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Supercritical Bifurcation: A mechanical model and analogy with phase transition

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Abstract. We consider a simple mechanical model which exhibits supercritical pitchfork bifurcation. We further discuss symmetry preserving and symmetry broken phases of the model. An analytical time dependent solution is then constructed which interpolates between these two phases. We also discuss the similarities of our model with coninuous phase transition. We argue that an analogue of critical exponents in our model can be constructed via a suitable Landau like expansion of the potential.

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1. INTRODUCTION

Bifurcation theory explains many natural phenomena. The purpose of this project is to analyse a simple mechanical model which exhibits supercritical bifurcation. It also shares some features of symmetry breaking and continuous phase transition.

The phenomena of bifurcation has one common cause: a specific physical parameter crosses a threshold and consequently it forces the system to organize itself to a new state. This specific state differs significantly from the original one. The states of a system generally correspond to the solutions of a nonlinear equation. A state can be observed if it is stable. However, if that state loses its stability when a parameter reaches a critical value, then the state is not observed. The system then generally organizes itself to a new stable state – causing a bifurcation from the original one.

A simple example of bifurcation is known as pitchfork bifurcation. Here the solution of the nonlinear equation bifurcates in pairs and generally the bifurcating state has less symmetry than the original one – often called a symmetry broken state. The simplest of such example is described by the solutions of the following equation:

$$x^3 - \lambda x = 0, \tag{1}$$

where λ is a parameter and x is real. Fo $\lambda \leq 0$, there is only one solution $x = 0$. However, for $\lambda > 0$, two new solutions appear at $x = \pm\sqrt{\lambda}$. It is possible to construct a 'potential' whose extrimization

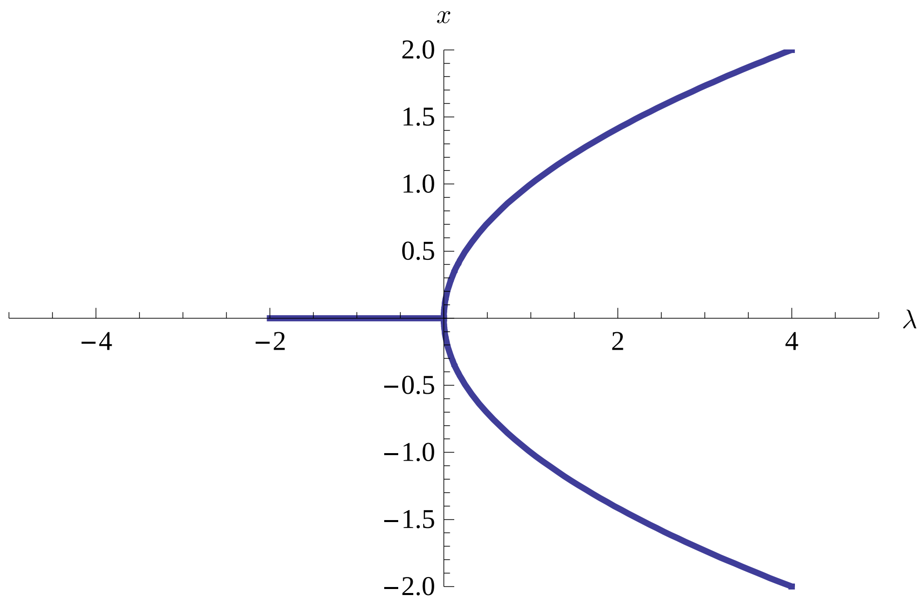


Figure 1. Solutions bifurcate in pairs when λ becomes positive

gives (1)

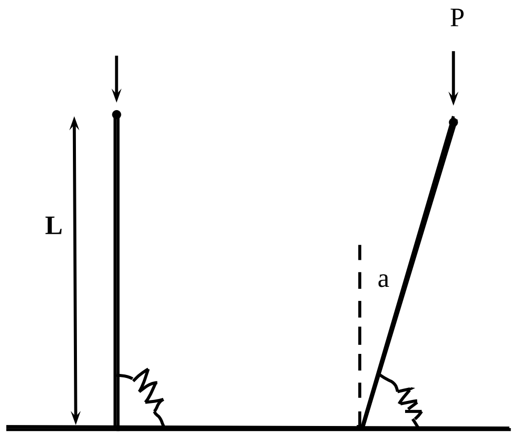
$$V(x, \lambda) = \int (x^3 - \lambda x) dx = \frac{1}{4}(x^4 - 2\lambda x^2). \tag{2}$$

