1}{2g_{\text{eff}}^2}\left(F_{tr}F^{tr} - \frac{1}{4} g_t^t F_{\rho\sigma}F^{\rho\sigma}\right) + T_t^{t,\ Matter}, \tag{3.19}$$

$$R_x^x - \frac{1}{2} g_x^x R = -\frac{1}{2g_{\text{eff}}^2}\frac{1}{4} g_x^x F_{\rho\sigma}F^{\rho\sigma} + T_x^{x,\ Matter}. \tag{3.20}$$

After subtracting Eq.(3.19) from Eq.(3.20), we get

$$\sqrt{-g}R_t^t - \sqrt{-g}R_x^x = \frac{1}{2g_{\text{eff}}^2}\sqrt{-g}F^{rt}F_{rt} + \sqrt{-g}(T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r)). \tag{3.21}$$

For the metric of the form in Eq.(A.2), following relations hold

$$\sqrt{-g}R_t^t = -\frac{d}{dr}\left(\frac{g_{xx}^{\frac{d-1}{2}}\frac{d}{dr}g_{tt}}{2g_{rr}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}}\right), \tag{3.22}$$

$$\sqrt{-g}R_x^x = -\frac{d}{dr}\left(\frac{g_{xx}^{\frac{d-3}{2}}g_{tt}^{\frac{1}{2}}}{2g_{rr}^{\frac{1}{2}}}\frac{d}{dr}g_{xx}\right), \tag{3.23}$$

which, after substituting in Eq.(3.21), we get,

$$\begin{aligned} -\frac{d}{dr}\left(\frac{g_{xx}^{\frac{d-1}{2}}}{2g_{rr}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}}\frac{d}{dr}g_{tt}\right) + \frac{d}{dr}\left(\frac{g_{x}^{\frac{d-3}{2}}g_{tt}^{\frac{1}{2}}}{2g_{rr}^{\frac{1}{2}}}\frac{d}{dr}g_{xx}\right) &= \frac{1}{2g_{\text{eff}}^2}\sqrt{-g}F^{rt}F_{rt} \\ &+ \sqrt{-g}(T_t^{t,Matter} - T_x^{x,Matter}). \end{aligned} \tag{3.24}$$

Upon further simplification, this reduces to

$$-\frac{d}{dr}\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right) = 2\kappa^2\rho\frac{d}{dr}A_t + 2\sqrt{-g}(T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r)). \tag{3.25}$$

Integrating above equation we get

$$\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right)\Bigg|_{r_h}^{r} = -2\kappa^2\rho A_t\Bigg|_{r_h}^{r} + 2\int_{r_h}^{r} dr\sqrt{-g}(T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r)). \tag{3.26}$$

Thus, if we impose the condition that

$$T_t^{t,\ Matter}(r) = T_x^{x,\ Matter}(r), \tag{3.27}$$

then we get

$$\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right)\Bigg|_{r_h}^{r} = -2\kappa^2\rho A_t\Bigg|_{r_h}^{r}, \tag{3.28}$$

which[29] is same as Eq.(3.17). Hence we have shown that, if the gravity background satisfies Eq.(3.27), then the dual gauge theory will satisfy Eq.(3.10). We suspect that whenever the boundary theory is in the Minkowski space, the condition imposed by Eq.(3.27) on the stress-energy tensor (barring the electromagnetic part) will hold true. This was also observed in [78, 79] in the context of proving the universality of shear viscosity. In the following section, we elaborate upon the above condition considering several examples.

3.3.1 Examples

In all of our examples in this section we will take the metric, gauge fields and other form fields as the functions of coordinate r only. It was observed in [78, 79] that if the scalar and other form fields are functions of the coordinate r only and if the boundary theory lives on the Minkowski space, then ${T_{\mu\nu}}^{\text{Matter}} \sim g_{\mu\nu}(\cdots)$, (where μ, ν are gauge theory indices) which in turn implies the condition given by Eq.(3.27). In what follows, in this section, we first discuss the boundary theories which live on Minkowski space-time where we will find explicitly that the Eq.(3.27) holds good. Next, we discuss one example where the boundary theory does not live on the Minkowski space-time, namely the asymptotically Lifshitz like space-time, where the condition in Eq.(3.27) does not hold.

- **Boundary theories living on Minkowski space-time**
  - **Conformal boundary theories:** Let us note that Reissner Nordström and R-charged black holes in various dimensions in asymptotically AdS space (as already checked in [54]) and in the previous section as well as any other background which satisfies Eq.(3.27), should satisfy Eq.(3.10).
  - **Non-conformal boundary theory:** Non-conformal theories such as gauge theory dual to charged Dp brane satisfies both Eq.(3.27) and Eq.(3.10).

[29] For the backgrounds which satisfies Eq.(3.27), it is interesting to note that, if we set $r \to \infty$, and use first law of thermodynamics as well as the fact that $sT_H = \frac{1}{2\kappa^2}\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right)\Bigg|_{r_h}$, we have $\epsilon + P = \frac{1}{2\kappa^2}\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right)\Bigg|_{r\to\infty}$ from Eq.(3.17). Let us note that we should add the Gibbons-Hawking term and counter terms (see [37]) in order to get finite values.

- **Boundary theory dual to charged Lifshitz like black hole:** For this case it was shown in previous chapter and in [54] that

$$\sigma_B \neq \sigma_H \Big(\frac{sT}{\epsilon + P}\Big)^2. \tag{3.29}$$

Now the above result can be understood easily. Let us consider the following action in $(d+2)$-dimensional space time (see for details in [70, 75])

$$S = \frac{1}{16\pi G_{d+2}} \int d^{d+2}x\sqrt{-g}(R - 2\Lambda - \frac{1}{4}F^2 - \frac{1}{2}m^2 A^2 - \frac{1}{4}F_1^2). \tag{3.30}$$

The corresponding equations of motion are given as follows,

$$\begin{aligned} \partial_\mu(\sqrt{-g}F^{\mu\nu}) &= m^2\sqrt{-g}A^\nu, \quad \partial_\mu(\sqrt{-g}F_1^{\mu\nu}) = 0, \\ R_{\mu\nu} &= \frac{2}{d}\Lambda g_{\mu\nu} + \frac{1}{2}F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{2}F_{1,\mu\lambda}F_{1,\nu}^\lambda + \frac{1}{2}m^2 A_\mu A_\nu \\ &\quad -\frac{1}{4d}F^2 g_{\mu\nu} - \frac{1}{4d}F_1^2 g_{\mu\nu}. \end{aligned} \tag{3.31}$$

From the above equation we can find the energy momentum tensor. Let us write it in the form $T_{\mu\nu}^{total} = T_{\mu\nu}^{E.M.} + T_{\mu\nu}^{Matter}$, where $T_{\mu\nu}^{E.M.}$ contains contribution from gauge field $F_{1,\mu\nu}$ whereas other fields contributes to $T_{\mu\nu}^{Matter}$. Let us note that the massive gauge field A_μ, was introduced to get the Lifshitz like scaling. If we take only non-vanishing components of gauge field to be A_t, then it is easy to see that

$$\begin{aligned} T_t^{t,Matter} - T_x^{x,Matter} &= \frac{1}{2}F_{tr}F^{tr} + \frac{1}{2}m^2 A_t A^t \\ &\neq 0, \end{aligned} \tag{3.32}$$

where $F_{rt} = \frac{d}{dr}A_t$ and also note that $g_t^t = g_x^x = 1$. This provides us with an explanation of Eq.(3.29).

3.4 Universality of electrical conductivity with multiple charges

Now we turn our attention to multiple charged black brane. For convenience we once again write down th equation that governs the perturbed gauge field. We have

$$\frac{d}{dr}(N_I \frac{d}{dr}\phi_I(r)) - \omega^2 N_I\ g_{rr}g^{tt}\phi_I(r) + \sum_{J=1}^{m} M_{IJ}\phi_J(r) = 0, \tag{3.33}$$

with

$$M_{IJ} = F_{rt}^I\sqrt{-g}G_{II}g^{xx}g^{rr}g^{tt}G_{JJ}F_{rt}^J. \tag{3.34}$$

Let us note that $M_{IJ} = M_{JI}$. One can show that, the solution to the Eq.(3.33) can be written as

$$\phi_i = \phi_i^0\Big(1 - \frac{\rho_i}{\epsilon + p}(A_t^i(r) - A_t^i(r=0))\Big) - \frac{A_t^i(r) - A_t^i(r=0)}{\epsilon + p}\sum_{J=1}^{m}\rho_j\phi_j^0, \quad (3.35)$$

where ϕ_i^0 is the boundary value of i'th perturbed gauge field and again the condition on bulk energy momentum tensor as stated as in Eq.(3.27), has to be satisfied[30].

Here we write the diagonal and off diagonal terms of the conductivity matrix.

$$\sigma_{ii} = \frac{1}{8\pi G} g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\Big[G_{ii}(r_h)(1 - \frac{2\mu_i\rho_i}{\epsilon + p}) + \rho_i^2\sum_{j=1}^{m}\frac{G_{jj}(r_h)\mu_j^2}{(\epsilon + p)^2}\Big], \quad (3.36)$$

and off diagonal components with $i \neq j$, we have

$$\sigma_{ij} = \frac{1}{16\pi G} g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\Big[-G_{ii}(r_h)\frac{2\mu_i\rho_j}{\epsilon + p} - G_{jj}(r_h)\frac{2\mu_j\rho_i}{\epsilon + p} + \rho_i\rho_j\sum_{j=1}^{m}\frac{G_{jj}(r_h)\mu_j^2}{(\epsilon + p)^2}\Big]. \quad (3.37)$$

One can now easily check that,

$$\rho_i\sigma_{ij}^{-1}\rho_j = \rho_i\sigma_{H,ii}^{-1}\rho_i\left(\frac{\epsilon + P}{sT}\right)^2, \quad (3.38)$$

as well as

$$\mu_i\sigma_{ij}\mu_j = \mu_i\sigma_{H,ii}\mu_i\left(\frac{sT}{\epsilon + P}\right)^2. \quad (3.39)$$

One can also find out the imaginary part of conductivity and it is given by

$$\Im(\lambda(\omega)_{ij}) = -\frac{i}{\omega}\left(\frac{g_{tt}}{g_{xx}}\right)_{r\to\infty}\frac{1}{16\pi G}\frac{\rho_i\rho_j}{\epsilon + P}. \quad (3.40)$$

3.5 Condition on energy momentum tensor

Let us consider a constant r hyper surface outside the horizon. The unit normal vector to that hyper surface is $n^\mu\partial_\mu = n^r\partial_r$, where $n^r = \sqrt{g^{rr}}$. One can define the extrinsic curvature $\Theta_{\mu\nu}$ of the hyper surface to be

$$\Theta_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu). \quad (3.41)$$

[30] To be more specific, the matter part of the energy momentum tensor that needs to satisfy Eq.(3.27), does not include the any of the $U(1)$ gauge field.

Using the form of the metric as in Eq.(A.2), we get

$$\Theta_{tt} = -\frac{1}{2}\sqrt{g^{rr}}\frac{d}{dr}g_{tt} \quad , \qquad \Theta_{xx} = -\frac{1}{2}\sqrt{g^{rr}}\frac{d}{dr}g_{xx}. \tag{3.42}$$

Using Eq.(3.20) and Eq.(3.19), we can write

$$\sqrt{g}R^t_t = \frac{d}{dr}(\sqrt{h}\Theta^t_t), \quad \sqrt{g}R^x_x = \frac{d}{dr}(\sqrt{h}\Theta^x_x), \tag{3.43}$$

where h is the determinant of the induced metric on the hyper surface. The induced metric on the constant r hyper surface is given by

$$\begin{aligned} ds^2_{\sum} &= h_{tt}dt^2 + h_{xx}\sum_{i=1}^{d-1}(dx^i)^2 \\ &= g_{tt}dt^2 + g_{xx}\sum_{i=1}^{d-1}(dx^i)^2. \end{aligned} \tag{3.44}$$

Let us define a tangent null vector $l^\mu\partial_\mu = \sqrt{-g^{tt}}\partial_t + \sqrt{g^{xx}}\partial_x$. Now we can write Eq.(3.21) and consequently Eq.(3.26) as

$$\begin{aligned} \sqrt{-g}R_{\mu\nu}l^\mu l^\nu &= \sqrt{-g}T^{Total}_{\mu\nu}l^\mu l^\nu \\ &= \sqrt{-g}T^{E.M.}_{\mu\nu}l^\mu l^\nu + \sqrt{-g}T^{Matter}_{\mu\nu}l^\mu l^\nu, \end{aligned} \tag{3.45}$$

$$\begin{aligned} \sqrt{-h}\Theta_{\mu\nu}l^\mu l^\nu\Big|_{r_h}^{r} &= \int_{r_h}^{r} dr\sqrt{-g}T^{E.M.}_{\mu\nu}l^\mu l^\nu + \int_{r_h}^{r} dr\sqrt{-g}T^{Matter}_{\mu\nu}l^\mu l^\nu \\ &= -\kappa^2\rho A_t\Big|_{r_h}^{r} + \int_{r_h}^{r} dr\sqrt{-g}T^{Matter}_{\mu\nu}l^\mu l^\nu, \end{aligned} \tag{3.46}$$

respectively. Upon using the Einstein equation (3.18) and the fact that for the metric of the form given in Eq.(A.2), the R_{xt} component of the Ricci tensor is zero, we get $T^{Matter}_{tx} = 0$, since $T^{E.M.}_{tx} = 0$. So the condition that we get on the energy momentum tensor[31] in Eq.(3.27) can be written as

$$T^{Matter}_{\mu\nu}l^\mu l^\nu = 0. \tag{3.47}$$

[31] According to null energy condition, $T^{total}_{\mu\nu}l^\mu l^\nu \geq 0$, with l^μ a null vector. Since $T^{E.M}_{\mu\nu}l^\mu l^\nu \geq 0$, the contribution from the matter part $T^{Matter}_{\mu\nu}l^\mu l^\nu$ may be negative as well. However it is interesting to note that, if we take a limit where charge of the black hole vanishes then $T^{E.M}_{\mu\nu}l^\mu l^\nu = 0$, so that null energy condition gives $T^{Matter}_{\mu\nu}l^\mu l^\nu \geq 0$. So if we are interested in the backgrounds where matter sector does not act as a source for electromagnetic field, it appears that $T^{Matter}_{\mu\nu}l^\mu l^\nu \geq 0$, irrespective of the presence of gauge fields.

