QFT: Problem Set 6

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1. Consider the integral

$$I(g) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2-\lambda x^4}. \tag{1}$$

This is a model integral, different variations of it has appeared in class lectures. Assume $\lambda << 1$. Write $I(g) \sim \sum_{n=0}^{\infty} \lambda^n I_n$ and show (for large n)

$$\lambda^n I_n \sim \Big(-\frac{\lambda n}{e}\Big)^n. \tag{2}$$

You may need the formula $n! \sim n^n e^{-n}$ for large n. Now, argue that no matter how small λ is, pertubation series diverges at order $\sim 1/\lambda$.

Ill-convergence of the series does not however imply that the perturbative approach fails. For that consider $\sum_{n=0}^{n_m} \lambda^n I_n$ and use the fact that

$$|e^{-\lambda x^4} - \sum_{n=0}^{n_m} \frac{(\lambda x^4)^n}{n!}| \leq \frac{(\lambda x^4)^{n_m+1}}{(n_m+1)!} \tag{3}$$

to estimate the error (for $n_m >> 1$)

$$\Big|I(\lambda) - \sum_{n=0}^{n_m} \lambda^n I_n\Big| \leq \lambda^{n_m+1}\Big|I_{n_m+1}\Big| \sim \Big(\frac{\lambda n_m}{e}\Big)^{n_m}. \tag{4}$$

At which value of n_m the error reaches its minimum?

2. Consider the Euclidean action in d-dimensional space

$$S[\phi] = \int d^d x\Big(\frac{1}{2}(\partial\phi)^2 + \frac{r}{2}\phi^2 + \lambda\phi^4\Big) \tag{5}$$

where ϕ is a scalar bosonic field. Find the length dimensions of ϕ, λ and r. Argue that any intrinsic length scale produced by the theory must scale as $r^{-1/2}$. Show that the free propagator

$$< \phi(\mathbf{x})\phi(0) > \sim \int \frac{d^d p}{(2\pi)^d} \frac{e^{-i\mathbf{p}.\mathbf{x}}}{p^2+r}. \tag{6}$$

Perform the integral in $d = 1, 2$ and 3 dimensions and show that the following results follow

$$\begin{aligned} < \phi(\mathbf{x})\phi(0) > &= \frac{e^{-\sqrt{r}|x|}}{2\sqrt{r}} \quad \text{for } d = 1 \\ &= -\frac{1}{2\pi}\ln\frac{\sqrt{r}|x|}{2} \quad \text{for } d = 2 \text{ and } |x| << 1/\sqrt{r} \\ &= \frac{1}{2}(2\pi\sqrt{r}|x|)^{-1/2} e^{-\sqrt{r}|x|} \text{ for } d = 2 \text{ and } |x| >> 1/\sqrt{r} \\ &= \frac{e^{-\sqrt{r}|x|}}{4\pi|x|} \quad \text{for } d = 3. \end{aligned} \tag{7}$$

From above results, we see that correlations generally decay exponentially, at a rate set by the correlation length $\xi \sim 1/\sqrt{r}$. What happens when $r \to 0$?