Note that V is symmetric under $x \to -x$. From the structure of the potential, one immediately realises that for $\lambda < 0$, V minimizes at $x = 0$. We call this a symmetric phase. However, this ground state becomes unstable when $\lambda > 0$ and two new minima appear at non-zero values of x symmetrically around $x = 0$ line. As soon as the system reaches one of this states, the $x \to -x$ symmetry gets broken. We call this a symmetry broken phase.

Our purpose is now to introduce a mechanical model which shows such bifurcation. This is what we do in the next section. Subsequently, we study time dependent solutions associated with the crossover from symmetric phase to the symmetry broken phase. We then study as to how our model brings out a simple analogy with second order phase transition. Second order phase transition appears in many thermodynamic systems - ferromagnetic material losing its magnetization with the increase of temperatue is one such example. We end our project with a discussion of our results.

2. THE MODEL

The system is an inverted pendulum where a rigid rod of negligible mass is pivoted about its lower end with a torsion spring [1]. This spring provides the restoring torque proportional to the angular displacement from the equilibrium [1]. See the figure for the details of the system.



A load of weight P is applied vertically on the top of the rod. The equation of motion that this rod of length L has to satisfy can be found from torque ballance equation. This is given by

$$I\frac{d^2a}{dt^2} = -\kappa a + PL\mathrm{sin}a, \tag{3}$$

where I is the moment of inertia of the rod about the axis of rotation and κ is the spring constant. The stable position, the right hand side of the above equation has to be zero. Hence the positions can be found from

$$\kappa a - PL\mathrm{sin}a = 0, \tag{4}$$

or equivalently,

$$\frac{\kappa}{PL}a - \mathrm{sin}a = 0. \tag{5}$$

We define $\frac{\kappa}{PL} = \tilde{\kappa}$. It can be shown that, besides $a = 0$, this equation always has a pair of non-zero solution for a when $\tilde{\kappa} < 1$. However, for $\tilde{\kappa} > 1$, we have $a = 0$ as the only solution. Note that $\tilde{\kappa}$ can be reduced by simply increasing the load P. Let us increase the load P such that it is just above $P_c = \kappa/L$. Now for this, since $\tilde{\kappa}$ is only a little less than one, we expect that non-trivial solutions of a will be close to zero. Hence the term $\mathrm{sin}a$ can be expanded in powers of a. This leads to

$$-(1-\tilde{\kappa})a + \frac{a^3}{6} = 0, \tag{6}$$

where we have neglected the heigher powers of a. This leads to the solution $a_\pm = \pm\sqrt{6(1-\tilde{\kappa})}$ besides the trivial one $a = 0$. We now note the similarity between equation (1) and (6). The role

[1] There are many other mechanical models which show similar bifurcations, see for example [2, 3].