We get a better understanding of the Eq.(3.27), by looking for simplest case of no black hole and uncharged solution. The Eq.(3.25) reduces to,

$$-\frac{d}{dr}\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{rr}^{\frac{1}{2}}}\frac{d}{dr}(g^{xx}g_{tt})\right) = 2\sqrt{-g}(T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r)). \tag{3.48}$$

So if we demand

$$T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r) = 0, \tag{3.49}$$

then we get $g^{xx}g_{tt} \sim -1$. This might be related to the fact that vacuum of dual gauge theory being Lorentz invariant.

3.6 Discussion

We have shown that, for $\mu \neq 0$, given that the form of Maxwell part of the action is

$$S = -\int d^{d+1}x\ \sqrt{-g}\frac{1}{4g_{eff}^2}F_{MN}F^{MN}, \tag{3.50}$$

the electrical conductivity at the boundary is given by

$$\begin{aligned}\sigma_B &= \frac{1}{g_{eff}^2}g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\frac{(sT)^2}{(\epsilon+P)^2}\\ &= \sigma_H\frac{(sT)^2}{(\epsilon+P)^2},\end{aligned} \tag{3.51}$$

where $\sigma_H = \frac{1}{g_{eff}^2}g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}$, is the electrical conductivity evaluated radially at the horizon. We can argue that once the real part of the conductivity is known, the imaginary part of conductivity is automatically fixed. To summarize, in the presence of chemical potential the electrical conductivity can be expressed as

$$\lambda = -\frac{i}{\omega}\left(\frac{g_{tt}}{g_{xx}}\right)_{r\to\infty}\frac{\rho^2}{\epsilon+P} + \frac{1}{g_{eff}^2}\ g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\frac{(sT)^2}{(\epsilon+P)^2}. \tag{3.52}$$

Let us mention here that the imaginary part of the conductivity has a pole at $\omega \to 0$ limit because of the translational invariance of the system. If one uses the Krammers-Kronig relation

$$\Im(\lambda(\omega)) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\Re(\lambda(\omega'))}{\omega'-\omega}d\omega', \tag{3.53}$$

then one can find that the real part of the conductivity contains a delta function if the imaginary part has a pole. As we have found a pole in the imaginary part of the conductivity, it follows that real part has a delta function singularity at $\omega = 0$. So, strictly speaking DC conductivity that we have computed is low frequency limit of AC conductivity or more precisely expression for conductivity is valid for $\omega \to 0^+$, see [32, 61] for a nice discussion.

It is interesting to note that the cutoff dependent conductivity can be computed and it interpolates smoothly between the results at the horizon and at the boundary. At any cutoff r_c the expression for electrical conductivity[32] can be written as

$$\lambda = -\frac{i}{\omega}\left(\frac{g_{tt}}{g_{xx}}\right)_{r_c}\left(\frac{\rho^2}{\epsilon+P}\right)_{r\to\infty} + \frac{1}{g_{eff}^2}\, g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\frac{(sT)^2}{(\epsilon+P)^2}\Big|_{r=r_c}, \tag{3.54}$$

where $r \to \infty$ is the boundary of the space time. It is interesting to compare our results with the results obtained from the membrane paradigm arguments. We have seen, that irrespective of the theory, the horizon conductivity is given by

$$\sigma_H = \frac{1}{g_{eff}^2}\, g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}, \tag{3.55}$$

whereas the universal conductivity of the membrane is given by

$$\sigma_{membrane} = \frac{1}{g_{eff}^2}\Big|_{r=r_h}. \tag{3.56}$$

So we conclude that the horizon conductivity is given by,

$$\sigma_H = \sigma_{mem} g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}. \tag{3.57}$$

We have also seen that for the backgrounds that satisfies Eq.(3.27), the boundary electrical conductivity can be related to horizon conductivity using thermodynamic quantities. More precisely we can write,

$$\begin{aligned}\sigma_B &= \sigma_H \frac{(sT)^2}{(\epsilon+P)^2}\\ &= \sigma_{mem}\, g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}\frac{(sT)^2}{(\epsilon+P)^2}.\end{aligned} \tag{3.58}$$

[32]Let us note that, at any radius r_c, the local temperature and the chemical potential can be given by $T_c = \frac{T_H}{\sqrt{g_{tt}(r_c)}}$ and $\mu_c = \frac{A_t(r_c)-A_t(r_h)}{\sqrt{g_{tt}(r_c)}}$ respectively. Assuming first law of thermodynamics $\epsilon(r_c) + P(r_c) = sT_c + \rho\mu_c$ to hold at and radius and using Eq.(3.13) we get

$$\frac{\phi(r_c)}{\phi(r_h)} = \frac{sT}{\epsilon+P}\Big|_{r=r_c},$$

and consequently Eq.(3.54).

Since mass dimension of electrical conductivity is $d-3$, one can understand the factor $g_{xx}^{\frac{d-3}{2}}$ as the converter of the length scale of the boundary to the proper length at the horizon [4, 57]. It would be very interesting to understand the meaning of extra factor $(\frac{sT}{\epsilon+P})^2$ that appears in the formula due to presence of chemical potential. At this moment it is not quite clear to us how to interpret it directly from the constraint Eq.(3.27) which appears to be related to Lorentz invariance of the vacuum of the field theory. Let us note that, at zero chemical potential

$$\begin{aligned} \sigma_B &= \sigma_H \\ &= \sigma_{mem}\; g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}, \end{aligned} \tag{3.59}$$

as was shown in [4].

In our result of the electrical conductivity, σ_H is given entirely in terms of the geometrical quantities evaluated at the horizon. A natural question that arises, whether it is possible to give an intrinsic meaning to the expression of conductivity in terms of field theory quantities? This will bring the formula for electrical conductivity in the same footing as celebrated universal result for $\frac{\eta}{s}$. Answer to this comes from the expression of thermal conductivity to viscosity ratio. As it was shown in [48] and will be discussed in the next chapter that, electrical conductivity can be expressed in terms of the field theory quantities alone.

4

Universality in thermal conductivity to viscosity ratio

4.1 Introduction

In the previous chapter we have shown that electrical conductivity can be expressed in terms of geometrical quantities evaluated at the horizon and thermodynamic variables. A natural question therefore arises: is it possible to give an expression for electrical conductivity solely in terms of boundary gauge theory variables? In this chapter we provide an affirmative answer to this question. With supports coming from various examples, we further conclude that the thermal conductivity also shows some universal behavior. More precisely, we propose that for a d dimensional strongly coupled gauge theory

$$\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2 = \frac{d^2}{d-2}\Big(\frac{c^{'}}{k^{'}}\Big) = 8\pi^2\frac{d-1}{d^3(d+1)}\frac{c}{k}, \tag{4.1}$$

where κ_T is the thermal conductivity (the heat current response to thermal gradient in the absence of electrical current), T is the temperature, μ the chemical potential, η the shear viscosity and c, k are central charges of dual gauge theory. The dimensionless constants $c^{'}$ and $k^{'}$ are roughly related to total and charged degrees of freedom and are related to to pressure and charge susceptibility of the system at equlibrium. We test our proposal against several examples. However a general proof of this result is still lacking. Using this universality we also find out electrical conductivity in terms of boundary thermodynamic variables.

This chapter is organized as follows. In the next section, thermal conductivity and thermal conductivity to viscosity ratio is computed for several examples. In the third section, after reviewing the standard result for viscosity to conductivity ratio at vanishing chemical potential, we show that Eq.(4.1) holds at $\mu = 0$.[33] Then, based on few examples, we conjecture that Eq.(4.1) holds true even for arbitrary

[33] Let us note that in the presence of equal number of positive and negative charges, chemical potential is zero.

nonzero chemical potential. Subsequently in section 4, using Eq.(4.1), we provide a way to compute electrical conductivity in terms of thermodynamical quantities alone even in the presence of non-zero chemical potential. In section 5, we compute thermal conductivity to viscosity ration for several non-conformal gauge theories and observe that they again behave universally. We end this chapter with a brief summary of the results.

4.2 Thermal conductivity

One of the aim of this section is to study thermal conductivity (κ_T) as defined in Eq.(1.113). In the following we shall start with the examples of computation of the thermal conductivity and thermal conductivity to viscosity ratio ($\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2$) for $R-$charged black holes in $4,5,7$ dimensions.

4.2.1 Single charge black hole

Note that for single charge black hole $\frac{1}{\rho_i \varkappa_{ij}^{-1}\rho_j} = \frac{\varkappa}{\rho^2}$. So that one gets

$$\kappa_T = \left(\frac{\epsilon + P}{\rho T}\right)^2 \varkappa = \left(\frac{\epsilon + P}{\rho}\right)^2 \frac{\sigma}{T} \tag{4.2}$$

Rather than providing details we here tabulate the thermal conductivity and dimension less ratio $\frac{\kappa_T \mu^2}{\eta T}$, where η is the shear viscosity.

Dimension	κ_T	$\frac{\kappa_T \mu^2}{\eta T}$
5	$\frac{(1+k)^2 N^2 T^2 \pi}{k(2+k)}$	$8\pi^2$
4	$\frac{2\sqrt{2}(1+k)^{3/2} N^{3/2} T\pi}{3k}$	$32\pi^2$
7	$\frac{8(1+k)^3 N^3 T^4 \pi^2}{3k(3+k)^3}$	$2\pi^2$

Table 4.1: Thermal conductivity to shear viscosity ratio for single charge black hole

4.2.2 Two charge black hole

In the ratio $\frac{\kappa_T \mu^2}{\eta T}$, μ^2 is replaced by $\mu_1^2 + \mu_2^2$. Note that $\mu_i \to -\mu_i$ is a symmetry [34] which implies reversing the sign of charge density.

Dimension	κ_T	$\frac{\kappa_T(\mu_1^2+\mu_2^2)}{\eta T}$
5	$\frac{N^2 T^2 \pi}{\left(2+\sum_{j=1}^{2} k_j\right)\left(\sum_{j=1}^{2} \frac{k_j}{(1+k_j)^2}\right)}$	$8\pi^2$
4	$\frac{(2N)^{\frac{3}{2}} T\pi}{\left(\sum_{j=1}^{2} \frac{3k_j}{(1+k_j)^2}\right)\sqrt{\prod_{i=1}^{2}(1+k_i)}}$	$32\pi^2$
7	$\frac{(2N)^3 T^4 \pi^2 \prod_{i=1}^{2}(1+k_i)}{\left(\sum_{j=1}^{2} \frac{3k_j}{(1+k_j)^2}\right)\left(3+\sum_{j=1}^{2} k_j - \prod_{j=1}^{2} k_j\right)^3}$	$2\pi^2$

Table 4.2: Thermal conductivity to shear viscosity ratio for two charge black hole

4.2.3 Three charge black hole

In this case the results are summarized below

4.2.4 Four charge black hole (4 Dimensional black hole)

The thermal conductivity is given by

$$\kappa_T = \frac{(2N)^{\frac{3}{2}} T\pi}{\left(\sum_{j=1}^{4} \frac{3k_j}{(1+k_j)^2}\right)\sqrt{\prod_{i=1}^{4}(1+k_i)}}, \tag{4.3}$$

and

$$\frac{\kappa_T(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2)}{\eta T} = 32\pi^2. \tag{4.4}$$

[34] The expression for κ_T in Eq.(1.113) is invariant under $SO(m)$ rotation among ρ_i's.

Dimension	κ_T	$\frac{\kappa_T(\mu_1^2+\mu_2^2+\mu_3^2)}{\eta T}$
5	$\frac{N^2T^2\pi}{\left(2+\sum_{j=1}^{3}k_j-\prod_{j=1}^{3}k_j\right)\left(\sum_{j=1}^{3}\frac{k_j}{(1+k_j)^2}\right)}$	$8\pi^2$
4	$\frac{(2N)^{\frac{3}{2}}T\pi}{\left(\sum_{j=1}^{3}\frac{3k_j}{(1+k_j)^2}\right)\sqrt{\prod_{i=1}^{3}(1+k_i)}}$	$32\pi^2$

Table 4.3: Thermal conductivity to shear viscosity ratio for three charge black hole

- $\underline{\mu \neq 0}$ **:** We observe that irrespective of number of chemical potential turned on, thermal conductivity to viscosity ratio shows same value although expression for thermal conductivity changes with number of chemical potentials.
- $\underline{\mu = 0}$ **:** We also observe that as $\mu \to 0$ i.e. $\rho \to 0$, thermal conductivity given in Eq.(4.2), diverges which implies finite, non decaying momentum. In spite of this divergence, we shall observe in the next section that thermal conductivity to viscosity ratio remains same as in the non zero chemical potential case. In the following we shall first concentrate at zero chemical potential case.

4.3 Universality in thermal conductivity to viscosity ratio

In the following we first review the relation between electrical conductivity and shear viscosity at vanishing chemical potential [3]. In a CFT, short distance physics is described by singularities of correlation functions where central charges of the theory appear explicitly (in this energy scale effects of temperature are irrelevant). For example let us consider correlation functions of energy momentum tensor $T_{\mu\nu}$ and $U(1)$ conserved current J_μ

$$\langle J(x)J(0)\rangle \sim \frac{k}{x^{2(d-1)}}, \quad \langle T(x)T(0)\rangle \sim \frac{c}{x^{2d}}, \tag{4.5}$$

where central charges c, k measure the number of total degrees of freedom and the number of charged degree of freedom of the system[35] respectively. We also know that at long distances physics is described by thermodynamics and transport

[35] So we expect $k \leq c$.

coefficients. In this scale, the effect of temperature becomes important. To describe equilibrium at $T \neq 0$, we look at pressure and charge susceptibility $\chi = \frac{\rho}{\mu}$ where $\rho(T, \mu)$ is the charge density . If T is the only scale in the theory [36], then

$$P = c^{'} T^{d}, \qquad\qquad \chi = k^{'} T^{d-2}, \tag{4.6}$$

where $c^{'}, k^{'}$ measure the number of total degree of freedom and number of charged degree of freedom at that scale. For $d > 2$, in general there is no relation between $c, c^{'}$ and $k, k^{'}$. But it was shown in [3] that for CFT's which admit gravity duals, there exist such relation and are given by

$$\frac{c^{'}}{c} = \frac{1}{4\pi^{\frac{d}{2}}} \Big(\frac{4\pi}{d}\Big)^{d} \frac{\Gamma(d/2)^3}{\Gamma(d)} \frac{d-1}{d(d-1)}, \qquad \frac{k^{'}}{k} = \frac{1}{2\pi^{\frac{d}{2}}} \Big(\frac{4\pi}{d}\Big)^{d-2} \frac{\Gamma(d/2)^3}{\Gamma(d)} \tag{4.7}$$

where[37] $d \geq 3$.

