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A method to solve Friedmann Equations for time dependent equation of state

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Abstract. Our present understanding of the evolution of the universe relies upon the Friedmann-Robertson-Walker cosmological models. We give a simple method to reduce Friedmann equations to a second order linear differential equation when it is supplemented with a *time dependent* equation of state. Furthermore, as illustrative examples, we solve this equation for some specific time dependent equation of states.

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1 INTRODUCTION

Our present understanding of the evolution of the universe relies upon the Friedmann-Robertson-Walker (FRW) cosmological models. This model is so successful that it is now being considered as the Standard Model of Cosmology [1]. One of the fundamental features of the standard model of cosmology is the expansion of the universe. This was discovered around 1920 from red shift measurements of galaxy spectra. It was also found that at large scale, our universe is isotropic and homogenous. The evidence of our universe being isotropic follows from uniformity of temperature of the Cosmic Microwave Background Radiation– commonly known as CMBR. Finally, the large scale homogeneity follows from the 'peculiar velocity field of the universe. See [1] for more detail.

Considering the importance of FRW cosmological models in understanding the time evolution of our universe, in this paper, we try to study some basic features of this model. In particular, we review the Friedmann equations for isotropic and homogenous universe following [2]. This is done in section 2 of the paper. Friedmann equations lead to a relation between the energy density of the universe with its size when an equation of state is supplemented. We review this behaviour for a general equation of state of the form $p = \omega\rho$. Here, ρ and p are energy density and pressure of the universe respectively. This equation of state includes the cases of radiation, matter and vacuum dominated era. In section 3, we simplify the Friedmann equations to a single linear second order differential equation for *time dependent* equation of state $p = \omega(t)\rho$. Here t parametrises the time. Time dependent equation of state arises in various models of inflations and also in string theory. Our method here is a generalization of [3]. In [3], a similar equation was constructed for *time* *independent* equation of state. In section 4, we solve our linear differential equation for some specific time dependent ω's. It should be noted that, though we give a few examples here, for large number of different time dependent $\omega(t)$, the differential equation can be explicitly solved. This, in turn, would allow us to explicitly figure out the time dependence of the radius of the universe - perhaps otherwise it would have been difficult. This paper ends with a brief summary of our results.

2 FRIEDMANN EQUATIONS

In this section, we review the Friedmann equations which is in the heart of the standard model of cosmology. We find out behaviour of the universe as a function of time when Friedmann equations are supplemented with time independent equation of state $p = \omega\rho$. In this section we closely follow [2].

Friedmann-Robertson-Walker metric for an isotropic and homogenous universe is given by

$$ds^2 = -dt^2 + R^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big], \tag{1}$$

where t, r, θ, ϕ are the coordinates and $R(t)$ is the scale factor. The parameter k takes values $1, 0, -1$ for closed, flat and open universe respectively. Friedmann equations relates the time dependence of the scale factor with the pressure (p) and the energy density (ρ) of the universe. These equations are given by:

$$\frac{1}{R^2}\Big(\frac{dR}{dt}\Big)^2 - \frac{8\pi G\rho}{3} = -\frac{k}{R^2}, \tag{2}$$

and

$$\frac{d^2R}{dt^2} = -\frac{4\pi G(\rho + 3p)R}{3}. \tag{3}$$

Now, given a relation between the energy density and pressure, it is possible to solve the Friedmann equations. This, in turn, tells us how the energy density of the universe changes as a function of the size of the universe. For matter dominated universe, one expects that as a sphere expands, the energy due to the presence of matter does not change. The matter only spreads. That leads to

$$\frac{d(R^3\rho_m)}{dt} = 0. \tag{4}$$

Or in other words,

$$\rho_m = \frac{\text{constant}}{R^3}. \tag{5}$$

However, Einstein mass-energy relation tells us (written in an unit such that $c = 1$)

$$dm = dE = -pdV. \tag{6}$$

Since $dm = 0$ for matter dominated universe and $dV \neq 0$, it follows $p_m = 0$. Substituting these in (2,3), we get for $k = 0$:

$$R \propto t^{\frac{2}{3}}. \tag{7}$$

Similarly, for radiation dominated universe, we know

$$p_\gamma = \frac{1}{3}\rho_\gamma. \tag{8}$$

Now since

$$\frac{d}{dt}(R^4\rho_\gamma) = \frac{d}{dt}[(R^3\rho_\gamma)R], \tag{9}$$

or we have,

$$\frac{d}{dt}(R^4\rho_\gamma) = R\frac{d}{dt}(R^3\rho_\gamma) + R^3\rho_\gamma\frac{d}{dt}R. \tag{10}$$