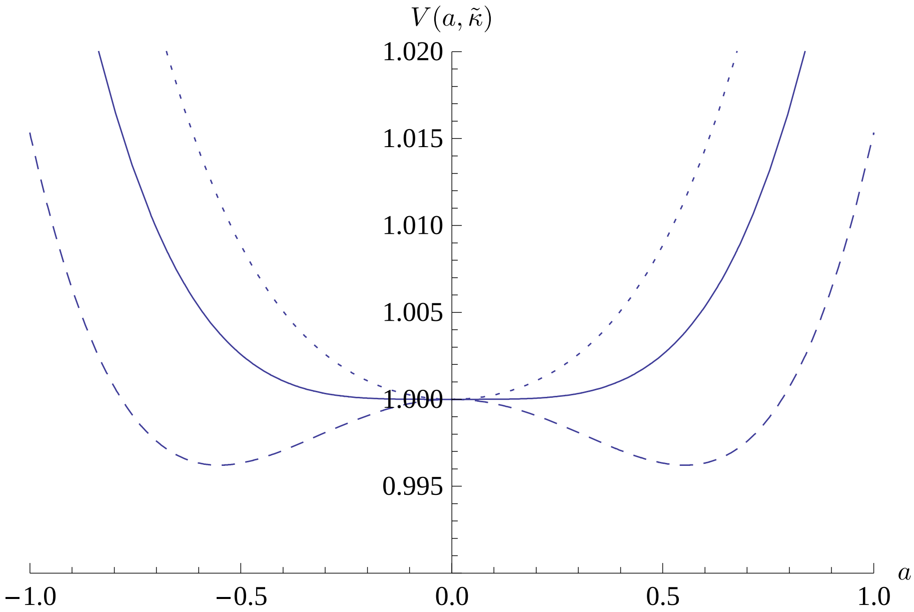


Figure 2. $V(a, \tilde{\kappa})$ for differnt values of $\tilde{\kappa}$. While the solid line is for the critical values of $\tilde{\kappa}$, namely $\tilde{\kappa} = 1$, representing $P = P_c$, the dotted and dashed curves are for $\tilde{\kappa} = 1.05$ and $.95$ respectively. For $\tilde{\kappa} > 1$, $a = 0$ is the minimum of the potential, This reperesents that the inverted pendulum is sitting staright making an angle zero with the verical axis. $\tilde{\kappa} > 1$, minima are at $a = a_{\pm}$. The pendulum settles at one of these angles, breaking $a \to -a$ symmetry. We refer to this as a symmetry broken phase.

of λ is played by $6(1 - \tilde{\kappa})$ and the role of x is played by a. Consequently, our model is a case of supercritical bifurcation. One can construct a potential whose extremaization gives (5). This is given by

$$V(a, \tilde{\kappa}) = \int da(\tilde{\kappa}a - \sin a) = \frac{\tilde{\kappa}a^2}{2} + \cos a. \tag{7}$$

The plot of the potential for different values of $\tilde{\kappa}$ is shown in figure (2). The stability of the system is decided from the minima of the potential and has been discussed in the caption of figure (2). In the next section we discuss some dynamical issues associated with the model.

3. TIME DEPENDENT INTERPOLATING SOLUTION

As we discussed in the previous section, for $\tilde{\kappa} > 1$ or in other words for $P < P_c$, the stable position is $a = 0$ and for $\tilde{\kappa} < 1$ or for $P > P_c$, $a = 0$ is a maximum and system settles down to either at a_+ or at a_-. The question that we ask in this section is the following: Suppose we suddenly reduce the load from $P > P_c$ to just below P_c, how the system rolls down from the symmetry preserving phase to the symmetry broken one? To analyze this issue in some detail, we go back to equation (5). First we note that this equation can be re-written as

$$\frac{d}{dt}\Big(I(\frac{da}{dt})^2 + \kappa a^2 + \frac{2PL}{I}\text{cos}a\Big) = 0, \tag{8}$$

Note that the experssion inside the big brackets is the total energy of the system. Hence (8) is a statement of energy conservation. A simple integration therfoe gives

$$I(\frac{da}{dt})^2 + \kappa a^2 + \frac{2PL}{I}\text{cos}a = C, \tag{9}$$

where C is a constant. This constants can be fixed by using the boundary condition that when $a = 0$, $\dot{a} = 0$. This leads to

$$C = \frac{2PL}{I}. \tag{10}$$

Substituting this back to (9) and integrating once more, we get

$$\int dt = \pm\sqrt{\frac{I}{PL}}\int \frac{da}{\sqrt{4\text{sin}^2\frac{a}{2} - \tilde{\kappa}a^2}}. \tag{11}$$

It turns out that this integral can only be performed exactly small a. This is only a good approximation when $\tilde{\kappa}$ is close to one, or equivalently P close to P_c. Keeping upto quartic terms in a, we get,

$$\int dt = \pm\sqrt{\frac{I}{PL}}\int \frac{da}{\sqrt{(1-\tilde{\kappa})a^2 - \frac{a^4}{12}}}. \tag{12}$$

This equation can be integrated easily with the result

$$a(\tilde{t}, \tilde{\kappa}) = a_{\pm}e^{\mp\sqrt{1-\tilde{\kappa}}(\tilde{t}-\tilde{t}_0)}. \tag{13}$$

In the above equation, we have defined $\tilde{t} = \sqrt{\frac{PL}{I}}t$, $\tilde{t}_0$ is an integration constant. In writing down the solution, we also used the fact that at $\tilde{t} = \tilde{t}_0$ $a(\tilde{t}) = a_{\pm}$. Note that above solution is real only for $\tilde{\kappa} < 1$. This indicates the fact that rolling down solution does not exist for $\tilde{\kappa} > 1$. We also note that the constant $\tilde{t}_0$ appears due to the time translational invariance of the equation (5). The interpolating solution is plotted in figure (3).