It is well known that, in this class of CFT's, even certain transport coefficients are determined in-terms of thermodynamical quantities (for example $\eta = \frac{s}{4\pi}$). Other such relation between viscosity and conductivity (σ) at vanishing chemical potential ($\mu = 0$) is

$$\frac{\eta}{\sigma T^2} = (d-2)\Big(\frac{c^{'}}{k^{'}}\Big) = 8\pi^2 \frac{(d-1)}{(d-2)d(d+1)} \frac{c}{k}. \tag{4.8}$$

Eq.(4.8) implies at vanishing chemical potential i.e. at $\mu = 0$, electrical conductivity can be computed in terms of central charges only. Using Eq.(4.8), (4.6) and $s = d\ c^{'} T^{d-1}$ one gets

$$\eta = \frac{d}{4\pi} c^{'} T^{d-1}, \qquad \sigma = \frac{1}{d-2} \frac{d}{4\pi} k^{'} T^{d-3}. \tag{4.9}$$

Since thermodynamics is determined by the central charges, we conclude that the momentum (η) and charge (σ) transport are fixed by thermodynamics[38]. Existence

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[36] to define χ, one can introduce small chemical potential and see the effect in ρ .

[37] In our notation d is the dimension of gauge theory.

[38] As an aside lets review membrane paradigm arguments. It was shown in [4] using membrane paradigm arguments that at $\mu = 0$, electrical conductivity can be determined in terms of geometry only. If we use the results in [4], we immediately reach at

$$\frac{\eta}{\sigma T^2} = \frac{1}{T^2} \frac{g_{d+1}^2}{16\pi G_N} g_{xx}(r_0). \tag{4.10}$$

As an example consider a CFT with the gravity dual given by AdS_{d+1} with $d \neq 3$, which has a metric

$$ds^2 = \frac{r^2}{R^2}\Big(- f(r) dt^2 + dx_1^2 + \cdots + dx_{d-1}^2 \Big) + \frac{R^2}{f(r) r^2} dr^2, \tag{4.11}$$

of such relations between thermodynamics and transport coefficient are interesting [39], since transport coefficients are characterized by inelastic collisions among thermally excited carriers (of energy $\sim T$) hence they are not fixed in terms of thermodynamics.

What we conclude from above discussion is that, at non zero temperature and at $\mu = 0$, certain transport coefficients are determined by thermodynamics. It is interesting to ask whether for $\mu \neq 0$ and at finite temperature, transport coefficients can be determined from thermodynamics. We note that in this case it is already known that $\frac{\eta}{s} = \frac{1}{4\pi}$ still holds i.e. momentum transport can be determined solely by thermodynamics. It would be interesting if one can express the electrical conductivity which encodes the charge transport, in terms of thermodynamics.

We now proceed to provide evidences in favor of Eq.(4.1). In what follows, we first derive equation for $\mu = 0$ and then provide support for cases with $\mu \neq 0$.

- **Derivation of Eq.(4.1) for** $\mu = 0$ **:** Let us consider theory at small (single) chemical potential and consider the ratio $\frac{\kappa_T}{\eta T}\mu^2$. Using the relation[40] $\kappa_T = \left(\frac{\epsilon+P}{\rho}\right)^2 \frac{\sigma}{T}$, one obtains

$$\frac{\kappa_T}{\eta T}\mu^2 = \left(\epsilon + P\right)^2 \frac{1}{\left(\frac{\rho}{\mu}\right)^2} \frac{1}{\left(\frac{\eta}{\sigma T^2}\right)} \frac{1}{T^4}. \tag{4.13}$$

Now taking $\mu \to 0$, using $\epsilon = (d-1)P$, $\chi = \frac{\rho}{\mu}$ we immediately get

$$\frac{\kappa_T}{\eta T}\mu^2 = \frac{d^2}{d-2}\left(\frac{c'}{k'}\right) = 8\pi^2 \frac{d-1}{d^3(d+1)} \frac{c}{k}. \tag{4.14}$$

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with $f(r) = 1 - (\frac{r_0}{r})^d$ and hawking temperature is $T_H = \frac{d}{4\pi}\frac{r_0}{R^2}$ where r_0 and R are the position of horizon and AdS curvature scale respectively. Using the above relations we obtain,

$$\frac{\eta}{\sigma T^2} = \frac{\pi}{d^2}\frac{R^2 g_{d+1}^2}{G_{d+1}} \tag{4.12}$$

which is same as reported in [3].

[39] We note that hydrodynamics description is valid in the energy range $\omega \ll T$ which is collision dominated regime [80, 81]

[40] Let us note that, at $\mu = 0$, the charge density vanishes such that $\frac{\rho}{\mu}$ remains finite. So, in this limit, the thermal conductivity diverges, which implies finite non-decaying momentum. Naively, one can understand this in the following way. At finite chemical potential, there is a net charge density. Now we imagine having a temperature gradient, under which there will be flow of charges from lower temperature to higher temperature region. This will imply a net current. So, one needs to apply voltage gradient in order to have zero current, which will effectively results in decaying momenta due to collision. In the case when there is no net chrage, there is no net current flow under temperature gradient and hence one does not require to apply a voltage gradient. This cause a finite but non-decaying momenta (see[80, 81], for further details).

- **Support for Eq.(4.1) for $\mu \neq 0$:** For non zero chemical potential, we recall some of the results already reported in the literature. In each case we show that they follow Eq.(4.1). Here we tabulate the results for strongly coupled gauge theories having gravity duals in the presence of single non zero chemical potential [32, 36, 48].

Gravity theory in $d+1$ dimension	$\frac{\kappa_T \mu^2}{\eta T}$	$\frac{d^2}{d-2}\left(\frac{c'}{k'}\right)$
R-charge B.H. in $4+1$ dim.	$8\pi^2$	$8\pi^2$
R-charge B.H. in $3+1$ dim.	$32\pi^2$	$32\pi^2$
R-charge B.H. in $6+1$ dim.	$2\pi^2$	$2\pi^2$
Reissner-Nordstrom B.H. in $3+1$ dim.	$4\pi^2\gamma^2$	$4\pi^2\gamma^2$

Table 4.4: Thermal conductivity to viscosity ratio at finite chemical potential

It was further reported in [48] that for the R-charged black holes in five, four and seven dimensions the appropriate ratio of thermal conductivity and viscosity, regardless of the number of charge contents, are $8\pi^2$, $32\pi^2$ and $2\pi^2$ respectively. Based on these observations we propose that, even in the presence of finite chemical potential (and arbitrary number of them) we can write

$$\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2 = \frac{d^2}{d-2}\left(\frac{c'}{k'}\right) = 8\pi^2\frac{d-1}{d^3(d+1)}\frac{c}{k}. \tag{4.15}$$

In the next section we use (4.15) to express electric conductivity in terms of the thermodynamical quantities alone.

4.4 Electrical conductivity

Let us first write down various expressions for thermodynamical quantities, transport coefficients such as viscosity, electrical conductivity in the presence of single chemical potential. In our definition, $\chi = \frac{\rho}{\mu}$. In case of nonzero chemical potential we expect different thermodynamical quantities and transport coefficients to get modified from that of Eq.(4.6), (4.9) . In general these can be written as

$$P = c'T^d f_p(m), \qquad \chi = k'T^{d-2}f_\chi(m), \tag{4.16}$$

and

$$\sigma = \frac{1}{d-2}\frac{d}{4\pi}k'T^{d-3}f_\sigma(m), \qquad \eta = \frac{d}{4\pi}c'T^{d-1}f_\eta(m), \tag{4.17}$$

where $m = \frac{\mu}{T}$ and $f(m)$'s are defined such that $f(m = 0) = 1$. Now using

$$\frac{\kappa_T \mu^2}{\eta T} = \left(\frac{(\epsilon + P)\mu}{\rho}\right)^2 \frac{\sigma}{\eta T^2} = \frac{d^2}{d-2}\left(\frac{c'}{k'}\right), \tag{4.18}$$

we get an important constraint between the function $f(m)$'s

$$\frac{f_p^2 f_\sigma}{f_\chi^2 f_\eta} = 1, \tag{4.19}$$

which gives $f_\sigma = \frac{f_\chi^2 f_\eta}{f_p^2}$. We then obtain expression for conductivity[41]

$$\sigma = \frac{1}{d-2}\frac{d}{4\pi} k' T^{d-3} \frac{f_\chi^2 f_\eta}{f_p^2}, \tag{4.20}$$

which is entirely fixed in terms of central charges (and thermodynamic quantities).

4.4.1 Examples

Here we present computations which led to the results of Table 3 in the previous section. We shall also illustrate with an example, how to use Eq.(4.20) to determine conductivity.

- **AdS$_4$ Reissner-Nordstrom blackhole:** The action is

$$S = \int d^4x \sqrt{-g}\Big[\frac{1}{2\kappa^2}(R + \frac{6}{L^2}) - \frac{1}{4g^2}F^2\Big]. \tag{4.21}$$

Metric is given by (for details see [32])

$$ds^2 = \frac{L^2}{r^2}(-f(r)dt^2 + \frac{dr^2}{f(r)} + dx^i dx^i). \tag{4.22}$$

Thermodynamical quantities are given by

$$T = \frac{1}{4\pi r_+}(3 - \frac{r_+^2 \mu^2}{\gamma^2}), \quad P = \frac{L^2}{2\kappa^2 r_+^3}(1 + \frac{r_+^2 \mu^2}{\gamma^2}) \tag{4.23}$$

and

$$S = \frac{2\pi}{\kappa^2}\frac{L^2}{r_+^2}, \quad \chi = \frac{\rho}{\mu} = \frac{2L^2}{\kappa^2}\frac{1}{r_+ \gamma^2} \tag{4.24}$$

[41] We may also write it as, $\sigma = \frac{d^2}{d-2}\left(\frac{c'}{k'}\right)\chi^2 \frac{\eta T^2}{(\epsilon+P)^2}$, where $\chi = \frac{\rho}{\mu}$.

where r_+ is the horizon radius and $\gamma^2 = \frac{2g^2L^2}{\kappa^2}$. To find out $c^{'}$ and $k^{'}$ best is to set μ to zero (then express r_+ in terms of T) and compare with Eq.(4.6). After doing this one finds

$$c^{'} = \frac{L^2}{2\kappa^2}(\frac{4\pi}{3})^3, \quad k^{'} = \frac{8\pi}{3}\frac{L^2}{\kappa^2\gamma^2}. \tag{4.25}$$

For this background with nonzero chemical potential, electrical conductivity is given by $\sigma = \frac{(sT)^2}{(\epsilon+P)^2}\frac{1}{g^2}$. Using this result we can find out thermal conductivity. On evaluating the ratio $\frac{\kappa_T\mu^2}{\eta T}$ one finds it to be equal to $4\pi^2\gamma^2$. Up on evaluating the ratio $\frac{d^2}{d-2}\left(\frac{c^{'}}{k^{'}}\right)$ we get the same result (note that here d=3). Let us note that, as for the special case of setting all the R-charges equal for R-charged black hole, one obtains Reissner-Nordstrom black hole with the identification $\gamma^2 = 8$ (see Appendix.B.). So we get $\frac{\kappa_T\mu^2}{\eta T} = 32\pi^2$ which matches with that written for R-charged black hole in four dimensions (see Table.4.4).

- **Five dimensional R-charged black hole:** Viscosity and various thermodynamical quantities are given by

$$T = \frac{2+\kappa_1}{2\sqrt{(1+\kappa_1)}}T_0\,, \tag{4.26}$$

$$\eta = \frac{\pi N^2T^3}{8}\frac{(1+\kappa_1)^2}{(1+\frac{\kappa_1}{2})^2}\,, \tag{4.27}$$

$$P = \frac{\pi^2N^2T^4}{8}\frac{(1+\kappa_1)^3}{(1+\frac{\kappa_1}{2})^4}\,. \tag{4.28}$$

where T_0 is the temperature at vanishing κ_1 i.e. at vanishing chemical potential. The charge density is given by

$$\rho = \frac{\pi N^2T_0^3}{8}\sqrt{2\kappa_1}(1+\kappa_1)^{1/2}\,. \tag{4.29}$$

The chemical potential conjugate to ρ is defined as

$$\mu = A_t(u)\Big|_{u=1} = \frac{\pi T_0\sqrt{2\kappa_1}}{(1+\kappa_1)}(1+\kappa_1)^{1/2}\,, \tag{4.30}$$

so that susceptibility is given by

$$\chi = \frac{\rho}{\mu} = \frac{N^2T^2}{8}\frac{(1+\kappa_1)^2}{(1+\frac{\kappa_1}{2})^2}, \tag{4.31}$$

where we have used Eq.(4.26) to express T_0 in terms of T.

Upon comparing Eq.(4.16) and Eq.(4.17) with Eq.(4.27), Eq.(4.28) and Eq.(4.31) we get

$$f_\chi(m) = \frac{(1+\kappa_1)^2}{(1+\frac{\kappa_1}{2})^2}, \quad f_\eta(m) = \frac{(1+\kappa_1)^2}{(1+\frac{\kappa_1}{2})^3}, \quad f_p(m) = \frac{(1+\kappa_1)^3}{(1+\frac{\kappa_1}{2})^4}, \tag{4.32}$$

and

$$c^{'} = \frac{\pi^2 N^2}{8}, \quad k^{'} = \frac{N^2}{8}. \tag{4.33}$$

Using Eq.(4.20) and f_χ, f_η, f_p and $k^{'}$ written in the above equations, we obtain

$$\sigma = N^2 T\Big(\frac{2+\kappa_1}{32\pi}\Big), \tag{4.34}$$

where κ_1 can be expressed in terms of m. This is same as the result reported in the literature [58, 48].