Furthermore, using (6) and (8), we get

$$\frac{d}{dt}(R^4\rho_\gamma) = 0. \tag{11}$$

Or, in other words, we have

$$\rho_\gamma = \frac{\text{constant}}{R^4}. \tag{12}$$

Substituting this in (3), we get

$$R \propto \sqrt{t}. \tag{13}$$

In general, the equation of state has the form

$$p = \omega\rho. \tag{14}$$

As we have seen previously, for matter dominated universe $\omega = 0$, for radiation dominated universe $\omega = 1/3$. Similarly, during the time of inflationary period, the universe was vacuum dominated and the value of ω during this period was -1. Thus, in general, during different epochs, ω takes different values. Proceeding as before, one gets, for flat universe,

$$\rho_\omega = \frac{1}{R^{3+3\omega}}, \tag{15}$$

and consequently, from (8), it follows

$$R \propto t^{\frac{2}{3+3\omega}}. \tag{16}$$

3 SIMPLIFYING FRIEDMAN EQUATIONS FOR TIME DEPENDENT EQUATION OF STATES

In this section, we will try to simplify the Friedmann equations for given time dependent equation of state of the form

$$p = \omega(t)\rho. \tag{17}$$

We will keep our study general enough to allow all three values of k. Beginning with the Friedmann Equations

$$\ddot{R} = -\frac{4\pi G(\rho + 3p)R}{3} \tag{18}$$

and

$$\dot{R} = \frac{8\pi G\rho R^2}{3} - k. \tag{19}$$

Now, multiplying (19) with a constant, say c, and adding to (18), we get

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc}{R^2} = \frac{-4\pi G(\rho + 3p)}{3} + \frac{8\pi G\rho c}{3}. \tag{20}$$

Using the equation of state (17), the above equation becomes

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc}{R^2} = \frac{-4\pi G(\rho + 3\omega\rho)}{3} + \frac{8\pi G\rho c}{3}. \tag{21}$$

We now choose c such that the right hand side of the above equation vanishes. This leads to

$$c = \frac{(1 + 3\omega)}{2}. \tag{22}$$

Hence, finally we have

$$\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc}{R^2} = 0 \tag{23}$$

To further simplify the above equation, we define η such that

$$R(\eta)d\eta = dt. \tag{24}$$

or,

$$\frac{d\eta}{dt} = \frac{1}{R(\eta)}. \tag{25}$$

All the t derivatives on R can now be re-written in terms of derivatives of η as

$$\dot{R} = \frac{dR}{dt} = \left(\frac{dR}{d\eta}\right)\left(\frac{d\eta}{dt}\right) = \frac{1}{R}\left(\frac{dR}{d\eta}\right). \tag{26}$$

This gives,

$$\ddot{R} = [\frac{-1}{R^3}\Big(\frac{dR}{d\eta}\Big)^2 + \frac{1}{R^2}\frac{d^2R}{d\eta^2}]. \tag{27}$$

Substituting the values of $\ddot{R}$ and $\dot{R}$ in the equation (23), we get

$$\frac{1}{R}\,[\frac{-1}{R^3}\Big(\frac{dR}{d\eta}\Big)^2 + \frac{1}{R^2}\frac{d^2R}{d\eta^2}] + \frac{c}{R^2}\,[\frac{1}{R}\frac{dR}{d\eta}]^2 + \frac{kc}{R^2} = 0 \tag{28}$$

Simplifying further, we get

$$\frac{1}{R}\,\frac{d^2R}{d\eta^2} + \frac{c-1}{R^2}\,\Big(\frac{dR}{d\eta}\Big)^2 + \frac{kc}{R^2} = 0. \tag{29}$$

Now we change variable as $u = \frac{1}{R}\frac{dR}{d\eta}$, such that

$$\frac{du}{d\eta} = \frac{d}{d\eta}\Big(\frac{1}{R}\frac{dR}{d\eta}\Big) = \frac{-1}{R^2}\,\Big(\frac{dR}{d\eta}\Big)^2 + \frac{1}{R}\,\frac{d^2R}{d\eta^2}. \tag{30}$$

Hence

$$\frac{1}{R}\frac{d^2R}{d\eta^2} = \frac{du}{d\eta} + [\frac{1}{R}\Big(\frac{dR}{d\eta}\Big)]^2 = \frac{du}{d\eta} + u^2 \tag{31}$$

