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Interpolating solution in a mechanical model under quench

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Abstract. We review a mechanical model which has qualitative similarities with ferromagnetic material in a magnetic field. We then study certain time dependent classical solutions of this model when it is appropriately quenched.

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1. INTRODUCTION

Simple mechanical models often provide insights into complicated physical processes in nature. Consider, for example, phase transition of ferromagnets above the Curie temperature. Below the Curie temperature, within the material, neighbouring magnetic spins are aligned parallel. As we increase the temperature towards the Curie point, the alignment (magnetization) within each domain decreases. Above the Curie temperature, the local magnetic dipoles are randomly oriented and therefore the material behaves as a paramagnet. Theoretical understanding of this phenomenon, including the system's behaviour at the Curie point requires the use of sophisticated techniques of field theory and renormalization group. One may inquire if there exists simple models which capture, at-least qualitatively, some essential features of this transition.

Indeed in [1], such a model was analyzed. It consists of a bead of mass m moving freely along a vertical loop. The loop is then made to rotate about a vertical axis passing through its center. It can be shown that if the loop rotates with a very small angular velocity ω, the bead stays at the bottom of the loop. However, as the angular velocity is increased, beyond a critical velocity ω_c, minimization of the potential energy requires the bead to sit at a non-zero θ (θ is shown in the figure). As we further increase ω, θ increases, reaching $\pi/2$ with $\omega \to \infty$. Note that the symmetry $\theta \to -\theta$, which was present initially, is spontaneously broken for $\omega > \omega_c$ by the equilibrium position of the bead. Similarities with ferromagnetic transition is now immediate. While the role of the temperature is played by the angular velocity ω, the position of the bead θ behaves similar to the order parameter, magnetization. Hence, the paramagnetic phase is analogous to the $\omega < \omega_c$ phase of the model. In literature this phenomena is known as a bifurcation. When a specific physical parameter crosses a threshold value, the system generally organizes itself to a new stable state causing a bifurcation from the original one.

What happens in ferromagnetic material if we quench the temperature from a low value to a value above the Curie temperature? Since temperature is tuned very fast, immediately after the quench, the system will still be in its unstable ferromagnetic state. However, slowly with time, the system will roll down to the stable paramagnetic state. We can ask similar question within the model we are discussing. Suppose we quench the angular velocity from a very low value to a higher one ($> \omega_c$), we should be able to find a time-dependent rolling down solution which will interpolate between $\theta = 0$ and a θ non-zero value. Indeed in [2], such a

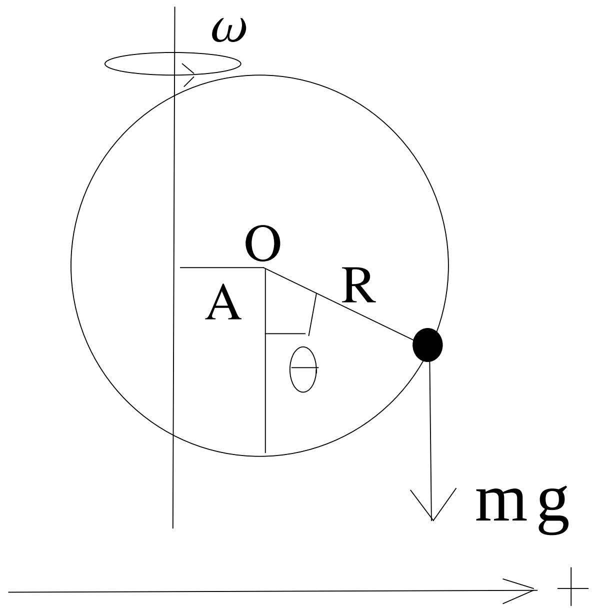

Figure 1. A vertical loop carrying a movable friction-lees bead is rotating about an off-center vertical axis, at a distance A from the center, with a constant frequency ω. The positive values of A and θ are shown by the horizontal axis at the bottom.

solution was explicitly constructed.