- **4 and 7 dimensional R-charge black holes:** In order to avoid repetition, here we just list values of $c^{'}$ and $k^{'}$ which were used in the Table 3. In four dimensions we have $k^{'} = \frac{N^{\frac{3}{2}}}{18\sqrt{2}}$, and $c^{'} = \frac{\sqrt{2}\pi^2}{3}\left(\frac{2}{3}\right)^3 N^{3/2}$. In seven dimensions we have $k^{'} = \left(\frac{2}{3}\right)^5 N^3\pi$, and $c^{'} = \frac{\pi^3}{2}\left(\frac{2}{3}\right)^7 N^3$.

4.5 Away from conformality

In the above discussion we considered cases where bulk geometries are asymptotically AdS. Now we turn our attention to the cases where bulk geometries are asymptotically non AdS. We show that, in this case as well the ratio $\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2$ is independent of number of chemical potential and same as uncharged cases. The examples that we have in mind is charged and uncharged Dp branes. The relevant details of geometry was discussed in the second chapter. We first discuss uncharged cases. The electrical conductivity is given by

$$\begin{aligned} \sigma &= \frac{1}{16\pi G}\frac{1}{g_{eff}^2} g_{xx}^{\frac{p-2}{2}}\Big|_{r_h} \\ &= \frac{1}{16\pi G}(g r_h)^{\frac{7-n}{2}}. \end{aligned} \tag{4.35}$$

It is easy to see that,

$$\begin{aligned} D_R &= \frac{\sigma}{\chi} \\ &= \frac{7-p}{8\pi T}, \end{aligned} \tag{4.36}$$

as was shown in [5], where

$$\begin{aligned} \chi &= \frac{\rho}{\mu} \\ &= \frac{1}{8\pi G} g^3 r_h^2. \end{aligned} \tag{4.37}$$

Let us note that, though ρ and μ go to zero separately for uncharged Dp brane, χ in Eq.(4.37), remains non-zero. Now using expression for thermal conductivity, $\kappa_T = \frac{(\epsilon+p)^2\sigma}{\rho^2 T}$, we get

$$\begin{aligned} \frac{\kappa_T}{\eta T}\mu^2 &= 4\pi \left(\frac{\sigma}{\chi}\right)^2 \frac{s}{\sigma} \\ &= \frac{4\pi^2}{g^2}. \end{aligned} \tag{4.38}$$

Note that, from Eq.(4.38), we see that thermal conductivity to viscosity ratio is same for any uncharged Dp brane. Also note, to match with charged $D1$ brane, replace η by bulk viscosity and $g = \frac{1}{L}$.

Our next aim is to see whether for charged non-conformal theories dual to charged Dp brane, thermal conductivity to viscosity ratio remains $\frac{4\pi^2}{g^2}$.

- **Single charge case:** Here we have

$$\sigma = \frac{1}{16\pi G} \frac{1}{X^2} g_{xx}^{\frac{p-2}{2}} \Big|_{r_h} \left(\frac{sT}{\epsilon + P}\right)^2. \tag{4.39}$$

  Next using the fact that,

$$\frac{\rho}{\mu} = \frac{1}{8\pi G} g^3 r_h^2 H(r_h), \tag{4.40}$$

  we get

$$\frac{K_T \mu^2}{\eta T} = \frac{4\pi^2}{g^2}, \tag{4.41}$$

  which is same as we get for uncharged case.

- **Multi charge case:** For multi charge case

$$\rho_i \sigma_{ij}^{-1} \rho_j = \rho_i \sigma_{H,ii}^{-1} \rho_i \left(\frac{\epsilon + P}{sT}\right)^2, \tag{4.42}$$

where $\sigma_{H,ii}^{-1}$ is the inverse of electrical conductivity evaluated at the horizon and only depends on geometrical quantities evaluated at the horizon. The expression for electrical conductivity at the horizon is given by,

$$\begin{aligned}\sigma_{H,ii} &= \frac{1}{16\pi G} G_{ii}(r)\ g_{xx}^{\frac{p-2}{2}}\Big|_{r=r_h} \\ &= \frac{1}{16\pi G}\frac{1}{X_i^2}\ g_{xx}^{\frac{p-2}{2}}\Big|_{r=r_h} \\ &= \frac{g^{\frac{7-n}{2}} r_h^3 H_i^2(r_h)}{16\sqrt{2m}\ \pi G}.\end{aligned} \tag{4.43}$$

Using this result, it can be easily shown that,

$$\frac{K_T \sum_{i=1}^{b} \mu_i^2}{\eta T} = \frac{4\pi^2}{g^2}. \tag{4.44}$$

For $D1$ brane η is replaced by $\frac{s}{4\pi}$ (which is same as bulk viscosity for single charge case or equally charged $D1$ brane case as shown in [61]).

4.6 Discussion

In this chapter we have discussed the universality of thermal conductivity to viscosity ratio at and away from conformality. We have proved this in the case of vanishing chemical potential, though general proof at non zero chemical potential is still lacking. At finite chemical potential, the ratio is

$$\frac{\kappa_T}{\eta T}\mu^2 = 8\pi^2 \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)} g_{xx}^{d-2}\Big|_{r=r_H} \frac{1}{(\frac{\rho}{\mu})^2}. \tag{4.45}$$

Right hand side of above equation should be independent of T, μ and some universal number. Although we have checked it against several examples, we could not provide a general proof of the result. Using the proposed universality in Eq.(4.1), we have also discussed how electrical conductivity can be expressed solely in terms of boundary data.

5

Universality of transport coefficients at extremality

5.1 Introduction

So far, our discussions on previous chapters concerned black holes away from extremality. This was assumed explicitly by considering only those metric whose component along the radial direction has single pole at the horizon. Our emphasize in this section will be on extremal black holes. This in turn means that we will study the behavior of various transport coefficients of gauge theories at zero temperature.

Extremal black holes are special in many ways. Often, various computations tend to break down as one tries to extract out results associated with extremal black holes via 'extremal limit' of non-extremal black holes. One such example recently has appeared in the calculation of shear viscosity (η) to entropy (s) ratio for gauge theory that is dual to extremal bulk geometry. In particular, in the low frequency limit ($\omega \to 0$ limit or in other words the IR limit of the boundary gauge theory), used for non extremal back ground in previous section , the perturbation in ω breaks down. In [82], a prescription was given which can be used to treat these extremal holes. Subsequently, in [83][42], following this prescription, η/s, conductivity (σ) was computed for four dimensional Reissner-Nördtstrom black holes in AdS. The result for $\frac{\eta}{s}$ turned out to be $1/(4\pi)$; same as their non-extremal partners. It was further argued that, regardless of the dimensions of space-time, the result would remain unchanged for Reissner-Nördtstrom black hole.

Encouraged by these developments, in the following we provide a computation of electrical conductivity (σ) and η/s for a generic extremal black hole in arbitrary dimensions having metric of the form

$$ds^2_{d+1} = g_{tt}dt^2 + g_{uu}du^2 + g_{ij}dx^i dx^j, \tag{5.1}$$

[42]For certain class of black holes on AdS_5, a discussion on η/s can be found in [84].

with

$$g_{tt} = -(1-u)^2\gamma_0(u), \quad g_{uu} = \frac{\gamma_u(u)}{(1-u)^2}. \tag{5.2}$$

In terms of coordinate u, the horizon is located at $u = 1$ while the boundary is at $u = 0$. We take functions $\gamma_0(u)$, $\gamma_u(u)$ to be finite on the horizon. Extremal nature of this geometry shows up in the double pole at the horizon. We assume that these gravity backgrounds have an associated gauge theory on the boundary. Among others, this class of metric includes asymptotically AdS spaces. Besides the metric in Eq.(5.1), there might be gauge fields and scalars. The detail forms of these quantities will not be required for the following discussion. As will be shown, the knowledge of the metric near the horizon is sufficient for determination of various quantities of interest. The geometry is characterized by the fact that its entropy is finite even though the temperature is zero [85]. We now proceed to compute η/s and the electrical conductivity associated with this geometry.

5.2 Shear viscosity to entropy density ratio at extremality

First, to compute the shear viscosity, one considers some specific fluctuations of the metric and uses Kubo formula as in [44, 36, 83]. This formula relates the viscosity to the correlation function of the stress-energy tensor at zero spatial momentum.Take the perturbation of the form $g^{'}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ with $g_{\mu\nu}$ given in Eq.(5.1), and Einstein equation leads to the following equation for $h^x{}_y$ (which turns out to be same as that of massless real scalar field. In what follows, we call it Φ.)

$$\partial_\mu\Big(\sqrt{-g}\; g^{\mu\nu}\partial_\nu\Big)\Phi = 0. \tag{5.3}$$

Further, using the ansatz $\Phi = e^{-i\omega t}\phi(u)$, we get

$$\partial_u^2\phi + \partial_u \mathrm{ln}(g^{uu}\sqrt{-g})\partial_u\phi - \frac{g^{tt}}{g^{uu}}\omega^2\phi = 0. \tag{5.4}$$

Using explicit forms of g^{tt}, g^{uu}, we finally reach at an equation of the form

$$\partial_u^2\phi + \partial_u \mathrm{ln}\Big(\frac{\sqrt{-g}(1-u)^2}{\gamma_u}\Big)\partial_u\phi - \frac{\omega^2\gamma_u}{(1-u)^4\gamma_0}\phi = 0. \tag{5.5}$$

We solve the above equation in the inner region (near the horizon) as well as in the outer region (away from the horizon). We then match both at the so called matching region [83]. We first look for solution in the inner region.

Due to the double pole singularity in $g_{uu}(u)$, the usual low frequency (ω) expansion of ϕ becomes subtle [82, 83]. One then defines ξ as $u = 1 - \omega/\xi$ and organizes the solution as an expansion in ω where $\omega \to 0$ and $\xi \to 0$ in such a way that $\omega/\xi \to 0$, see [82, 83, 87, 88, 86] for details. The Eq.(5.5) then simplifies to (keeping only zeroth order in ω)

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\gamma_u}{\gamma_0}\phi = 0, \tag{5.6}$$

Next, defining $\alpha = \frac{\gamma_u}{\gamma_0}\xi$ above equation reduces to standard form in AdS_2

$$\frac{\partial^2 \phi}{\partial \alpha^2} + \phi = 0. \tag{5.7}$$

The incoming wave solution is then

$$\phi_{in} = a_I^0 e^{i\alpha} \sim a_I^0(1 + i\alpha) = a_I^0 + \frac{g(\omega)}{1-u} a_I^0. \tag{5.8}$$

with

$$g(\omega) = i\sqrt{\frac{\gamma_u}{\gamma_0}}\omega. \tag{5.9}$$

In Eq.(5.8), a_I^0 is a constant. Since Eq.(5.8) represents incoming solution near the horizon, this is often called the solution in the inner region or the solution in the IR of the boundary gauge theory.

As for the solution in the outer region or in other words away from the horizon, we go back to Eq.(5.5). Note that in this case, to zeroth order in ω, we get

$$\partial_u^2 \phi + \partial_u \ln(g^{uu}\sqrt{-g})\partial_u \phi = 0. \tag{5.10}$$

Integrating once,

$$\partial_u \phi = c_1 \frac{g_{uu}}{\sqrt{-g}}, \tag{5.11}$$

where c_1 is a constant. This implies

$$\phi_{out} = c_2 + c_1\Big[\frac{\gamma_u(u)}{(1-u)\sqrt{-g}} - \int \Big(\frac{\gamma_u(u)}{\sqrt{-g}}\Big)' \frac{1}{1-u}\, du\Big], \tag{5.12}$$

In Eq.(5.12), c_2 is an integration constant. In order to get the complete low frequency profile of ϕ, we need to match Eq.(5.12) and Eq.(5.8) at $u \to 1$. Outer region solution gives

$$\phi_{out} = c_2 + c_1 B + c_1 \frac{\gamma_u}{(1-u)\sqrt{-g(u=1)}}, \tag{5.13}$$

with

$$B = \Big[- \int \Big(\frac{\gamma_u(u)}{\sqrt{-g}}\Big)' \frac{1}{1-u}\, du\Big]_{u\to 1} \tag{5.14}$$

Now comparing Eq.(5.13) with Eq.(5.8), we get

$$c_1 = \frac{\sqrt{-g}g(\omega)}{\gamma_u}a_I^0, \quad c_2 = a_I^0\Big(1 - \frac{B\sqrt{-g}g(\omega)}{\gamma_u}\Big). \tag{5.15}$$

Substituting these constants in Eq.(5.12)

$$\begin{aligned}\phi_{out} &= a_I^0\Big(1 - \frac{B\sqrt{-g}g(\omega)}{\gamma_u}\Big) + a_I^0 B\frac{\sqrt{-g}g(\omega)}{\gamma_u} \\ &+ a_I^0\frac{\sqrt{-g}g(\omega)}{\gamma_u}\frac{\gamma_u}{(1-u)\sqrt{-g(u=1)}}.\end{aligned} \tag{5.16}$$

Hence

$$\partial_u\phi_{out} = a_I^0\frac{\sqrt{-g}g(\omega)}{\gamma_u}\frac{g_{uu}}{\sqrt{-g}}. \tag{5.17}$$

Now it is straightforward to compute the boundary action and then the correlation function following [44]. As for the boundary action, we get

$$S_{\text{boundary}} = -\frac{1}{2}\frac{1}{16\pi G}\Big[g^{uu}\sqrt{-g}\phi_{out}\partial_u\phi_{out}\Big]_{u=\epsilon\to 0} = -\frac{g(\omega)\sqrt{-g}(a_I^0)^2}{32\pi G\gamma_u}. \tag{5.18}$$

Hence, to first order in ω

$$G_{xy,xy} = \frac{\partial S_{boundary}}{\partial a_I^0\partial a_I^0} = -\frac{g(\omega)\sqrt{-g}}{16\pi G\gamma_u} = -\frac{i\omega}{16\pi G}\Big[\sqrt{\frac{-g}{g_{tt}g_{uu}}}\Big]_{u=1}. \tag{5.19}$$

