QFT: Problem Set 2

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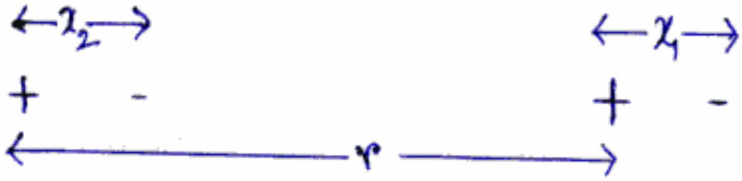
1. *Vacuum Cleaner?*

The quantum vacuum fluctuates. One example of vacuum fluctuation (Casimir effect) was done in the class. Here we consider another – the van der Walls effect: nautral atoms attract each other by a potential that at small seperation r, scales like r^{-6}.



Consider two hydrogen atoms (see figure) seperated by a distance r. In an atom, electron is bound by harmonic oscillator force to heavy proton.

(a) Show that in the limit $x_1, x_2 << r$, the Hamiltonian can be approximated as

$$\hat{H} = \frac{1}{2m}\hat{p}_1^2 + \frac{1}{2m}\hat{p}_2^2 + \frac{1}{2}m\omega_0^2\hat{x}_1^2 + \frac{1}{2}m\omega_0^2\hat{x}_2^2 - \frac{e^2}{2\pi}\frac{\hat{x}_1\hat{x}_2}{r^6}. \tag{1}$$

(b) Show that the interaction term (the last term) in the Hamiltonian causes a shift in the zero point energy by an amount

$$-\frac{e^4}{32\pi^2 m^2\omega_0^2 r^6}. \tag{2}$$

2. *All about Lorentz*

Under Lorentz transformation (LT), $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$, where Λ is such that it preserves the interval x^2 between x^μ and the origin.

(a) Show that

$$g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{3}$$

(b) For an infinitesimal transformation, we write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu. \tag{4}$$

Using (a), show that

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu}. \tag{5}$$

Do we conclude from here that there are six independent infinitesimal LT?

(c) Inverse of Λ is defined through

$$(\Lambda^{-1})^\rho{}_\nu \Lambda^\nu{}_\sigma = \delta^\rho{}_\sigma. \tag{6}$$

From here, show that

$$(\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho. \tag{7}$$

Further, argue that the above equation leads to det $\Lambda = \pm 1$. LT with det $\Lambda = 1$, and with $\Lambda^0{}_0 \geq 1$ is called proper and orthochronous transformation. Note that (7) belongs to proper transformation. Proper transformation forms a subgroup of LT.

(d) In quantum theory, symmetry is usually represented by unitary transformation. Cosider an unitary operator $U(\Lambda)$ for a proper orthochronous transformation Λ which satisfies

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda). \tag{8}$$

Now writing the infinitesimal trasformation as

$$U(\Lambda) = U(1 + \delta\omega) = I + \frac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}, \tag{9}$$

show that

$$U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma}. \tag{10}$$

Further writing $\Lambda = I + \delta\omega$, find $[M^{\mu\nu}, M^{\rho\sigma}]$.

(e) Finally, defining angular momentum operator $J_i = \epsilon_{ijk}M^{jk}/2$ and the boost operator $K_i = M^{0i}$, show that

$$\begin{aligned} [J_i, J_k] &= i\hbar\epsilon_{ijk}J_k, \\ [K_i, K_j] &= -i\hbar\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\hbar\epsilon_{ijk}K_k. \end{aligned} \tag{11}$$

Note that the first equation is the usual relation that angular momentum satisfies. The second one tells us that a series of boosts can be equivalent to rotation and the thrid one shows that K_i transforms as a component of three vector under rotation.