In this paper, we discuss the same model when it is rotated about a vertical axis which *does not* pass through the center. This is explicitly shown in figure (1). As analyzed in [1], this model depicts some features of ferromagnetic material in an external magnetic field. Here, for the ferromagnet, the rotational symmetry is broken by external field itself. Similarly, by choosing off-center axis of rotation, we break the $\theta \to -\theta$ symmetry in our model right from the beginning. We will describe the model in brief in the next section. Our primary aim of this work is to construct explicit rolling down solution as we suddenly shift the axis of rotation of the loop parallely. This is what we discuss in the third section. The last section summarizes the results.

2. THE LAGRANGIAN AND THE EQUATION OF MOTION

As discussed in [1], the model has an effective Lagrangian description. Let us assume that at any instant of time the mass is at a position $\theta(t)$ The Lagrangian then reads [1]

$$L = \text{kinetic energy} - \text{potential energy}. \tag{1}$$

While the kinetic energy is given by

$$KE = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(R\sin^2\theta + A)^2, \tag{2}$$

the potential energy is

$$PE = -mgR\cos\theta. \tag{3}$$

Therefore the total Lagrangian is

$$L = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\omega^2(R\sin^2\theta + A)^2 + mgR\cos\theta. \tag{4}$$

This allows us to have a description of the system in terms of an *effective* potential

$$V' = -\frac{1}{2}m\omega^2(R\sin^2\theta + A)^2 - mgR\cos\theta. \tag{5}$$

Or, in other words, we can study the equilibrium position of the bead by analyzing the minima of the potential

$$V = \frac{V'}{mgR} = -\cos\theta - \frac{1}{2}(\sin\theta + \alpha)^2, \tag{6}$$

where we have defined

$$\alpha = \frac{A}{R}, \quad \text{and} \;\; \beta = \frac{\omega^2 R}{g}. \tag{7}$$

Note that, because of the presence of α, the potential does not have a $\theta \to -\theta$ symmetry.

In the following, we will study the effective potential in the range $\beta > 1$ and for all positive α. Notice that, for $\alpha = 0$, it has a maximum at $\theta = 0$ with two symmetric minima at

$$\theta_0 = \pm\cos^{-1}(1/\beta). \tag{8}$$

Let the bead be in one of the degenerate minima. We choose the negative one. Now we increase α. This means that, in figure (1), we move the axis of rotation to the right. For very large α, we can neglect the $\sin\theta$ term in the potential. It then easily follows that there is only one minimum at approximately

$$\theta = \tan^{-1}(\beta\alpha) \tag{9}$$

So at this high α, the bead must be resting at a positive θ value. To find at what value of α, the transition from negative to positive θ occurred, it is instructive to search if there is an inflection point associated with the effective potential. Indeed there is one and that can be found by setting first and second θ derivative to zero. It occurs at

$$\alpha = \alpha_c = (1 - \beta^{-2/3})^{3/2}, \;\; \cos\theta_c = \frac{1}{\beta^{1/3}} \tag{10}$$

The dependence of equilibrium angle on alpha is shown in Fig (refdiagram1) by a bifurcation diagram [4]. Further, the behaviour of the potential is shown in figure (3).

Now we would like to address the following question. Suppose, for a fixed β, we suddenly change α from a value less then α_c to a value greater than α_c, how is the bead going to relax from a wrong ground state (at negative θ) to the right one (in positive θ)? To address this question, we need to find out a rolling down solution of θ as a function of *time*. We address this issue in the next section.

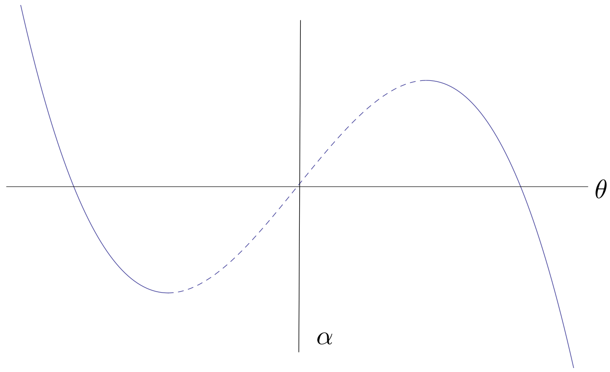

Figure 2. Bifurcation Diagram for the model. The dotted lines show the points of unstable equilibrium; whereas the solid lines represent points of stable equilibrium. As soon as α is slightly decreased from the maximum (call it α_c) – which is the transition point with saddle-point bifurcation –, the particle sitting on the left false vacua, should slip down to the true vacuum at the right.

