P R A Y A S
Students' Journal
of Physics

Analogy between Mechanical systems, Phase Transitions and possible Experimental Implementation

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Abstract. Following earlier works, we attempt to bring out analogies between some mechanical models and phase transitions in thermodynamical systems. Keeping possible obstacles in setting up such an experiment in mind, we discuss a generalisation of such models where we introduce forces that closely mimic static frictional force. The system is then studied by constructing an effective potential. Analogies between this effective potential and Landau's theory of phase transition are establised. We also make few comments on setting this up as an experiment in our laboratory.

Communicated by L. Satpathy

1. INTRODUCTION

Phase transitions are central to our life and therefore understanding them is one of the prime task of physicists. Phase transitions of water are matters of every day experience . Examples include boiling of water, the formation of frost during winter time, melting of ice cubes and so on. In all these above examples water makes transition from one phase to another. These crossovers involve absorption or emission of latent heat and are examples of the first order phase transition. Such emission or absorption of heat provide us with an indication of radical change in structure of the material around the phase transition point. Phase transition may also occur where state of a system changes continuously. These are examples of continuous phase transition and unlike the previous cases, crossover from one phase to the other does not involve latent heat. Some examples of this kind will be discussed at a later stage of this report. Though there are no latent heat in such transitions, certain detivatives of thermodynamic quantities become discontinuous around the phase transition point. Specific heat is such a quantity in a second order phase transition. In this paper, our aim will be to understand both kinds of transitions in terms of simpler models provided by some mechanical systems. As we discuss later, we believe that such models can be set up in our laboratory. In the rest of this section, we review a novel way to undersand several features of phase transition. In literature, this is called the Landau theory of phase transition. In the later sections, the framework will be used extensively.

Many insights of phase transition can be gained from Landau theory of phase transition [1]. In this theory, phases are characterised by an order parameter. Typically, this parameter is zero at high temperature and non-zero at low temperature. Magnetisation of ferromagnet, density for water vapour transition are examples of this kind. For magnetic systems, in the absence of external field, at a temperature T above a critical temperature T_c, magnetisation is zero, while for $T < T_c$, it is non-zero. It is a parameter that distinguishes different phases. To describe phase transition with in the framework of Landau Theory, one constructs a Landau potential V for the system. This is an expansion in terms of order parameter η. In general, this has the form

$$V(\eta, T, P) = a_0(T, P) + a_1(T, P)\eta + a_2(T, P)\eta^2 + a_3(T, P)\eta^3 + a_4(T, P)\eta^4 + ... \quad (1)$$

Here a_is are the coefficients that are in general non-zero and functions of thermodynamic variables such as temperature T, pressure P or chemical potential μ. While equilibrium states come from the solutions of

$$\frac{\partial V}{\partial \eta} = 0, \quad (2)$$

stability of the states can be inferred from $\partial^2 V/\partial\eta^2$. Typically, in a second order phase transition, the Landau function has a form [1]

$$V(\eta, T, P) = a_0(T, P) + a_2(T, P)\eta^2 + a_4(T, P)\eta^4 + ... \quad (3)$$

with $a_4 > 0$. We will have occasions to discuss this in some details at a later part of these report.

In this paper, we look into a mechanical system which provides us with a direct analogue of a second order phase transition [2], [3]. This is described in section 2 of the report. Our interest is to set up such an experiment in laboratory. However, we find that before we do so we need to address few other issues. Among them the most crucial one is to analyse the same mechanical system in the presence of a friction. This is our prime focus in this report. In the next section, we first introduce the mechanical system closly following [3] and bring out an analogy with second order phase transition within the framework of Landau theory. In section 3, we discuss the possibilities of implementing this as an experiment in our laboratory and also discuss the role of static friction. It turns out that, in general, it is quite a complicated problem. We therefore simplify the model to take care of some effects of the friction.We find that, after including these measures, the second order phase transition turns in to a first order one. We then again analyse the phases in terms of a Landau function. Finally, we end this report with a discussion of our results, possible generalisations, and also discuss the scope of implementing our model as an experiment.