Here G is the $d+1$ dimensional Newton's constant. In the last line we have used the form of $g(\omega)$ given in Eq.(5.9). Now the Kubo formula gives us the shear viscosity [43]

$$\eta = \frac{1}{16\pi G}\left[\sqrt{\frac{-g}{g_{uu}g_{tt}}}\right]_{u=1}. \tag{5.20}$$

Since the entropy density of black hole is given by

$$s = \frac{\sqrt{\det g_{ij}}\Big|_{u=1}}{4G} = \frac{1}{4G}\Big[\sqrt{\frac{-g}{g_{uu}g_{tt}}}\Big]_{u=1} \tag{5.21}$$

we get

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{5.22}$$

In the following we give a different derivation of the $\frac{\eta}{s} = \frac{1}{4\pi}$ in the same spirit of [4]). Let us note that in [4], the single pole structure of the metric played a

[43]We observe that the form of η is same as that obtained in the non-extremal cases [4]. The structural similarity leads us to speculate that there might be a Iqbal-Liu like prescription[86] for extremal black holes having non-zero entropy.

crucial role in determining transport coefficient, where as we are considering the double pole structure. In spite of this difference, as we will see below, one can apply argument similar to that in [4]. Consider the bulk action for a massless scalar Φ:

$$S_{\text{bulk}} = \frac{1}{2}\int d^{d+1}x\sqrt{-g}\frac{\partial_A\Phi\partial^A\Phi}{16\pi G} \tag{5.23}$$

Using linear response theory one can write the transport coefficient as

$$\chi = \lim_{\omega\to 0}\lim_{u\to 0}\left(\frac{\Pi_\Phi(u,\omega)}{i\omega\Phi(u,\omega)}\right), \tag{5.24}$$

where $\Pi_\Phi(u,t) = \frac{\partial\mathcal{L}_{\text{bulk}}}{\partial(\partial_u\Phi)}$[4]. Note that $\Pi_\Phi(u,\omega)$ is the Fourier transform of the function $\Pi_\Phi(u,t)$. If we take $\Phi(u,t) = h^x_y$, then we get η as the transport coefficient. Following our previous discussion, we note that the field momentum is of the form

$$\Pi_\Phi(u,\omega) = \frac{\sqrt{-g}}{16\pi G}g^{uu}\partial_u\phi. \tag{5.25}$$

Now using the fact that

$$\partial_u\phi_I = i\frac{\partial\alpha}{\partial u}\phi_I = i\frac{\omega}{(1-u)^2}\sqrt{\frac{\gamma_u}{\gamma_t}}\phi_I, \tag{5.26}$$

and Eq.(5.17), we see

$$\eta = \lim_{\omega\to 0}\lim_{u\to 0}\left(\frac{\Pi_\Phi(u,\omega)}{i\omega\Phi(u,\omega)}\right) = \lim_{\omega\to 0}\lim_{u\to 1}\left(\frac{\Pi_\Phi(u,\omega)}{i\omega\Phi(u,\omega)}\right) = \frac{1}{16\pi G}\sqrt{\frac{-g}{g_{uu}g_{tt}}}\Big|_{u\to 1}. \tag{5.27}$$

This is same as what we got previously Eq.(5.20). To evaluate the above expression, we have used Eq.(5.26) for ϕ in $u\to 1$ region and Eq.(5.17) for $u\to 0$ region. So in spite of double pole nature of the geometry, membrane paradigm like argument gives the same result.

5.2.1 Radial independence of the response function

We have seen that the response function ($\chi(u,\omega) = \frac{\Pi(u,\omega)}{i\omega\phi(u,\omega)}$) for shear viscosity evaluates to same value whether one computes it at the horizon or at the boundary. In fact one can convince oneself that the response function is independent of radial direction. To show that, let us define $\Sigma(u,\omega) = \frac{1}{16\pi G}\sqrt{\frac{-g}{g_{uu}g_{tt}}}$. Now following [4] and using Eq.(**??**) we can write

$$\partial_u\chi = i\omega\sqrt{\frac{g_{uu}}{g_{tt}}}\Big(\frac{\chi^2}{\Sigma_\phi} - \Sigma_\phi\Big). \tag{5.28}$$

Near the horizon we have already checked that $\chi = \Sigma$ where as away from horizon because of explicit ω dependence in the above equation, in the limit $\omega \to 0$, we get $\partial_u \chi = 0$, and hence radial independence. To strenthen the argument further let us compute the response function in the outer region at arbitrary radial position.

Using Eq.(5.9,5.17) one obtains

$$\Pi = a_I^0 \sqrt{\frac{-g}{g_{uu}g_{tt}}}\Big|_{u\to 1} + O(\omega^2) \tag{5.29}$$

and using Eq.(**??**) we get

$$\omega\phi = \omega a_I^0 + O(\omega^2) \tag{5.30}$$

and hence

$$\chi = \frac{1}{16\pi G}\sqrt{\frac{-g}{g_{uu}g_{tt}}}\Big|_{u\to 1} \tag{5.31}$$

and radially independent.

5.3 Conductivity for extremal black hole

In this section we compute electrical conductivity for extremal background. We shall first give some examples which motivate us to determine conductivity for more general cases. At extremality metric in the vicinity of horizon takes the form

$$g_{tt} = -(1-u)^2\gamma_0, \quad g_{uu} = \frac{\gamma_u}{(1-u)^2}, \tag{5.32}$$

where $\gamma_0 = \gamma_0(u=1)$ and $\gamma_u = \gamma_u(u=1)$. Near the horizon Eq.(2.10) reduces to

$$\frac{d^2}{du^2}\phi_i(u) - \frac{2}{1-u}\frac{d}{du}\phi_i(u) + \frac{\gamma_u}{\gamma_0}\frac{\omega^2}{(1-u)^4}\phi_i(u) - \frac{c_i}{(1-u)^2}\frac{(\sum\limits_{j=1}^{m} d_j\phi_j(u))}{\gamma_0} = 0 \tag{5.33}$$

Note that $c_i = F^i_{ut}(u=1)$ and $d_j = G_{jj}(u)F^j_{ut}(u)$ at $u=1$. Following [82, 83] let us define $u = 1 - \frac{\omega}{\xi}$. In this coordinate system Eq.(5.29) reduces to

$$\frac{d^2}{d\xi^2}\phi_i(\xi) + \frac{\gamma_u}{\gamma_0}\phi_i(\xi) - \frac{c_i}{\xi^2}\frac{(\sum\limits_{j=1}^{m} d_j\phi_j(\xi))}{\gamma_0} = 0 \tag{5.34}$$

Above equation is in general a complicated coupled differential equation. To solve it we observe that

$$\frac{\frac{d^2}{d\xi^2}\phi_i(\xi) + \frac{\gamma_u}{\gamma_0}\phi_i(\xi)}{c_i} = \frac{(\sum\limits_{j=1}^{m} d_j\phi_j(\xi))}{\gamma_0\xi^2} \tag{5.35}$$

In the case when more than one field is present then we get

$$\frac{\frac{d^2}{d\xi^2}\phi_1(\xi)+\frac{\gamma_u}{\gamma_0}\phi_1(\xi)}{c_1}=\frac{\frac{d^2}{d\xi^2}\phi_2(\xi)+\frac{\gamma_u}{\gamma_0}\phi_2(\xi)}{c_2}=.... \tag{5.36}$$

We take solution of the form

$$\frac{\phi_1(\xi)}{c_1}=\frac{\phi_2(\xi)}{c_2}=.... \tag{5.37}$$

Plugging Eq.(5.33) in Eq.(5.30) one obtains

$$\frac{d^2}{d\xi^2}\phi_i(\xi)+\frac{\gamma_u}{\gamma_0}\phi_i(\xi)-\frac{(\sum\limits_{j=1}^{m}d_jc_j)}{\gamma_0\xi^2}\phi_i(\xi)=0 \tag{5.38}$$

Introduce $\eta=\sqrt{\frac{\gamma_u}{\gamma_0}}\,\xi$ and $a=\frac{(\sum\limits_{j=1}^{m}d_jc_j)}{\gamma_0}$, so that one gets (from Eq.(5.34))

$$\frac{d^2}{d\eta^2}\phi_i(\eta)+\phi_i(\eta)-\frac{a}{\eta^2}\phi_i(\eta)=0 \tag{5.39}$$

The incoming solution to Eq.(5.35) takes the form

$$\phi_i(\eta)=CH^1_\nu(\eta), \tag{5.40}$$

where $H^1_\nu(\eta)$ is Henkel function and $\nu=\frac{\sqrt{1+4a}}{2}$. Taking $\eta\to 0$ limit one gets

$$\lim_{\eta\to 0}\phi_i(\eta)=\eta^{\frac{1}{2}+\nu}2^{-\nu}(\frac{1}{\Gamma[1+\nu]}-i\frac{\cos(\pi\nu)\Gamma[-\nu]}{\pi})-i\eta^{\frac{1}{2}-\nu}2^{\nu}\frac{\Gamma[\nu]}{\pi} \tag{5.41}$$

Using $\eta=\sqrt{\frac{\gamma_u}{\gamma_0}}\frac{\omega}{(1-u)}$, and some properties of Gamma functions as well as doing some re scaling one finds

$$\phi_i(u\to 1)=A_0\Big[\frac{1}{(1-u)^{\frac{1}{2}-\nu}}+(\sqrt{\frac{\gamma_u}{\gamma_0}})^{2\nu}(\frac{\omega}{2})^{2\nu}\frac{\pi(i-\cot(\nu\pi))}{\Gamma[1+\nu]\Gamma[\nu]}\frac{1}{(1-u)^{\frac{1}{2}+\nu}}\Big]. \tag{5.42}$$

Again using properties of Gamma functions we get

$$\begin{aligned}\phi_i(u\to 1) &= A_0\Big[\frac{1}{(1-u)^{\frac{1}{2}-\nu}}-(\sqrt{\frac{\gamma_u}{\gamma_0}})^{2\nu}(\frac{\omega}{2})^{2\nu}\frac{\Gamma[1-\nu]}{\Gamma[1+\nu]}\frac{e^{-i\nu\pi}}{(1-u)^{\frac{1}{2}+\nu}}\Big]\\ &= A_0\Big[\frac{1}{(1-u)^{\frac{1}{2}-\nu}}+g(\omega)\frac{1}{2\nu(1-u)^{\frac{1}{2}+\nu}}\Big],\end{aligned} \tag{5.43}$$

where for notational simplicity we introduced

$$g(\omega) = -2\nu e^{-i\nu\pi}(\sqrt{\frac{\gamma_u}{\gamma_0}})^{2\nu}(\frac{\omega}{2})^{2\nu}\frac{\Gamma[1-\nu]}{\Gamma[1+\nu]}. \tag{5.44}$$

Following the standard procedure, we obtain conductivity to be proportional to

$$\sigma \propto \lim_{\omega\to 0}\frac{1}{\omega}\Im[g(\omega)] \propto (\omega)^{2\nu-1}, \tag{5.45}$$

where

$$\begin{aligned} 2\nu &= \sqrt{1+4a} \\ &= \sqrt{1+(\frac{4}{\gamma_0})\sum_{j=1}^{m} d_j c_j} \\ &= \sqrt{1+(\frac{4}{\gamma_0})\sum_{j=1}^{m} G_{jj}(F^j_{ut})^2}. \end{aligned} \tag{5.46}$$

In the above expression every quantity is calculated at the horizon ($u = 1$). Hence, we see only way to get non-zero conductivity in the limit $\omega \to 0$ at extremality is $\nu \leq \frac{1}{2}$ where as $\sigma \to 0$ if $\nu > \frac{1}{2}$.

- To obtain above form of $g(\omega)$, we have only assumed that extremal black hole exhibits double pole. So the expression for operator dimension in general follows only from criteria of extremality i.e. it is independent of particular background. In all the examples considered below (see Appendix for details about bulk geometry) we find $\nu = \frac{3}{2} \Rightarrow \delta = \nu + \frac{1}{2} = 2$. There are other classes of black hole as well (dialatonic black hole [89, 90, 91, 92, 93, 94, 95]) where one finds $\delta = 2$.

- ***R-charged black brane in four dimension:*** In this case

$$2\nu = \sqrt{1+4\frac{\prod_{i=1}^{4}(1+k_i)}{3+\sum_{j=1}^{4}k_j+\prod_{i=1}^{4}k_i}\Big(\sum_{j=1}^{4}\frac{k_i}{(1+k_i)^2}\Big)} \tag{5.47}$$

Using extremality condition[44](see appendix) we get $2\nu = 3$.

[44] $k_1 = \frac{3+2(k_2+k_3+k_4)+k_2(k_3+k_4)+k_3k_4}{k_2k_3k_3-2-k_2-k_3-k_4}$

- ***R-charged black brane in five dimension:*** In this case

$$2\nu = \sqrt{1 + 4\frac{\prod_{i=1}^{3}(1+k_i)}{1+\prod_{i=1}^{3}k_i}\Big(\sum_{j=1}^{3}\frac{k_i}{(1+k_i)^2}\Big)} \tag{5.48}$$

  Using extremality condition[45] one finds $2\nu = 3$. Which implies $\delta = \nu + \frac{1}{2} = 2$. Note that above result also applicable for 5d Reissner-Nordstrom black hole (for which $k_1 = k_2 = k_3$) considered in other places [82].

- ***R-charged black brane in seven dimension:*** In this case

$$2\nu = \sqrt{1 + 4\frac{4(1+k_1)(1+k_2)}{3+k_1k_2}(\frac{k_1}{(1+k_1)^2} + \frac{k_2}{(1+k_2)^2})} \tag{5.49}$$

  Now extremality condition implies $k_1 = \frac{3+k_2}{k_2-1}$. So one gets $2\nu = 3$.

- Above results implies that for black hole at extremality obeys

$$(\frac{1}{\gamma_0})\sum_{j=1}^{m} G_{jj}(F_{ut}^j)^2 = 2. \tag{5.50}$$

  It would be interesting to find out under what conditions extremal backgrounds obeys this relation.

What we observe is that, form of conductivity is insensitive to the details of geometry and mostly determined by the fact that the metric has double pole or zero.