Substituting the values in equation (29), we arrive at

$$\frac{du}{d\eta} + c(\eta)u^2 + kc(\eta) = 0. \tag{32}$$

This is known as Riccati equation. By using suitable variable, it is possible to bring this to a linear differential equation. Now defining y such that $u = \frac{1}{c}\frac{y'}{y}$ we get

$$y' = \frac{dy}{d\eta}, \tag{33}$$

$$\frac{du}{d\eta} = \frac{d}{d\eta}\,[\frac{1}{c}\,\frac{y'}{y}], \tag{34}$$

$$\frac{du}{d\eta} = \frac{y'}{y}\,\frac{d}{d\eta}\,\Big(\frac{1}{c}\Big) + \frac{1}{c}\,\frac{d}{d\eta}\,\Big(\frac{y'}{y}\Big), \tag{35}$$

$$\frac{du}{d\eta} = \frac{y'}{y}\,\frac{d}{d\eta}\,\Big(\frac{2}{1+3\omega}\Big) + \frac{1}{c}\,[\frac{-y'^2}{y^2} + \frac{y''}{y}]. \tag{36}$$

Upon simplifying the above equation, we get

$$\frac{du}{d\eta} = \frac{1}{y}\,\frac{-6y'\omega'}{(1+3\omega)^2} + \frac{1}{c}\left(\frac{y''}{y} - \frac{y'^2}{y^2}\right). \tag{37}$$

Substituting the value of $\frac{du}{d\eta}$ in equation (32) we get

$$\frac{1}{y}\,\frac{-6y'\omega'}{(1+3\omega)^2} + \frac{1}{c}\left(\frac{y''}{y} - \frac{y'^2}{y^2}\right) + \frac{1}{c}\,\frac{y'^2}{y^2} + kc = 0 \tag{38}$$

Simplifying the above equation, we get the final expression as

$$y'' - \frac{3\omega'}{1+3\omega}y' + \frac{k}{4}\left(1+3\omega\right)^2 y = 0. \tag{39}$$

For equation of state of the form $p = \omega\rho$, $c(\eta)$ reduces to a constant. In that case (39) simplifies to

$$y'' + kc^2 y = 0. \tag{40}$$

This case was studied in [3]. Solutions of this equation for $k = 1$ is

$$y(\eta) = \sin(c\,\eta). \tag{41}$$

This leads to

$$R(\eta) = \sin(c\eta)^{\frac{2}{1+3\omega}}, \tag{42}$$

So for $k = 1$, universe starts from a singularity at $\eta = 0$, reaches to a maximum size and ends at a singularity.

Similarly for $k = -1$, we get

$$y'' - c^2 y = 0. \tag{43}$$

The solution is

$$y(\eta) = c_1 \sinh(c\eta). \tag{44}$$

Proceeding exactly same as before we end up with a solution for R as

$$R = \sinh(c\,\eta)^{\frac{2}{1+3\omega}}. \tag{45}$$

Here, unlike the $k = 1$ models, universe starts out with a zero size at $\eta = 0$ and keeps growing to an infinite size at late time.

What we have learned so far is the following. For time dependent equation of state (17), it is possible to bring Friedmann equations to a single linear second order differential equation. This equation is now much easier to handle. In the next section, we solve this equation for some specific time dependent ω. Though we only work out a couple of examples for the purpose of illustration, it should be noted that large number of different cases can be solved starting from (39). This would have been difficult to achieve starting from complicated coupled Friedmann equations given in (2) and (3).

We will now try to solve equation (39) for some given form of ω. Solution of y as a function of η will allow us to find exact η dependence of the radius of the universe R. We will only consider open and closed universe as for flat universe, (39) reduces to a much simplified form.

First example

The first example that we would solve is where

$$\omega = \sqrt{\frac{1}{\eta}} - \frac{1}{3}. \tag{46}$$

As η runs from 0 to ∞, ω takes the value $\infty \geq \omega \geq -1/3$.