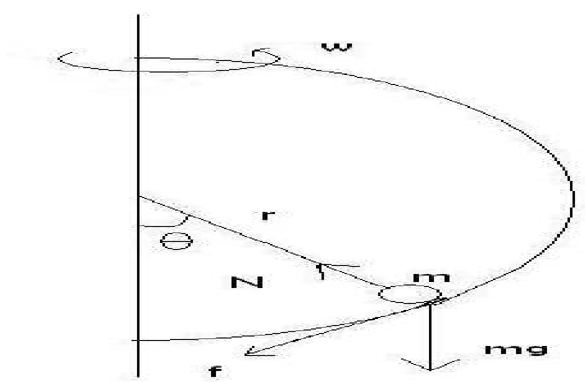

Figure 1. A half circular loop of radius r, kept vertical on a horizontal plane, is rotating with angular frequency ω. A ball of mass m is allowed to move along the frictionless half-loop

2. A MECHANICAL MODEL AND SECOND ORDER PHASE TRANSION

Consider a half circlular loop of radius r which is kept vertical on a horizontal plane. Now we rotate it along its diameter with an angular frequency ω. This is shown in the figure 1. A ball of mass m is free to move on this

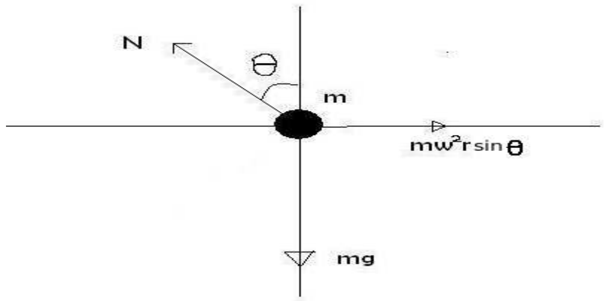

Figure 2. THe free body diagram

frictionless half-loop. The equilibrium position of the ball can be inferred from the free-body diagram which is shown in figure 2. The determining equations are given

$$N\cos\theta = mg, \quad N\sin\theta = m\,\omega^2 r\sin\theta. \tag{4}$$

In the above equations, N is the normal reaction and g is the gravitational constant. Solving this two equations, we get

$$\sin\theta = \beta\sin\theta\cos\theta, \tag{5}$$

where, we have defined $\beta = \omega^2 r/g$. The solutions of this equation are as follows:

$$\begin{aligned}\theta &= 0, \quad \text{for} \;\; \beta < 1, \\ &= \theta_0 = \cos^{-1}\Big[\frac{1}{\beta}\Big], \quad \text{for} \;\; \beta \geq 1.\end{aligned} \tag{6}$$

As in [3], the behaviour can be described by an effective potential that follows from 5. It is given by

$$\begin{aligned}V(\theta) &= \int d\theta\Big(\sin\theta - \beta\sin\theta\cos\theta\Big), \\ &= -\cos\theta - \frac{\beta}{2}\sin^2\theta.\end{aligned} \tag{7}$$

The behaviour of the effective potential for different values of β is shown in the figure 3. First of all, we note that, for $\beta < \beta_c = 1$, the potential has a minimum at $\theta = 0$. However for $\beta > \beta_c$, $\theta = 0$ is no longer a minimum, rather, the location depends on β. Infact, the location changes continuously from zero as we increase β. This provides an analogy between θ and the order parameter in continuous phase transition.

The potential (7) can be expanded in powers of θ near $\theta = 0$ and has the structure

$$V(\theta) = -1 + \Big(\frac{1}{2} - \frac{\beta}{2}\Big)\theta^2 + \Big(\frac{\beta}{6} - \frac{1}{24}\Big)\theta^4 + ... \tag{8}$$

As expected, we have an expansion in terms of θ. This should be compared with η in (3). Furthermore we note, comparing (3) and (8), that a_is are functions of β. It immediately suggests that β plays the role of the temperature. Furthermore, $\beta = \beta_c$ is the analogue of the critical temperature (T_c).

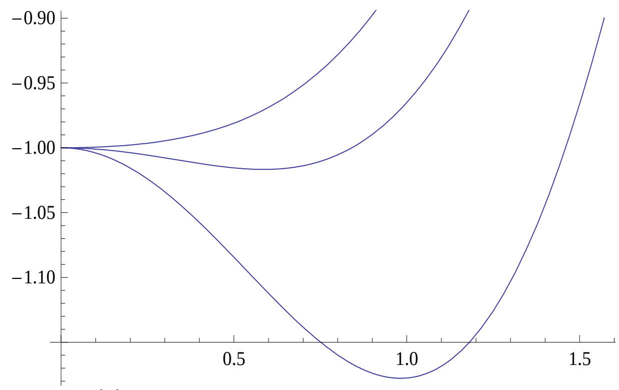

Figure 3. Behaviour of $V(\theta)$ for various values of β. The upper one is for $\beta = .9$, the next one below is for $\beta = 1.2$ and the last one is for $\beta = 1.8$. Note that if we interprete θ as an order parameter, the above diagram represents a second order phase transition with θ changing continuously as β changes from below $\beta_c = 1$ to $\beta > \beta_c$