Having analysed the time dependent solution, we now turn our attention to some equilibrium properties of our model. In particular, we would be interested in bringing out some analogies between continuous phase trantion and our model of inverted pendulum.

In the next section, we discuss various static properties of our model and we try argue that our model serves as a crude analogy with second order phase transition.

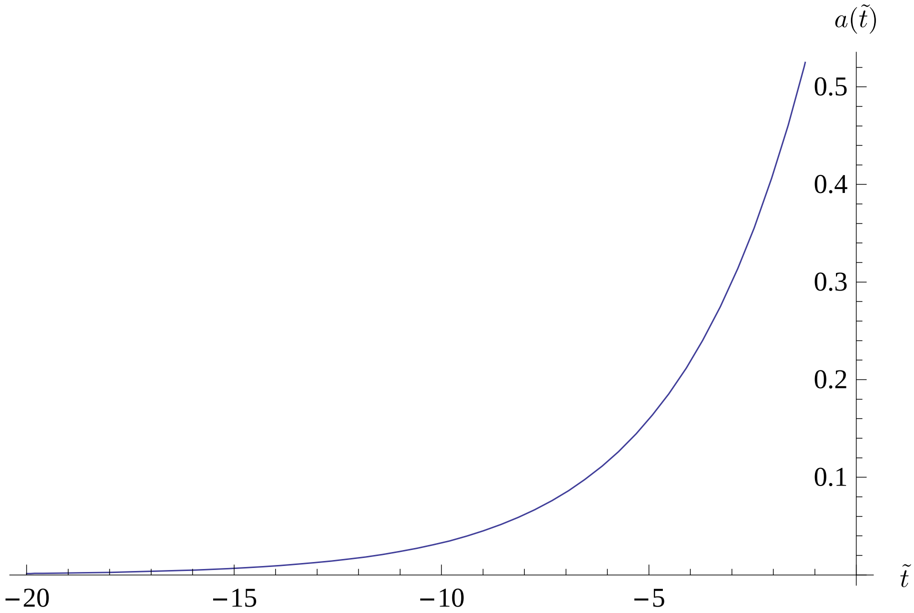


Figure 3. The interpolating solution $a(\tilde{t}, \tilde{\kappa}) = a_+ e^{\sqrt{1-\tilde{\kappa}}(\tilde{t}-\tilde{t}_0)}$ for $\tilde{\kappa} = .9$ and $\tilde{t}_0 = 0$. At very early time the system is at $a = 0$, while at $\tilde{t} = 0$, it reaches a finite value for a.

4. ANALOGY WITH CONTINUOUS PHASE TRANSITION

Let us note that for very small a (that is for P close to P_c), we can expand the potential in (7) as

$$V(a) = \frac{\kappa}{2}a^2 + PL\Big(1 - \frac{a^2}{2} + \frac{a^4}{24}\Big) + ... \tag{14}$$

or as

$$\tilde{V}(a, \tilde{\kappa}) = \frac{V - PL}{PL} = \frac{a^2}{2}(\tilde{\kappa} - 1) + \frac{a^4}{24} + ... \tag{15}$$

Depending on the value of $\tilde{\kappa}$, it shows three distinct behaviours similar as in figure (2). The occurance of only even powers is a consequence of $a \to -a$ symmetry of the potential. We would like to compare (15) with the expansion of free energy function in terms of order parameter near the vecinity of second order phase transition within the framework of the Landau theory of phase transition [4]. In Landau theory, the phase of a system is characterized by an order parameter. This is a measurable quantity. It is generally zero in the disordered or high temperature phase and acquires a non-zero value in the ordered or the low temperature phase. A common example is the ferromanget. In the absence of any external field, the magnetization of a ferromagnet is zero above a critical temperature T_c. However, for $T < T_c$, its magnetization is non-zero. Therefore, here, magnetization is generally used as the order parameter which distiguishes the high and low temperature phases of a ferromagnet.