To proceed, all we will have to do is to find $\frac{3\omega'}{1+3\omega}$ and $\frac{k}{4}(1+3\omega)^2$ and substitute in equation (39). We then get

$$y'' + \frac{1}{2\eta}\, y' + \frac{9k}{4}\, \frac{1}{\eta}\, y = 0. \tag{47}$$

First, we consider the open universe with $k = -1$. We therefore have

$$y'' + \frac{1}{2\eta}\, y' - \frac{9}{4}\, \frac{1}{\eta}\, y = 0. \tag{48}$$

The solution of this equation is given by

$$y[\eta] = c_1 \cosh[3\sqrt{\eta}] + c_2 \sinh[3\sqrt{\eta}]. \tag{49}$$

For simplicity, we choose $c_1 = 1$ and $c_2 = 0$

$$y[\eta] = \cosh[3\sqrt{\eta}]. \tag{50}$$

Thus

$$y'[\eta] = \frac{3}{2\sqrt{\eta}} \sinh[3\sqrt{\eta}] \tag{51}$$

and

$$c = \frac{3}{2\sqrt{\eta}}. \tag{52}$$

We know that $u = \frac{-1}{c}\ \frac{y'}{y}$. Therefore

$$u = -\tanh[3\sqrt{\eta}]. \tag{53}$$

Our interest is to find R and it is defined as

$$u = \frac{1}{R}\frac{dR}{d\eta}. \tag{54}$$

We therefore get after an integration

$$\ln R = -(\eta + \frac{2}{3}\sqrt{\eta}\,\mathrm{Log}[1 + exp(-6\sqrt{\eta})] - \frac{1}{9}\mathrm{PolyLog}[2, exp(-6\sqrt{\eta})]), \tag{55}$$

which can, in turn, be written as

$$R(\eta) = \mathrm{Exp}[-(\eta + \frac{2}{3}\sqrt{\eta}\,\mathrm{Log}[1 + \mathrm{Exp}(-6\sqrt{\eta})] - \frac{1}{9}\mathrm{PolyLog}[2, \mathrm{Exp}(-6\sqrt{\eta})])] \tag{56}$$

Behaviour of R is shown in figure (1).

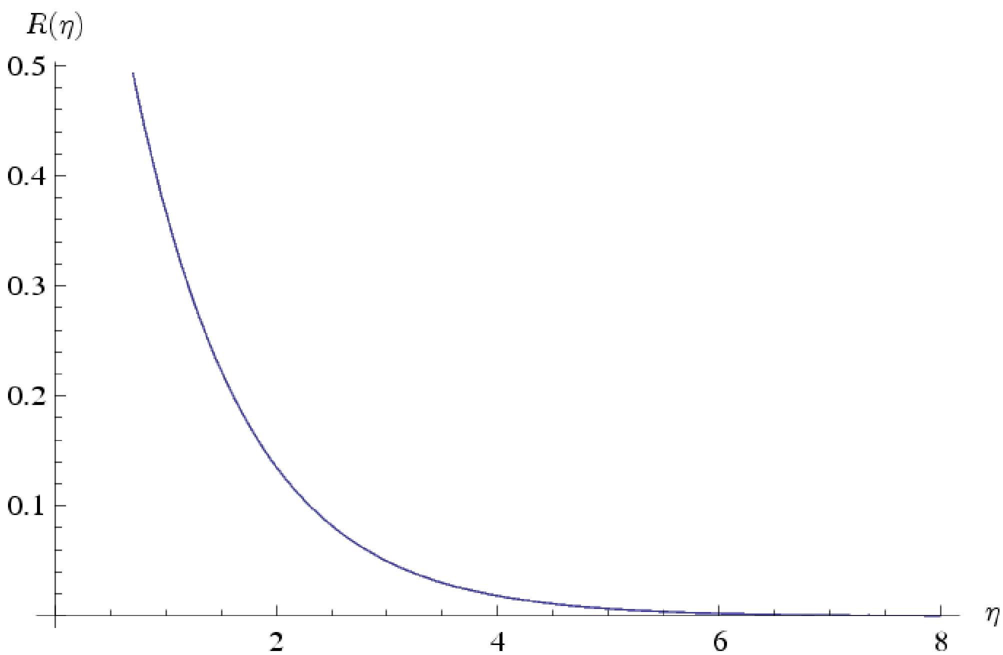

Figure 1. The behaviour of R as a function of η is shown above. This follows from equation (56).

It turns out that when ω is given by (46), even though there are real solutions of (39) for closed universe, one does not get a real solution for R. So, we believe that for $k = 1$, given (46), closed universe would not exist.