3. MODEL INCLUDING STATIC FRICTION

The question that we now ask ourselves is: can this simple model be constructed in our laboratory? Even if it is possible, the immediate problem we face is the need to incorporate static friction. It is rather difficult to construct a model which does not have a friction between the ball and the semicircular ring. In this section we try to address the changes that may occur one we take into account such a phenomenon. It turns out that a complete treatment is beyond the scope of present report. Static equations analogus to (5) is not enough to capture the complete ground states. We comment more on that at the end of this section. Here, we rather assume that there is an additional force μN along the direction of decreasing θ. Please see figure 4.. Here N is the normal reaction force and μ is the analogue of 'frictional coefficient'. Due to this addition of a friction like term what we find is the following. The analogue of the continuous phase transition of the previous turns into an analogue of first ouder phase transition. In what follows, we will see that across the phase transition point, the order parameter θ changes discontinuously.

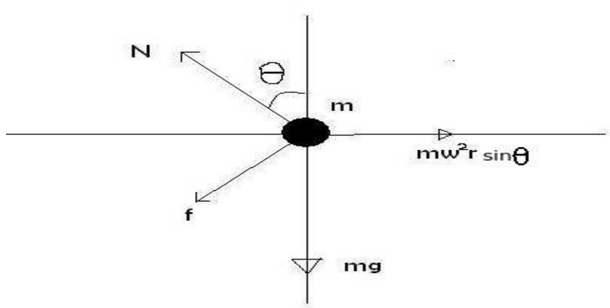

Figure 4. Free body diagram in the presence of static friction

The free body diagram in this case is shown in figure 4. One can immediately write the force balance equations. They are as follows:

$$
\begin{aligned}
&N\cos\theta - \mu N\sin\theta = mg, \\
&N\sin\theta + \mu N\cos\theta = m\omega^2 r\sin\theta. \qquad (9)
\end{aligned}
$$

Above equations can be simplified to a form

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \beta\sin\theta. \tag{10}$$

This can be rewritten as

$$\tan(\theta + \psi) = \beta\sin\theta. \tag{11}$$

Here we have introduced ψ which is related to μ as

$$\mu = \tan\psi. \tag{12}$$

As before, it is perhaps easiest to solve the equation in terms of an effective potential. In the following, we will assume μ is small and expand (11) in ψ. The resulting equation is:

$$\sin\theta + \psi\sec\theta = \beta\sin\theta\cos\theta \tag{13}$$

An effective potential, from which the above equation follows by minimisation, is given by

$$\begin{aligned} V(\theta) &= \int d\theta(\sin\theta + \psi\sec\theta - \beta\sin\theta\cos\theta) \\ &= -\cos\theta - \frac{\beta}{2}\sin^2\theta - 2\psi\tanh^{-1}\Big(\tan(\frac{\theta}{2})\Big). \end{aligned} \tag{14}$$

We notice that (14) reduces to (7) when we set ψ to zero. This is equivalent to setting friction to zero. For small θ, we can expand (14) and get

$$V(\theta) = -1 + \psi\,\theta + \Big(\frac{1}{2} - \frac{\beta}{2}\Big)\,\theta^2 + \frac{\psi}{6}\,\theta^3 + \Big(-\frac{1}{24} + \frac{\beta}{6}\Big)\theta^4 + ... \tag{15}$$

This has a general structure of (1). Comparing (1) and (15), it is now easy to read out the coefficients a_is in (1). We note that besides β, some of the a_is now also depend on ψ. A plot of V for various β and for fixed ψ is shown in the figure 5. As can be seen from the figure, for small β, it has a minimum

at $\theta = 0$ and the ball will stay there. As we increase β, another local minimum develops away from $\theta = 0$. As we increase β further, this minimum becomes the global minimum. If we disturb the ball a little which is sitting at $\theta = 0$, it will reach the global minimum and settle there. Therefore, as we cross a critical value of angular velocity ω, there will be a discontinuous change in the position of the ball. This is analogus to a first order phase transition, where the order parameter changes discontinuously around the transition point. For a water-vapour transition, for example, this order parameter is identified with the density. $V(\theta)$ also reminds us of a magnetisation versus free energy curve. In a magnetic system, when the temperature is below T_c, the magnetisation (M) is non-zero. If a magnetic field is now applied in the direction of M, the magnetisation will increase. However, if the magnetic field is applied in the opposite direction, the magnetisation will discontinuously change from negative to positive (assuming that there is no hystresis). This is an example of first order phase transition. Here, magnetisation is the relevant order parameter for the system. To summarise, due to the inclusion of a friction like term, we see that the order of the transition changes while θ continues to behave as an order parameter.