We consider metric with near horizon behavior to be

$$ds^2 = -(1-u)^2\gamma_0 dt^2 + \frac{\gamma_u}{(1-u)^2}du^2 + \gamma_x\sum_{i=1}^{d-1}(dx^i)^2, \tag{5.51}$$

and gauge coupling has no zero or pole as we approach horizon. For this bulk back ground, temperature is zero but the entropy is finite.

The Einstein equation is given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= T_{\mu\nu}^{E.M.} + T_{\mu\nu}^{Matter} \\ &= \frac{1}{2g_{\text{eff}}^2}\left(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) + T_{\mu\nu}^{Matter}, \end{aligned} \tag{5.52}$$

[45] $k_3 = \frac{2+k_1+k_2}{k_1k_2-1}$

where $T_{\mu\nu}^{Matter}(u)$, will include all the other stuffs which may come from scalar fields, cosmological constant or any other fields present in the theory. Since only $A_t(u)$ is non-zero, we have $F_{ut} \neq 0$. Using Eq.(5.48), we can write

$$R_t^t - \frac{1}{2}g_t^t R = \frac{1}{2g_{\text{eff}}^2}\left(F_{tu}F^{tu} - \frac{1}{4}g_t^t F_{\rho\sigma}F^{\rho\sigma}\right) + T_t^{t,\ Matter}, \tag{5.53}$$

$$R_x^x - \frac{1}{2}g_x^x R = -\frac{1}{2g_{\text{eff}}^2}\frac{1}{4}g_x^x F_{\rho\sigma}F^{\rho\sigma} + T_x^{x,\ Matter}. \tag{5.54}$$

After subtracting Eq.(5.49) from Eq.(5.50), we get

$$\sqrt{-g}R_t^t - \sqrt{-g}R_x^x = \frac{1}{2g_{\text{eff}}^2}\sqrt{-g}F^{ut}F_{ut} + \sqrt{-g}(T_t^{t,\ Matter}(u) - T_x^{x,\ Matter}(u)). \tag{5.55}$$

For the metric of the form in Eq.(A.2), following relations hold

$$\sqrt{-g}R_t^t = -\frac{d}{du}\left(\frac{g_{xx}^{\frac{d-1}{2}}\frac{d}{du}g_{tt}}{2g_{uu}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}}\right), \tag{5.56}$$

$$\sqrt{-g}R_x^x = -\frac{d}{du}\left(\frac{g_{xx}^{\frac{d-3}{2}}g_{tt}^{\frac{1}{2}}}{2g_{uu}^{\frac{1}{2}}}\frac{d}{du}g_{xx}\right), \tag{5.57}$$

which, after substituting in Eq.(5.51), we get,

$$\begin{aligned}-\frac{d}{du}\left(\frac{g_{xx}^{\frac{d-1}{2}}}{2g_{uu}^{\frac{1}{2}}g_{tt}^{\frac{1}{2}}}\frac{d}{du}g_{tt}\right) + \frac{d}{du}\left(\frac{g_{xx}^{\frac{d-3}{2}}g_{tt}^{\frac{1}{2}}}{2g_{uu}^{\frac{1}{2}}}\frac{d}{du}g_{xx}\right) &= \frac{1}{2g_{\text{eff}}^2}\sqrt{-g}F^{ut}F_{ut} \\ &+ \sqrt{-g}(T_t^{t,Matter} - T_x^{x,Matter}).\end{aligned} \tag{5.58}$$

If we impose the condition that

$$T_t^{t,\ Matter}(u) = T_x^{x,\ Matter}(u), \tag{5.59}$$

then we get

$$-\frac{d}{du}\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{uu}^{\frac{1}{2}}}\frac{d}{du}(g^{xx}g_{tt})\right) = \frac{1}{g_{\text{eff}}^2}\sqrt{-g}F^{ut}F_{ut}. \tag{5.60}$$

In the near horizon limit we get

$$\begin{aligned}\frac{1}{g_{\text{eff}}^2(u=1)}\frac{F_{ut}^2(u=1)}{\gamma_0} &= -\frac{\gamma_u}{\sqrt{-g}}\frac{d}{du}\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}}g_{uu}^{\frac{1}{2}}}\frac{d}{du}(g^{xx}g_{tt})\right)\Bigg|_{u=1} \\ &= 2.\end{aligned} \tag{5.61}$$

So we have proved that

$$\sigma \sim \omega^2$$

for the metric with double pole in g_{uu} and double zero in g_{tt}. The case of multiply charged extremal black brane is totally analogues and can be shown that under the same condition on the energy momentum tensor of bulk space time, the form of conductivity is again ω^2. Let us note that, the condition on the energy momentum tensor Eq.(5.55), has the interpretation that dual gauge theory vacuum is Lorentz invariant as was the case for non extremal case.

5.3.1 Imaginary part of the conductivity

We can even find out the imaginary part of conductivity. This is given by

$$\Im(\sigma) = -\frac{1}{\omega}\frac{\rho^2}{\epsilon + P} = -\frac{1}{\omega}\frac{\rho}{\mu}. \tag{5.62}$$

Let us note that, this is very similar to finite temperature case and has a pole as ω goes to zero.

5.4 Discussion

We have shown that the viscosity to entropy ratio as well as the electrical conductivity are insensitive to many details of the extremal black brane geometry. For our computation, we only required the double pole nature of g_{uu} and double zero of g_{tt} at the horizon. Rest of the quantities associated with the metric are only assumed to be finite and non-zero on the horizon. Given these information, we argued that electrical conductivity goes as ω^2. We have also seen that analytic expression for shear viscosity and the viscosity to entropy ratio remain same as that of many non-extremal black holes where near horizon geometry is radically different. We have also observed that a analog of Iqbal-Liu like arguments for computation of the shear viscosity go through in the extremal case with double pole nature of metric, even though the computations of [4] seem to depend crucially on the the single pole nature of the geometry.

6
Summary

The gauge/gravity duality allows us to gain insights into various properties of strongly coupled gauge theories both at zero and non-zero temperature. In particular, the transport coefficients of strongly coupled gauge theories, which are hard to compute otherwise, can now be computed using gauge/gravity duality. Furthermore, for many cases, in the low frequency limit, at the level of linear response, the horizon geometry of the gravity dual determines the behavior of the gauge theory. This can, in particular, be used to show that the shear viscosity to entropy density ratio for strongly coupled gauge theories at finite temperature with a gravity dual is universal and takes value $\frac{1}{4\pi}$. One can further show that, the electrical conductivity of the gauge theory at finite temperature but zero chemical potential can be determined in terms of geometrical quantities evaluated at the horizon. This is so because the response function in the low frequency limit evolves in a very simple manner as we go away from the horizon along the radial direction. However, the introduction of a chemical potential primarily brings in several non-trivialities in the evolution of response function from the horizon to the boundary. Although the shear viscosity can still be computed solely in terms of horizon data, for the computation of electrical conductivity, horizon data is not enough. Nevertheless, our analysis reveals that if the stress-energy tensor related to the matter content of the bulk satisfies a compact relation among its space and time components, the boundary conductivity at low frequencies is universal and can be written in terms of geometrical quantities evaluated at the horizon and thermodynamic quantities. In this thesis, we also have shown that at any radial position out side the horizon, the conductivity is given by a simple expression which interpolates smoothly between the one computed at the horizon and at the boundary. We also computed the electrical conductivity in the presence of more than one chemical potentials for several models. What we observe is that, in the presence of multiple chemical potentials, there is a nontrivial mixing between current operators which, from the bulk point of view, can be understood to be arising because of the interactions through graviton. We have also shown that one can write a general expression for conductivity matrix in the presence of multiple chemical potentials provided dual gravity background satisfies some constraints. By using the relation with electrical conductivity, we have also computed the thermal conductivity and observed that thermal conductivity to shear viscosity ratio ($\frac{\kappa_T \sum_{i=1}^{n} \mu_i^2}{\eta T}$) is independent of the number of chemical potentials turned on. This ratio remains same even in the limit of zero chemical potential. We also discussed, how for CFT's with gravity dual, this ratio can be expressed in terms of central charges of the CFT. Using these results, we could express the electrical conductivity solely in terms of the thermodynamic quantities of the gage theory. We then turn our attention to study of transport coefficients of gauge theories at zero temperature which corresponds to extremal black hole in the bulk. We have shown that electrical conductivity goes as ω^2. We have also seen that analytic expression for shear viscosity and the viscosity to entropy ratio remain same as that of many non-extremal black holes where near horizon geometry is radically different.

We hope that our explorations regarding the universalities of various transport coefficients will be useful in understanding generic behaviour of the strongly coupled quantum field theories at zero and non-zero temperature.

A
Membrane paradigm

To an external observer, a black hole appears as dynamical fluid membrane sitting at the horizon, with mechanical and electrical properties. They also show dissipation and one can compute quantities such as conductivity, shear viscosity. In the following we shall give a brief introduction to membrane paradigm in the spirit of [4, 96]. See [5] and [97, 98, 99, 100, 101, 102] for discussion on same topic. Classically an outside observer does not see inside the horizon. Effectively, for an external observer one can write

$$S_{\text{eff}} = S_{\text{out}} + S_{\text{surf}}, \tag{A.1}$$

where S_{out} is the part of action defined out side the horizon where as S_{surf} represents effectively the effect of black hole to external universe. S_{surf} is a boundary term to the horizon, and can be determined by demanding S_{eff} to be stationary with respect to solution to the equation of motion. Rather than putting the membrane exactly at the horizon, one can put it slightly away and thus avoiding complexity that arises due to null hypersurface. In the following we shall discuss briefly electrical and mechanical properties of the membrane.

A.1 Electrical properties of the membrane

Let us consider the metric of the form

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)\sum_{i=1}^{d-1}(dx^i)^2, \tag{A.2}$$

where r is the radial coordinate. We have assumed full rotational symmetry in x^i directions so that $g_{ij} = g_{xx}\delta_{ij}$, where i, j run over all the indices except r, t. We also assume that metric components depend on radial coordinate only. We shall work with the metric which has an event horizon, where g_{tt} has a first order zero and g_{rr} has a first order pole. We also assume that all the other metric components are finite as well as non vanishing at the horizon. Consider a bulk $U(1)$ gauge field for which the action is of the form

$$S_{\text{Out}} = -\int_{r>r_h} d^{d+1}x \ \ \sqrt{-g}\frac{1}{4g_{d+1}^2(r)}F_{MN}F^{MN}. \tag{A.3}$$

Now varying this action we get,

$$\begin{aligned} \delta S_{\text{out}} &= -2\int_{r>r_h} d^{d+1}x \ \ \sqrt{-g}\frac{1}{4g_{d+1}^2(r)}\delta F_{MN}F^{MN} \\ &= -4\int_{r>r_h} d^{d+1}x \ \ \sqrt{-g}\,\nabla_M\,(\frac{1}{4g_{d+1}^2(r)}\delta A_N F^{MN}) \\ &+ \ 4\int_{r>r_h} d^{d+1}x \ \ \sqrt{-g}\delta A_N\,\nabla_M\,(\frac{1}{4g_{d+1}^2(r)}F^{MN}). \end{aligned} \tag{A.4}$$

Using Maxwell equation

$$\nabla_M(\frac{1}{4g_{d+1}^2(r)}F^{MN}) = 0, \tag{A.5}$$

and the fact that for any vector V^A, we have

$$\nabla_M V^M = \frac{1}{\sqrt{-g}}\partial_A(\sqrt{-g}V^A), \tag{A.6}$$

we get

$$\delta S_{\text{out}} = -\int d^dx\sqrt{-g}\frac{1}{g_{d+1}^2}\delta A_M F^{rM}\Big|_{r=r_h}^{r\to\infty}. \tag{A.7}$$

Using the fact that at the boundary $\delta A_B = 0$, and staying slightly away from the horizon we get,

$$\begin{aligned} \delta S_{\text{out}} &= -\int d^dx\sqrt{-h}\Big(\frac{\sqrt{-g}}{\sqrt{-h}}\frac{1}{g_{d+1}^2}\delta A_M F^{rM}\Big)_{r=r_h+\epsilon} \\ &= -\int d^dx\sqrt{-h}\delta A_M J^M_{\text{membrane}}(x). \end{aligned} \tag{A.8}$$

where $h_{\mu\nu}$ is the induced metric at the stretched horizon and

$$J^B_{\text{membrane}} = \frac{\sqrt{g_{rr}}}{g_{d+1}^2}F^{rB}\Big|_{r=r_h+\epsilon}. \tag{A.9}$$

In order to have a well defined variational principle, we need to cancel the boundary term. For that purpose we add S_{Surf} such that

$$\delta S_{\text{Surf}} = -\delta S_{\text{Out}}. \tag{A.10}$$

One can write S_{Surf}, as

$$S_{\text{Surf}} = \int d^d x \sqrt{-h}\delta A_M J^M_{\text{membrane}}(x). \tag{A.11}$$

Let us note that Maxwell equation can be written as

$$\begin{aligned} \nabla_M(\sqrt{-g}\frac{1}{g^2_{d+1}} F^{rM}) &= 0 \\ \Rightarrow \nabla_M J^M_{\text{membrane}} &= 0, \end{aligned} \tag{A.12}$$

where J^M_{membrane} can be interpreted as the membrane current. Total integral of J^0_{membrane} over the horizon will give charge of the black hole. The spatial component of the membrane current is given by

$$J^i_{\text{membrane}} = \frac{\sqrt{g_{rr}}}{g^2_{d+1}} F^{ri}\Big|_{r=r_h+\epsilon}. \tag{A.13}$$

In order to proceed further, let us choose the gauge $A_r = 0$. Since horizon is a regular place for an in falling observer, the A_i should be regular at the horizon. This implies that, gauge field should only depend on a non singular combination v with

$$dv = dt + \sqrt{\frac{g_{rr}}{-g_{tt}}} dr. \tag{A.14}$$

This gives,

$$\begin{aligned} (\partial_r - \sqrt{\frac{g_{rr}}{-g_{tt}}}\partial_t)A_i &= 0 \\ \Rightarrow F_{ri} = \sqrt{\frac{g_{rr}}{-g_{tt}}} F_{ti}. \end{aligned} \tag{A.15}$$

Plugging it in Eq.(A.13) we get,

$$J^i_{mem} = \frac{1}{g^2_{d+1}} \sqrt{-g^{tt}} F^i_t = \frac{1}{g^2_{d+1}} \widehat{E^i}, \tag{A.16}$$

where $\widehat{E^i}$ is the electric field measured in an orthonormal frame of a physical observer hovering just outside of the black hole. J^i_{mem} can be interpreted as the response of the membrane to electric field $\widehat{E^i}$. Now comparing with $\overrightarrow{J} = \sigma \overrightarrow{E}$ we get

$$\sigma_{mem} = \frac{1}{g^2_{d+1}(r_h)}, \tag{A.17}$$

where σ is the electrical conductivity of the membrane.