Second Example

The second case that we would solve is where

$$\omega = \frac{1}{\eta} - \frac{1}{3}. \tag{57}$$

Again, we first focus on to the case $k = -1$. For $\omega = \frac{1}{\eta} - \frac{1}{3}$.

$$\frac{3\omega'}{1+3\omega} = \frac{-1}{\eta}. \tag{58}$$

and

$$(1+3\omega)^2 = \frac{9}{\eta^2}. \tag{59}$$

We take the help of (39) to write our equation in the form as given below,

$$y''(\eta) + \frac{1}{\eta}y'(\eta) - \frac{9}{4\eta^2}y(\eta) = 0. \tag{60}$$

The solution of the above equation is of the form

$$y(\eta) = c_1 \cosh[\frac{3}{2}log(\eta)] + c_2 \sinh[\frac{3}{2}log(\eta)]. \tag{61}$$

Assuming for simplicity $c_1 = 1$ and $c_2 = 0$ we get

$$y(\eta) = \cosh[\frac{3}{2}log(\eta)]. \tag{62}$$

So $u = \frac{-1}{c}\frac{y'}{y}$ now is,

$$u = -\tanh[\frac{3}{2}log(\eta)]. \tag{63}$$

To know the η dependence of R we make use of the earlier definition,

$$u = \frac{1}{R}\frac{dR}{d\eta}. \tag{64}$$

On solving for R we get

$$\ln R = [-(\eta - \frac{2\mathrm{ArcTan}[\frac{-1+2\eta}{\sqrt{3}}]}{\sqrt{3}} + \frac{2}{3}\log[1+\eta] + \frac{1}{3}\log[1-\eta+\eta^2])]. \tag{65}$$

So finally we get,

$$R = \mathrm{Exp}[-(\eta - \frac{2\mathrm{ArcTan}[\frac{-1+2\eta}{\sqrt{3}}]}{\sqrt{3}} + \frac{2}{3}\log[1+\eta] + \frac{1}{3}\log[1-\eta+\eta^2])]. \tag{66}$$

The right hand side of the above equation is plotted in Figure (2).

As before, for $k = 1$, here also there does not exist a real solution for $R(\eta)$.

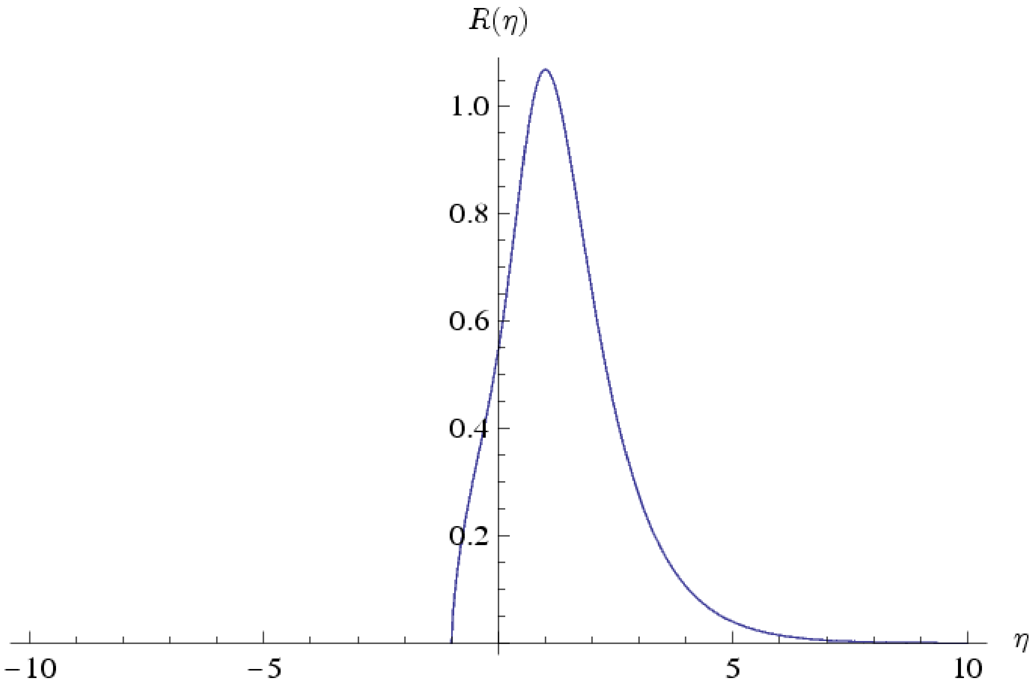

Figure 2. The behaviour of R as a function of η is shown above. This follows from equation (66).

5 CONCLUSION

To conclude, in this paper, we have briefly reviewed the Friedmann equations of standard model of cosmology. We have also reviewed a way to solve these equations when they are supplemented with the equation of state of the form $p = \omega\rho$. Furthermore, we generalized this method for time dependent equation of state of the form $p = \omega(t)\rho$. As some illustrative examples, we figured out the time evolution of the universe for some explicit time dependent ω's.

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