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Rolling down solution in a simple mechanical model

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Abstract. We explicitly construct a *time-dependent* rolling down solution from symmetry preserving phase to the symmetry broken one within a simple mechanical system. It consists of a particle which is free to move on a loop. The loop, attached at the top to a support, rotates about the vertical axis passing through its center. As one tunes the frequency, this model provides a toy example of spontaneous symmetry breaking and continuous phase transition.

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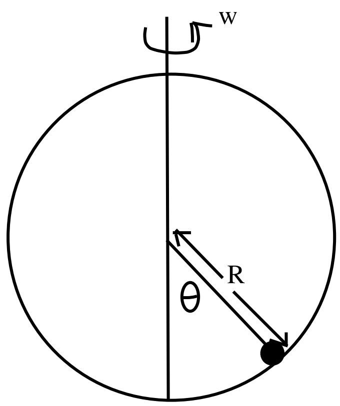
1. INTRODUCTION

Often simple mechanical systems provide instructive insights into complicated physical processes that occur in our nature. Consider, for example, the Alben model [1]. Very simple and exactly solvable, yet this model captures various features of phase transition. Another system, that is of our interest in this work, is discussed in [2]. It considers motion of a particle of mass m constrained to move on a frictionless circular loop of radius R. The loop is rotated about a vertical axis passing through its center as in the figure below. The analysis of [2] shows that when the loop rotates with a small angular frequency ω, the particle sits at the bottom of the loop. This is the symmetry preserving phase of the model. However, as one increases the frequency beyond a critical value, which we call ω_c, the ball settles at a non-zero value of θ. As soon as this happens, the particle breaks $\theta \to -\theta$ symmetry, and consequently, a symmetry broken phase is reached. If we increase ω further, θ increases continuously to a limiting value of $\pi/2$. This model provides a close analogy with the second order phase transition where θ plays the role of the order parameter and ω the temperature. Indeed, it is possible to construct a Landau potential in powers of the oreder parameter [3].



The generalisation of the above model to include first order transition was discussed in [3]. Behaviour of this model in presence of friction was partially analysed in [4]. The purpose of this work is to understand the transition from the symmetric phase to the symmetry broken phase in more detail. In particular, we explicitly construct the *time-dependent* solution for θ which shows how the ball goes from the unstable position $\theta = 0$ to the stable $\theta \neq 0$ one when $\omega > \omega_c$.

In passing, we note that our analysis might provide an analogy with the inflationary phase of our universe. Inflation is triggered by rolling down of a scalar field called inflaton. As inflaton rolls down from unstable phase to its true minimum, it releases energy. This energy, in turn, causes exponential growth of the universe. In the model of [2], θ mimics the inflaton field and the time dependent solution that we construct is the analogue of rolling down solution of the inflaton [5].

In paper is structured as follows. In the next section, we review the model within the Lagrangian framework [3]. Then we solve the classical equation of motion with required boundary condition. This leads to a rolling down solution from unstable to the stable phase. Finally, in the last section of this paper, we summerise our results.

2. THE LAGRANGIAN AND THE EQUATION OF MOTION

As discussed in [3], the model has an effective Lagrangian description. Let us assume that at any instant of time the mass is at a position $\theta(t)$ The Lagrangian then reads [3]

$$L = \text{kinetic energy} - \text{potential energy}. \tag{1}$$

While the kinetic energy is given by

$$KE = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta, \tag{2}$$

the potential energy is

$$PE = -mgR\cos\theta. \tag{3}$$

Therefore the total Lagrangian is

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta. \tag{4}$$

This can be re-written as

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - (-(\frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta)) \tag{5}$$

This allows us to have a description of the system in terms of an *effective* potential

$$V = -(\frac{1}{2}mR^2\omega^2\sin^2\theta + mgR\cos\theta) \tag{6}$$

Note that V is symmetric under $\theta \to -\theta$. The nature of the effective potential is shown in figure (1). The stable positions correspond to the extrema of this effective potential