A.2 Mechanical properties of the membrane

Fluctuation of gravitational field will induce energy momentum tensor $T^{\mu\nu}$ in the membrane. To illustrate this with an example, let us consider a metric fluctuation $h^1_2(x)$. The action of this to the quadratic order is that of a free mass less scalar field ,

$$S^{grav}_{out} = \frac{1}{2}\int d^{d+1}x\sqrt{-g}\frac{1}{16\pi G_N}(\nabla\phi)^2, \tag{A.18}$$

with $\phi = h^1_2$. Following previous discussion, we need to add a surface term

$$S_{\text{surf}} = \int_{\text{horizon}} d^d x\sqrt{-h}\frac{\Pi_r(x)}{\sqrt{-h}}\phi(x), \tag{A.19}$$

with $\Pi_r = \frac{\sqrt{-g}g^{rr}\partial_r\phi}{16\pi G_N}$. This will induce a current $J(x)$ in the membrane $J(x) \propto T^1_2$. Regularity implies

$$\partial_r\phi = \sqrt{\frac{g_{rr}}{-g_{tt}}}\partial_t\phi, \tag{A.20}$$

so that one can write

$$\begin{aligned}\Pi_{\text{mem}} &= \frac{\Pi_r(x)}{\sqrt{-h}}\Big|_{r_h} \\ &= -\frac{1}{\sqrt{g_{tt}}}\frac{1}{16\pi G_N}\partial_t\phi \\ &= -\frac{1}{16\pi G_N}\partial_{\hat{t}}\phi.\end{aligned} \tag{A.21}$$

In the last line, we again have passed to ortho-normal basis. As in the electromagnetic case (see Eq.(A.11)), we can interpret Π_{mem} in Eq.(A.18) as the membrane response of the field ϕ, with response function $\eta = \frac{1}{16\pi G}$, the shear viscosity since $\Pi_{\text{mem}} = (T_{\text{mem}})^x_y$. Since the entropy density (s) per unit volume of membrane fluid is $s_{\text{mem}} = \frac{1}{4G}$, we get

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{A.22}$$

So we see that one can consider horizon as fluid with response fluctuations such as η_{mem}, σ_{mem}. Let us note that computation done using gauge gravity duality for boundary fluid also shows

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{A.23}$$

B

R-charged black holes in various dimensions

Here we collect all the relevant information about four and seven dimensional R-charged black hole [33]. The case of five dimensional black hole was already discussed in the introduction. The R-charged black hole solutions in asymptotically AdS_4 and AdS_7 can be obtained by doing dimensional reduction of rotating $M2$ brane and $M5$ branes on S^7 and S^4 respectively. The relevant part of the Lagrangian is

$$\frac{\mathcal{L}}{\sqrt{-g}} = R - \frac{1}{4} G_{ij} F^i_{\mu\nu} F^{\mu\nu\, j} - G_{ij} \partial_\mu X^i \partial^\mu X^j + ... \tag{B.1}$$

B.1 Four dimensional black hole

Metric and gauge fields in this case are

$$ds_4^2 = \frac{16(\pi T_0 L)^2}{9u^2} \mathcal{H}^{1/2} \left(-\frac{f}{\mathcal{H}} dt^2 + dx^2 + dz^2 \right) + \frac{L^2}{fu^2} \mathcal{H}^{1/2}\, du^2\ , \tag{B.2}$$

$$A_t^i = \frac{4}{3} \pi T_0 \sqrt{2\kappa_i \prod_{i=1}^{4} (1+\kappa_i)}\ \frac{u}{H_i}\ , \qquad H_i = 1 + k_i u\ , \tag{B.3}$$

$$\mathcal{H} = \prod_{i=1}^{4} H_i, \qquad f = \mathcal{H} - \prod_{i=1}^{4} (1+\kappa_i) u^3. \tag{B.4}$$

Thermodynamic quantities are given by

$$\epsilon = \sqrt{2}\,\pi^2 \left(\frac{2}{3}\right)^4 N^{3/2}\, T_0^3\, \prod_{i=1}^{4}(1+\kappa_i)\ , \qquad P = \frac{\sqrt{2}\,\pi^2}{3}\left(\frac{2}{3}\right)^3 N^{3/2}\, T_0^3\, \prod_{i=1}^{4}(1+\kappa_i)\ , \tag{B.5a}$$

$$s = \sqrt{2}\pi^2 \left(\frac{2}{3}\right)^3 N^{3/2}\, T_0^2\, \prod_{i=1}^{4}\sqrt{1+\kappa_i}\ , \qquad T = \frac{T_0\left(3 + 2\sum\limits_{j=1}^{4} k_i + \sum\limits_{j>i,i,j=1}^{4} k_i k_j - \prod\limits_{i=1}^{4} k_i\right)}{3\sqrt{\prod\limits_{i=1}^{4}(1+\kappa_i)}} \tag{B.5b}$$

$$\rho_i = \sqrt{2}\,\pi \left(\frac{1}{3}\right)^3 N^{3/2}\, T_0^2 \sqrt{2\,k_i \prod_{j=1}^{4}(1+\kappa_j)}\ , \quad \mu_i = \frac{4\pi\, T_0}{3}\,\frac{1}{1+k_i}\sqrt{2\,\kappa_i \prod_{i=1}^{4}(1+\kappa_i)}\ , \tag{B.5c}$$

Other relevant expressions are

$$G_{ij} = \frac{L^2}{2}\mathrm{diag}\left[(X^1)^{-2},\,(X^2)^{-2},\,(X^3)^{-2},\,(X^4)^{-2}\right] \qquad X^i = \frac{\mathcal{H}^{1/4}}{H_i(u)}. \tag{B.6}$$

and $\frac{1}{16\pi G_4} = \frac{N^{\frac{3}{2}}}{24\sqrt{2}L^2}$. As was discussed in section (1.5.2), in this case as well, one can go to a case where one has diagonal $U(1)$ of the group $U(1)^4$. In this case, all the scalar field vanishes and one is left with the action of the form

$$S_4 = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}(R\ -\frac{1}{4}F^2 + ...) \tag{B.7}$$

which is exactly same as with Resinner-Nordstrom black hole in four dimension. Now comparing this action Eq.(**??**) gives us $\frac{1}{2\kappa^2} = \frac{1}{16\pi G_4}$ and $\gamma^2 = 8$.

B.2 Seven dimensional black hole

$$ds_7^2 = \frac{4(\pi T_0 L)^2}{9u}\mathcal{H}^{1/5}\left(-\frac{f}{\mathcal{H}}dt^2 + dx_1^2 + \cdots + dx_4^2 + dz^2\right) + \frac{L^2}{4fu^2}\mathcal{H}^{1/5}\, du^2\ , \tag{B.8}$$

$$A_t = \frac{2}{3}\pi T_0\sqrt{2\kappa_i \prod_{i=1}^{2}(1+\kappa_i)}\,\frac{u^2}{H_i}\ , \qquad H_i = 1+\kappa_i u^2\ , \tag{B.9}$$

$$H_i = 1 + \kappa_i u^2 \ , \qquad \mathcal{H} = \prod_{i=1}^{2} H_i, f = \mathcal{H} - \prod_{i=1}^{2}(1+\kappa_i)u^3 \ , \tag{B.10}$$

Thermodynamic quantities are given by

$$\epsilon = \frac{5\,\pi^3}{2}\left(\frac{2}{3}\right)^7 N^3\, T_0^6\, \prod_{i=1}^{2}(1+\kappa_i)\ , \qquad P = \frac{\pi^3}{2}\left(\frac{2}{3}\right)^7 N^3\, T_0^6\, \prod_{i=1}^{2}(1+\kappa_i)\ , \tag{B.11}$$

$$s = 3\,\pi^3\left(\frac{2}{3}\right)^7 N^3\, T_0^5\, \sqrt{\prod_{i=1}^{2}(1+\kappa_i)}\ , \qquad T = \frac{T_0\,(3+\kappa_1+\kappa_2-\kappa_1\kappa_2)}{3\sqrt{\prod_{i=1}^{2}(1+\kappa_i)}}\ , \tag{B.12}$$

$$\rho_i = \pi^2\left(\frac{2}{3}\right)^6 N^3\, T_0^5\, \sqrt{2\,\kappa_i\prod_{i=1}^{2}(1+\kappa_i)}\ , \quad \mu_i = \frac{2\pi\,T_0}{3(1+\kappa_i)}\sqrt{2\,\kappa_i\prod_{i=1}^{2}(1+\kappa_i)}\ . \tag{B.13}$$

Other relevant results are

$$G_{ij} = \frac{L^2}{2}\mathrm{diag}\left[(X^1)^{-2},\,(X^2)^{-2}\right], \qquad X^i = \frac{\mathcal{H}^{2/5}}{H_i(u)}, \tag{B.14}$$

and $\frac{1}{16\pi G_7} = \frac{N^3}{6\pi^3 L^5}$.

B.3 R-charged black holes at extremality

Above black holes at extremality was constructed in [85]. Take

$$\bar{g}_{tt} = -f(u)A_1(u),\ \ \bar{g}_{uu} = A_2(u)f^{-1}(u),\ \ f(u) = (1-u)^2V(u). \tag{B.15}$$

Here we just give relevant information about f.

Dimension	Extremality condition	$V(u)$
5	$2+\kappa_1+\kappa_2+\kappa_3-\kappa_1\kappa_2\kappa_3=0$	$(1+\kappa_1\kappa_2\kappa_3 u)$
4	$3+\sum_{j=1}^{4} k_i + \sum_{i<j,i,j=1}^{4} k_i k_j - \prod_{i=1}^{4} k_i = 0$	$(1+(2+\sum_{j=1}^{4} k_i)u + \prod_{i=1}^{4} k_i u^2)$
7	$3+\kappa_1+\kappa_2-\kappa_1\kappa_2=0$	$(1+2u+\kappa_1\kappa_2 u^2)$

Bibliography

[1] G. Policastro, D. T. Son and A. O. Starinets, "From AdS/CFT correspondence to hydrodynamics," JHEP **0209**, 043 (2002) [arXiv:hep-th/0205052].

[2] P. Kovtun, D. T. Son and A. O. Starinets, "Viscosity in strongly interacting quantum field theories from black hole physics," Phys. Rev. Lett. **94** (2005) 111601 [arXiv:hep-th/0405231].

[3] P. Kovtun and A. Ritz, "Universal conductivity and central charges," Phys. Rev. D **78**, 066009 (2008) [arXiv:0806.0110 [hep-th]].

[4] N. Iqbal and H. Liu, "Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm," Phys. Rev. D **79**, 025023 (2009) [arXiv:0809.3808 [hep-th]].

[5] P. Kovtun, D. T. Son and A. O. Starinets, "Holography and hydrodynamics: Diffusion on stretched horizons," JHEP **0310**, 064 (2003) [arXiv:hep-th/0309213].

[6] B. Zwiebach, "A first course in string theory," Cambridge, UK: Univ. Pr. (2004) 558 p.

[7] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge, UK: Univ. Pr. (1998) 402 p.

[8] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge, UK: Univ. Pr. (1998) 531 p.

[9] K. Becker, M. Becker, J. H. Schwarz, "String theory and M-theory: A modern introduction," Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p.

[10] M. B. Green, J. H. Schwarz, E. Witten, "Superstring Theory. Vol. 1: Introduction," Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

[11] M. B. Green, J. H. Schwarz, E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).

[12] E. Bergshoeff, C. M. Hull and T. Ortin, "Duality in the type II superstring effective action," Nucl. Phys. B **451** (1995) 547 [arXiv:hep-th/9504081].

[13] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," Phys. Rev. Lett. **75** (1995) 4724 [arXiv:hep-th/9510017].

[14] J. Polchinski, "Tasi lectures on D-branes," arXiv:hep-th/9611050.

[15] J. M. Maldacena, “ The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2** (1998) 231–252, [arXiv:hep-th/9711200] .

[16] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “ Gauge theory correlators from non-critical string theory,” Phys. Lett. B **428**, 105 (1998) 105–114, [arXiv:hep-th/9802109].

[17] E. Witten, “ Anti-de sitter space and holography,” Adv. Theor. Math. Phys **2** (1998) 253–291 [arXiv:hep-th/9802150] .

[18] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, “Gauge/String Duality, Hot QCD and Heavy Ion Collisions,” [arXiv:1101.0618 [hep-th]].

[19] R. Grimm, M. Sohnius and J. Wess, “Extended supersymmetry and gauge theories”, Nucl. Phys. **B133** (1978) 275.

[20] I. R. Klebanov, “World volume approach to absorption by nondilatonic branes,” *Nucl. Phys.* **B496** (1997) 231, [arXiv:hep-th/9702076].

[21] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, “String theory and classical absorption by three-branes,” *Nucl. Phys.* **B499** (1997) 217, [arXiv:hep-th/9703040].

[22] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept.**323** (2000) 183-386, [arXiv:hep-th/9905111].

[23] C. G. . Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson and X. Wu, “Quantizing string theory in AdS(5) x S**5: Beyond the pp wave,” Nucl. Phys. B **673**, 3 (2003) [arXiv:hep-th/0307032].

[24] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen, “The mass spectrum of chiral $N = 2$ $d = 10$ supergravity on S^5,” *Phys. Rev.* **D32** (1985) 389.

[25] M. Gunaydin and N. Marcus, “The Spectrum of the S^5 Compactification of the Chiral $N = 2, D = 10$ Supergravity and the Unitary Supermultiplets of $U(2, 2/4)$.,” *Class. Quant. Grav.* **2** (1985) L11.