In our discussion above, we only tried to mimic the effect of static friction by considering a force μN pointing tangentially in the direction of reducing θ. We believe that a complete study of the effect of static friction will be much more complicated and, perhaps, are not captured only by equations analogus to (11). To see this,

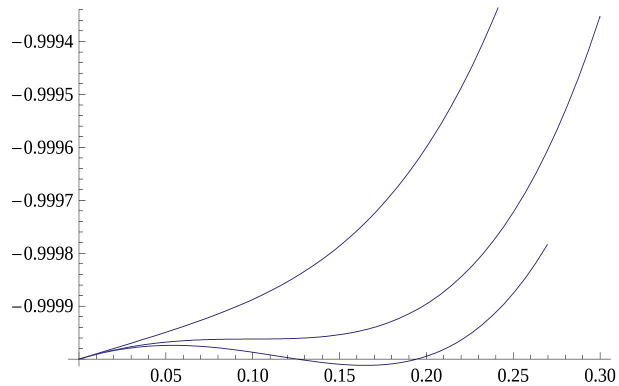

Figure 5. Behaviour of V for various β. We have set $\psi = .001$. The upper curve is for $\beta = 1$, One below is for $\beta = 1.015$ and the lowest one is for $\beta = 1.02$. Clearly, as β increases from 1, θ jumps from zero value to a non-zero value. As the second minima appear at a different value which is not continuously connected to $\theta = 0$, there a discontinuous change in θ around the crossover. This is analogus to a first order phase transition.

let us consider the ball left at a non-zero large value of θ. The ball will then try to come down and the tangential frictional force will be along the direction of increasing θ. Therefore, equation governing its equilibrium position would be different from the one in (11). So, in general, behaviour of the ball would, depend on the boundary conditions. We leave a detail analysis of various possible boundary conditions and a subsequent study of possible equilibrium positions for a future study.

We also notice that instead of working with a semi-circle, we could have studied the simlar problem with a complete circle as well [3]. Conclusions of section 2 would have been similar with only difference being that in this case θ could take positive or negative values. However, introduction of friction complicates the scenario. Various possiblities then come up depending on the initial position and perhaps the initial velocity of the ball as well.

4. DISCUSSION

In this paper, we first analysed a mechanical model closely following the work of [3]. In [3], the author has worked with a rotating circular ring with a ball that is allowed to move along the circumference of the ring. We rather worked with a semicircle and worked out different equilibrium positions of the ball. Calculation is similar to that of [3]. The system provides a very close analogy with continuous phase transition.

Can this model be constructed in our laboratory? We do not see any porblem a priori except that we take care of the effects due to friction. Construction of a semi-circular channel is not difficult. It can also be made to rotate with certain angular velocity via a motor. Current in the motor can be adjusted in order to change the angular velocity. We can then make a ball to move along the channel. The equilibrium positions of the ball (as we change ω) can then be studied. However, what makes thing a bit more complicated is the friction between the ball and the channel surface. Though, friction can be reduced to a low value, it can not be made zero.

Our attempt is to initiate a theoretical study of the system in the presence of static friction. It turned out that the inclusion of friction, in complete generality, is difficult. Different boundary conditions come in to play. We studied in this project only one among such scenarios. We used a force that mimics the frictional force which acts tangentially at the point where the ball is touching the surface of the semicircle. Furthermore, it acts along the direction of decreasing θ. However, let us imagine the following possibility. The ball is dropped near $\theta = \pi/2$.

The ball then will try to fall down. Consequently, the frictional force will act along the direction of increasing θ. The equation of for the equilibrium position will change from (11) to

$$\tan(\theta - \psi) = \beta \sin \theta. \tag{16}$$

We can similarly consider other scenarios. In that sense, in our work we have focussed into one such case.

Interestingly, we have found that even if we include a small friction like term, the qualitative features change considerably. We have seen in the previous section, the model then shows close resemblance with first order phase transition. Real test of our calculation can only be made in the laboratory and we look forward to constructing such a model in the laboratory. We do not see any hindrance a priori.

Acknowledgements: This work was done under supervision of Sudipta Mukherji, Institute of Physics, Bhubaneswar. I thank him and Souvik Banerjee for discussions.

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