$$\frac{dV}{d\theta} = 0. \tag{7}$$

This gives

$$\sin\theta(g/R - \omega^2\cos\theta) = 0. \tag{8}$$

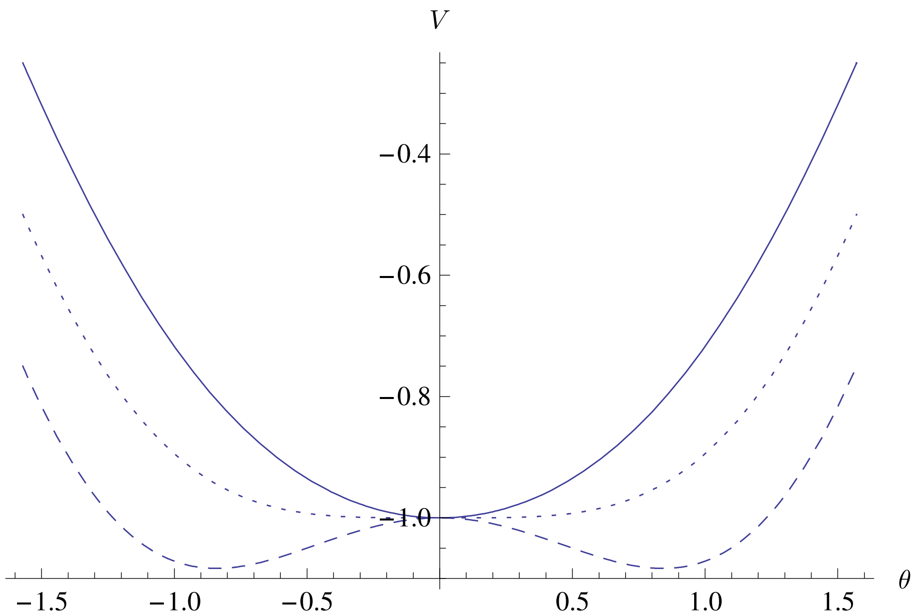


Figure 1. Effective potential for various values of ω. We have taken $g/R = 1$ for this plot. The solid, dotted and dashed curves are for $\omega = .5, 1$ and 1.5 respectively.

The solutions are

$$\theta = 0, \text{ or, } \theta = \cos^{-1}\frac{g}{R\omega^2}. \tag{9}$$

Since $\cos\theta \leq 1$, the second condition holds only when $\omega > \omega_c$ where we have defined

$$\omega_c = \sqrt{\frac{g}{R}}. \tag{10}$$

Consequently, for $\omega < \omega_c$, the particle remains at $\theta = 0$. However, as soon as we increase the frequency beyond ω_c, the second solution in (9) becomes the minimum and therefore, the mass settles at $\theta = \cos^{-1}\frac{g}{R\omega^2}$. This, in turn, breakes $\theta \to -\theta$ symmetry spontaneously and we reach a symmetry broken phase. Our aim is now to explicitly find the time dependent solution for $\theta(t)$ which represents a rolling down solution from $\theta = 0$ to some non-zero stable value for $\omega > \omega_c$. To proceed, we first write down the Eular-Lagrange equation for the mass m. This follows from

$$\frac{d}{dt}(\frac{\partial L}{\partial\dot{\theta}}) - \frac{\partial L}{\partial\theta} = 0, \tag{11}$$

giving

$$mR^2\ddot{\theta} - \omega^2\sin\theta\cos\theta + {\omega_c}^2\sin\theta = 0. \tag{12}$$

This equation can be rewritten as

$$\frac{d}{dt}(\frac{1}{2}\dot{\theta}^2 + \frac{1}{4}\omega^2\cos 2\theta - {\omega_c}^2\cos\theta) = 0. \tag{13}$$

This is noting but the energy conservation equation. Upon integrating, we get

$$\frac{1}{2}\dot{\theta}^2 + \frac{1}{4}\omega^2\cos 2\theta - {\omega_c}^2\cos\theta) = c, \tag{14}$$

where c is a constant. Now c can be fixed by appropriate boundary condition. In particular, we use $\dot{\theta} = 0$ at $\theta = 0$. This gives

$$c = \frac{1}{4}\omega^2 - {\omega_c}^2. \tag{15}$$

Substituting c in (14), we get

$$\frac{1}{2}\dot{\theta}^2 + \frac{1}{4}\omega^2\cos 2\theta - {\omega_c}^2\cos\theta) = \frac{1}{4}\omega^2 - {\omega_c}^2. \tag{16}$$

Further simplifying, we reach at

$$\dot{\theta}^2 = \omega^2(1-\cos\theta)(1+\cos\theta - \frac{2}{\omega^2}{\omega_c}^2). \tag{17}$$