[26] Tanay Kumar Dey, “Phase transitions in Asymptotically AdS black holes and gauge theory duals,” Thesis, Nov 2007.

[27] I. R. Klebanov and E. Witten, “AdS / CFT correspondence and symmetry breaking,” Nucl. Phys. B **556** (1999) 89 [arXiv:hep-th/9905104].

[28] P. Breitenlohner and D. Z. Freedman, "Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity," *Phys. Lett.* **B115** (1982) 197.

[29] P. Breitenlohner and D. Z. Freedman, "Stability in Gauged Extended Supergravity," *Ann. Phys.* **144** (1982) 249.

[30] L. Mezincescu and P. K. Townsend, "Stability at a Local Maximum in Higher Dimensional Anti-de Sitter Space and Applications to Supergravity," *Ann. Phys.* **160** (1985) 406.

[31] S. W. Hawking and D. N. Page, "Thermodynamics of black holes in anti-de Sitter space," *Commun. Math. Phys.* **87** (1983) 577.

[32] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," Class. Quant. Grav. **26**, 224002 (2009) [arXiv:0903.3246 [hep-th]].

[33] M. J. Duff, "TASI lectures on branes, black holes and Anti-de Sitter space," [arXiv:hep-th/9912164].

[34] K. Behrndt, M. Cvetic and W. A. Sabra, "Non-extreme black holes of five dimensional N = 2 AdS supergravity," Nucl. Phys. B **553**, 317 (1999) [arXiv:hep-th/9810227].

[35] M. Cvetic *et al.*, "Embedding AdS black holes in ten and eleven dimensions," Nucl. Phys. B **558**, 96 (1999) [arXiv:hep-th/9903214].

[36] D. T. Son and A. O. Starinets, "Hydrodynamics of R-charged black holes," JHEP **0603**, 052 (2006) [arXiv:hep-th/0601157].

[37] A. Batrachenko, J. T. Liu, R. McNees, W. A. Sabra and W. Y. Wen, "Black hole mass and Hamilton-Jacobi counterterms," JHEP **0505**, 034 (2005) [arXiv:hep-th/0408205].

[38] M. Cvetic and S. S. Gubser, "Phases of R-charged black holes, spinning branes and strongly coupled gauge theories," JHEP **9904**, 024 (1999) [arXiv:hep-th/9902195].

[39] S. S. Gubser, "Thermodynamics of spinning D3-branes," Nucl. Phys. B **551**, 667 (1999) [arXiv:hep-th/9810225].

[40] S. S. Gubser and I. Mitra, "Instability of charged black holes in anti-de Sitter space," arXiv:hep-th/0009126.

[41] S. Jain, S. Mukherji and S. Mukhopadhyay, "Notes on R-charged black holes near criticality and gauge theory," arXiv:0906.5134 [hep-th], JHEP 0911:051,2009

[42] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, "Charged AdS black holes and catastrophic holography," Phys. Rev. D **60**, 064018 (1999) [arXiv:hep-th/9902170].

[43] G. Policastro, D. T. Son, and A. O. Starinets, "The shear viscosity of strongly coupled N = 4 supersymmetric yang-mills Âăplasma," Phys. Rev. Lett. **87** (2001) 081601, [arXiv:hep-th/0104066].

[44] D. T. Son and A. O. Starinets, "Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications," JHEP **0209**, 042 (2002) [arXiv:hep-th/0205051].

[45] M. Rangamani, "Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence," Class. Quant. Grav. **26** (2009) 224003 [arXiv:0905.4352 [hep-th]].

[46] L. D. Landau and E. M. Lifshitz, "Textbook on Theoretical Physics. Vol. 1: Mechanics,". , Elsevier (1976).

[47] S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, "Nonlinear Fluid Dynamics from Gravity," *JHEP* **0802** (2008) 045, [arXiv:0712.2456 [hep-th]].

[48] S. Jain, "Holographic electrical and thermal conductivity in strongly coupled gauge theory with multiple chemical potentials," JHEP **1003**, 101 (2010) [arXiv:0912.2228 [hep-th]].

[49] D. T. Son and A. O. Starinets, "Viscosity, black holes, and quantum field theory," [arXiv: 0704.024 [hep-th]].

[50] P. Arnold, G. D. Moore, and L. G. Yaffe, "Transport coefficients in high temperature gauge theories: (I) Leading-log results," *JHEP* **11** (2000) 001, [hep-ph/0010177].

[51] P. Arnold, G. D. Moore, and L. G. Yaffe, "Transport coefficients in high temperature gauge theories. II: Beyond leading log,", *JHEP* **05** (2003) 051, [hep-ph/0302165].

[52] C. P. Herzog and D. T. Son, " Schwinger-keldysh propagators from AdS/CFT correspondence," JHEP **03** (2003) 046, [arxiv: hep-th/0212072].

[53] S. Caron-Huot, P. Kovtun, G. D. Moore, A. Starinets, and L. G. Yaffe, "Photon and dilepton production in supersymmetric Yang- Mills plasma," JHEP **12** (2006) 015, [arXiv:hep-th/0607237].

[54] S. Jain, "Universal thermal and electrical conductivity from holography," JHEP **1011**, 092 (2010) [arXiv:1008.2944 [hep-th]].

[55] S. K. Chakrabarti, S. Chakrabortty, S. Jain, "Proof of universality of electrical conductivity at finite chemical potential," JHEP **1102**, 073 (2011), [arXiv:1011.3499 [hep-th]].

[56] S. Jain, "Universal properties of thermal and electrical conductivity of gauge theory plasmas from holography," JHEP **1006**, 023 (2010) [arXiv:0912.2719 [hep-th]].

[57] R. C. Myers, M. F. Paulos and A. Sinha, "Holographic Hydrodynamics with a Chemical Potential," JHEP **0906**, 006 (2009) [arXiv:0903.2834 [hep-th]].

[58] K. Maeda, M. Natsuume and T. Okamura, "Dynamic critical phenomena in the AdS/CFT duality," Phys. Rev. D **78**, 106007 (2008) [arXiv:0809.4074 [hep-th]].

[59] S. A. Hartnoll and C. P. Herzog, "Ohm's Law at strong coupling: S duality and the cyclotron resonance," Phys. Rev. D **76**, 106012 (2007) [arXiv:0706.3228 [hep-th]].

[60] C. P. Herzog, P. Kovtun, S. Sachdev and D. T. Son, "Quantum critical transport, duality, and M-theory," Phys. Rev. D **75**, 085020 (2007) [arXiv:hep-th/0701036].

[61] J. R. David, M. Mahato, S. Thakur and S. R. Wadia, "Hydrodynamics of R-charged D1-branes," JHEP **1101** (2011) 014 [arXiv:1008.4350 [hep-th]].

[62] M. Cvetic, H. Lu and C. N. Pope, "Consistent sphere reductions and universality of the Coulomb branch in the domain-wall / QFT correspondence," Nucl. Phys. B **590**, 213 (2000) [arXiv:hep-th/0004201].

[63] T. Harmark and N. A. Obers, "Thermodynamics of spinning branes and their dual field theories," JHEP **01** (2000) 008 [arXiv:hep-th/9910036].

[64] M. Cvetic, H. Lu and C. N. Pope, "Consistent Kaluza-Klein sphere reductions," Phys. Rev. **D62** (2000) 064028 [arXiv: hep-th/0003286].

[65] S. Kachru, X. Liu and M. Mulligan, "Gravity Duals of Lifshitz-like Fixed Points," Phys. Rev. D **78**, 106005 (2008) [arXiv:0808.1725 [hep-th]].

[66] U. H. Danielsson and L. Thorlacius, "Black holes in asymptotically Lifshitz spacetime," JHEP **0903**, 070 (2009) [arXiv:0812.5088 [hep-th]].

[67] R. B. Mann, "Lifshitz Topological Black Holes," JHEP **0906**, 075 (2009) [arXiv:0905.1136 [hep-th]].

[68] G. Bertoldi, B. A. Burrington and A. Peet, "Black Holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent," arXiv:0905.3183 [hep-th].

[69] G. Bertoldi, B. A. Burrington and A. W. Peet, "Thermodynamics of black branes in asymptotically Lifshitz spacetimes," arXiv:0907.4755 [hep-th].

[70] M. Taylor, "Non-relativistic holography," arXiv:0812.0530 [hep-th].

[71] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, "Analytic Lifshitz black holes in higher dimensions," JHEP **1004**, 030 (2010) [arXiv:1001.2361 [hep-th]].

[72] T. Azeyanagi, W. Li and T. Takayanagi, "On String Theory Duals of Lifshitz-like Fixed Points," JHEP **0906**, 084 (2009) [arXiv:0905.0688 [hep-th]].

[73] W. Li, T. Nishioka and T. Takayanagi, "Some No-go Theorems for String Duals of Non-relativistic Lifshitz-like Theories," JHEP **0910**, 015 (2009) [arXiv:0908.0363 [hep-th]].

[74] E. J. Brynjolfsson, U. H. Danielsson, L. Thorlacius and T. Zingg, "Holographic Superconductors with Lifshitz Scaling," arXiv:0908.2611 [hep-th].

[75] D. W. Pang, "On Charged Lifshitz Black Holes," JHEP **1001**, 116 (2010) [arXiv:0911.2777 [hep-th]].

[76] D. W. Pang, "Conductivity and Diffusion Constant in Lifshitz Backgrounds," JHEP **1001**, 120 (2010) [arXiv:0912.2403 [hep-th]].

[77] I. Bredberg, C. Keeler, V. Lysov and A. Strominger, "Wilsonian Approach to Fluid/Gravity Duality," arXiv:1006.1902 [hep-th].

[78] A. Buchel and J. T. Liu, "Universality of the shear viscosity in supergravity," Phys. Rev. Lett. **93**, 090602 (2004) [arXiv:hep-th/0311175].

[79] A. Buchel, "On universality of stress-energy tensor correlation functions in supergravity," Phys. Lett. B **609**, 392 (2005) [arXiv:hep-th/0408095].

[80] M. Vojta, Y. Zhang, and S. Sachdev, Intl. Jour. of Mod. Phys. B **14**, 3719 (2000).,

[81] M. Mueller, L. Fritz, S. Sachdev and J. Schmalian Graphene: Relativistic transport in a nearly perfect quantum liquid, [arXiv.org:0910.5680]

[82] T. Faulkner, H. Liu, J. McGreevy and D. Vegh, "Emergent quantum criticality, Fermi surfaces, and AdS2," arXiv:0907.2694 [hep-th].

[83] M. Edalati, J. I. Jottar and R. G. Leigh, "Transport Coefficients at Zero Temperature from Extremal Black Holes," arXiv:0910.0645 [hep-th].

[84] E. Imeroni and A. Sinha, "Non-relativistic metrics with extremal limits," JHEP **0909**, 096 (2009) arXiv:0907.1892 [hep-th].

[85] H. Lu, J. w. Mei, C. N. Pope and J. F. Vazquez-Poritz, "Extremal Static AdS Black Hole/CFT Correspondence in Gauged Supergravities," Phys. Lett. B **673**, 77 (2009) arXiv:0901.1677 [hep-th].

[86] S. K. Chakrabarti, S. Jain and S. Mukherji, "Viscosity to entropy ratio at extremality," JHEP **1001** (2010) 068 [arXiv:0910.5132 [hep-th]].

[87] M. F. Paulos, "Transport coefficients, membrane couplings and universality at extremality," JHEP **1002** (2010) 067 [arXiv:0910.4602 [hep-th]].

[88] R. G. Cai, Y. Liu and Y. W. Sun, "Transport Coefficients from Extremal Gauss-Bonnet Black Holes," JHEP **1004** (2010) 090 [arXiv:0910.4705 [hep-th]].

[89] K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, "Holography of Charged Dilaton Black Holes," arXiv:0911.3586 [hep-th].

[90] K. Goldstein, N. Iizuka, S. Kachru, S. Prakash, S. P. Trivedi and A. Westphal, "Holography of Dyonic Dilaton Black Branes," JHEP **1010**, 027 (2010) [arXiv:1007.2490 [hep-th]].

[91] G. T. Horowitz and M. M. Roberts, "Zero Temperature Limit of Holographic Superconductors," JHEP **0911**, 015 (2009) [arXiv:0908.3677 [hep-th]].

[92] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis and R. Meyer, "Effective Holographic Theories for low-temperature condensed matter systems," JHEP **1011**, 151 (2010) [arXiv:1005.4690 [hep-th]].

[93] R. Meyer, B. Gouteraux, B. S. Kim, "Strange Metallic Behaviour and the Thermodynamics of Charged Dilatonic Black Holes," [arXiv:1102.4433 [hep-th]].

[94] B. Gouteraux, B. S. Kim, R. Meyer, "Charged Dilatonic Black Holes and their Transport Properties," [arXiv:1102.4440 [hep-th]].

[95] N. Iizuka, N. Kundu, P. Narayan and S. P. Trivedi, "Holographic Fermi and Non-Fermi Liquids with Transitions in Dilaton Gravity," arXiv:1105.1162 [hep-th].

[96] M. Parikh and F. Wilczek, "An Action for black hole membranes," Phys. Rev. D **58** (1998) 064011, [arXiv:gr-qc/9712077].

[97] T. Damour and M. Lilley, "String theory, gravity and experiment," arXiv:0802.4169 [hep-th].

[98] T. Damour, "Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs"; Thèse de Doctorat d'Etat, Université Pierre et Marie Curie, Paris VI, 1979.

[99] T. Damour, "Black Hole Eddy Currents," Phys. Rev. D **18**, 3598 (1978).

[100] T. Damour, "Surface effects in Black Hole Physics," in *Proceedings of the second Marcel Grossmann Meeting on general Relativity*, Ed. R.Ruffini, North-Holland, 1982.

[101] K. S. . Thorne, R. H. . Price and D. A. . Macdonald, "Black Holes: the Membrane Paradigm," *New Haven, USA: Yale Univ. Pr. (1986) 367p.*

[102] R. H. Price and K. S. Thorne, "Membrane viewpoint on black holes: properties and evolution of the stretched horizon," Phys. Rev. D **33**, 915 (1986).