When a parameter of the system changes, a system may go through a phase transition. For the case just discussed, this parameter is the temperature. During a phase transition process, order parameter changes either continuously or abruptly. While for a first order transition, the change of the order parameter is disconinuous, for a second order transition, it changes smoothly around T_c. Ferromagnetic transition is an example of a second order phase transition.

To describe a second order phase transition, Landau constructed a free energy function of a system near its critical point. This function, for a ferromagnetic system, has a general form

$$F(M,T) = a_1(T - T_c)M^2 + a_4M^4 + ..., \tag{16}$$

where, M, the oder parameter, is the avarage magnetization. We note that since $M \to -M$ is expected to be a symmetry of the system, all odd powers of M are absent. Secondly, considering the system close to the critical temperature, we neglect higher powers of M. The equilibrium state of the system is determined by the extimization condition,

$$\frac{\partial F}{\partial M} = 0. \tag{17}$$

Note that the above equation tells us that for $T > T_c$, the only real solution for M is $M = 0$. However, for $T < T_c$ two more real solution develops:

$$M = \pm\sqrt{\frac{a_1(T_c - T)}{2a_4}}, \tag{18}$$

besides $M = 0$. Taking a second derivative on F, one realizes that (18) represents the stable points, while the solution $M = 0$ becomes unstable. From here we see that that the order parameter becomes nonzero and grows at $(T_c - T)^{1/2}$ for temperature below T_c. This leads to a critical exponent β, characterising the phase transition, defined as

$$M \sim (T_c - T)^{\beta}. \tag{19}$$

We get $\beta = 1/2$ for the system in consideration. Similarly, we can define susceptibility as

$$\chi^{-1} = \left(\frac{\partial^2 F}{\partial M^2}\right)_{T,M\to 0}. \tag{20}$$

Taking second derivative of (16) with respect to M, keeping T constant and $T > T_c$, we get

$$\chi^{-1} \sim (T - T_c) \tag{21}$$

and for $T < T_c$,

$$\chi^{-1} \sim (T_c - T). \tag{22}$$

These lead to two other critical exponents

$$\gamma = \gamma' = 1. \tag{23}$$

Now we turn our attention to the model we are discussing. We notice the following similarities with ferromagnets.

• F is similar to V in (15).

• Like M in ferromagnet, a behaves as order parameter in our model.

• The load P behaves as temperature which we tune externally.

• The critical temperature T_c is similar to the critical load P_c.

• It can easily be checked that the order parameter in our case behaves near the critical point as

$$a \sim (P - P_c)^{\frac{1}{2}}, \tag{24}$$

giving $\beta = 1/2$.
• Likewise the susceptibility, defined as

$$\chi^{-1} = \Big(\frac{\partial^2 V}{\partial a^2}\Big)_{T, a \to 0}. \tag{25}$$

leads to

$$\chi^{-1} \sim (P - P_c), \quad \text{for} \;\; P > P_c, \tag{26}$$

and

$$\chi^{-1} \sim (P_c - P), \quad \text{for} \;\; P < P_c. \tag{27}$$

Hence, we get

$$\gamma = \gamma' = 1. \tag{28}$$

Though there are quite a few similarities between our model and ferromagnets, there are many differences also. To mention a few, we note that unlike our case, ferromagnetic transition is temperature driven. Morover, M is a local order parameter for ferromagnets. This means that M may change from point to point inside a ferromagnetic material. For our case, a is only a global parameter.

5. CONCLUSION

In this work, we discussed a simple model showing supercritical bifurcation. We constructed a time dependent solution interpolating the symmetric phase and the symmetry broken phase. We also tried to bring out some similarities between our model and ferromagnetic material near its critical temperature using Landau theory.

It would be interesting to set up an experiment to test our results. However, before we do so, the model has to be generalised to take care of other effects. Firstly, we have to do the analysis including the weight of the rod. Secondly, we need to include friction into the system by introducing a term $\beta\dot{a}$ into the equation (5). We hope to look into these areas in the future.

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