Defining $\frac{2}{\omega^2}{\omega_c}^2 = a$, the above equation becomes

$$\dot{\theta}^2 = \omega^2(1-\cos\theta)(1+\cos\theta - a). \tag{18}$$

Equation (18) can now be easily solved or can be read off from a similar equation appeared in [6] in a different context. The solution is

$$\theta = \pm 2\tan^{-1}\Big(\sqrt{\frac{2-a}{a}}\mathrm{sech}\{\sqrt{\omega(\omega - \frac{a\omega}{2})}(t-t_0)\}\Big). \tag{19}$$

Substituting the value of a, we get

$$\theta = \pm 2\tan^{-1}\Big(\sqrt{\frac{\omega^2 - {\omega_c}^2}{{\omega_c}^2}}\mathrm{sech}\{(\sqrt{\omega^2 - {\omega_c}^2})(t-t_0)\}\Big). \tag{20}$$

In the above equation, t_0 is an arbitrary constant. This appears due to the time translational invariance of the differential equation (18). To fix the integration constant, we have used the boundary condition that at $t = -\infty$ the particle is at $\theta = 0$. Note that equation (20) is only real for $\omega > \omega_c$. This is what we expect. For $\omega < \omega_c$, only $\theta = 0$ is a stable minimum of the effective potential. Furthermore, we expect that as we increase ω beyond ω_c, the rate at which the particle rolls down would be more. This is indeed the case as can be seen from figure (2) where we have plotted the negative θ part of the solutions (20). We also notice that, it requires infinite time to reach finite θ value from zero. Consequently, it would never reach the symmetry preserving phase again. Note that the non-zero stable θ value in the figure is $\theta = -\cos^{-1}(\frac{\omega_c^2}{\omega^2})$. Its absolute value increases as we increase ω.

3. SUMMARY

To conclude, in this paper, we have constructed a time-dependent rolling down solution from symmetry preserving phase to the symmetry broken one within the context of a mechanical model [2]. This model, though very simple, captures various features of spontaneous symmetry breaking and second order phase transition. We hope our results will also serve as an analogy to more complicated scenarios including the models of infation.

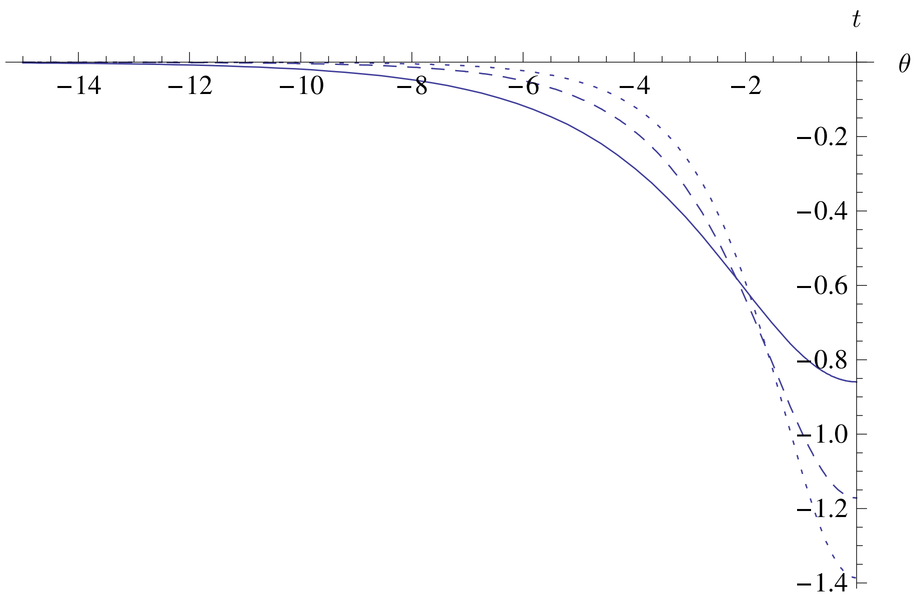


Figure 2. This is plot of $\theta(t)$ with time for $\omega > \omega_c$. The mass rolls down from unstable $\theta = 0$ position to some non-zero (negative) θ in time. The solid, dashed and dotted lines are for $\omega = 1.1, 1.2$ and 1.3 respectively. ω_c has been set to 1. We see that the rate of rolling down increases as we increase $\omega - \omega_c$.

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