3. INTERPOLATING SOLUTION

We start with by writing down the Euler-Lagrange equation that follows from (4). This is given by

$$\frac{d^2\theta}{dt^2} - \omega^2 \sin\theta \cos\theta - \frac{\omega^2 A}{R} \cos\theta + \frac{g}{R} \sin\theta = 0 \tag{11}$$

By defining $\tilde{t} = \omega t$, we can re-write the equation as

$$\frac{d^2\theta}{d\tilde{t}^2} - \sin\theta \cos\theta - \alpha \cos\theta + \frac{1}{\beta} \sin\theta = 0. \tag{12}$$

Integrating this equation once, we get

$$\frac{1}{2}\left(\frac{d\theta}{d\tilde{t}}\right)^2 + \frac{1}{4} \cos 2\theta - \alpha \sin\theta - \frac{1}{\beta} \cos\theta = c, \tag{13}$$

where, c is an integration constant to be determined by the boundary condition. Note that the above equation is just a statement of energy conservation.

To this end, let us consider the following situation. Suppose we start with $\alpha = 0$ and $\beta > 1$. The particle is sitting in one of the degenerate minima given by

$$\theta_+ = \cos^{-1}\left(\frac{1}{\beta}\right), \text{ or } \theta_- = 2\pi - \cos^{-1}\left(\frac{1}{\beta}\right). \tag{14}$$

Let us take the second one. Now we suddenly increase α to a value greater than α_c. Since θ_- is no longer a minimum of the effective potential, particle is expected to roll down from its unstable position with zero initial velocity. This condition allow us to fix the constant appearing in (13).

$$\frac{d\theta}{d\tilde{t}} = 0, \quad \text{at } \theta = \theta_-, \tag{15}$$

With this value of the constant, one can search for a time dependent solution for θ simply by integrating (13). This exercise can be performed exactly, but the solution is a bit messy. We rather illustrate here with specific

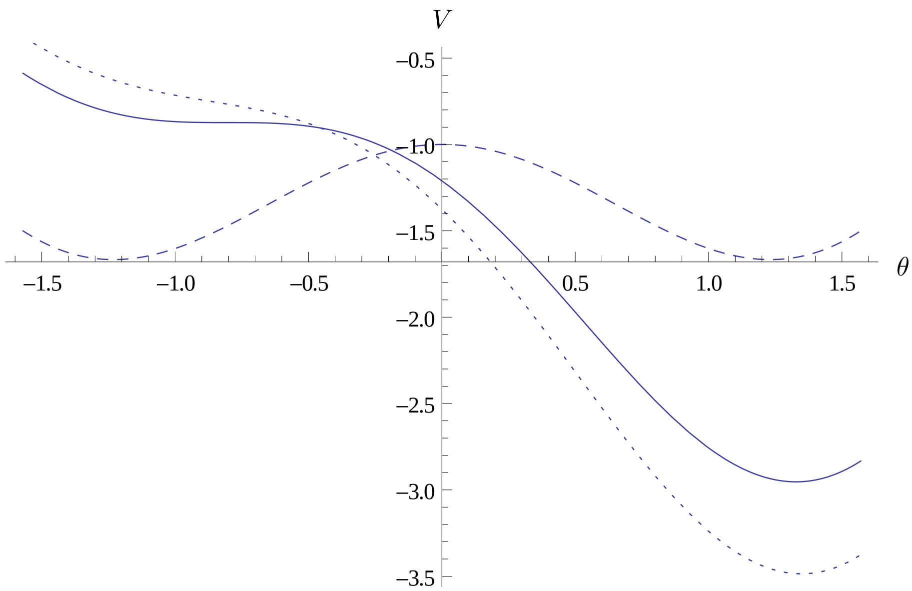

Figure 3. Effective potential for fixed $\beta = 3$ and for various α. For $\alpha = 0$ (dashed line), there are degenerate minima. The solid line is for $\alpha = \alpha_c = (1 - \beta^{-2/3})^{3/2} = .374$ which shows the inflection point arising from the left minimum. For $\alpha = .5$, single minimum is shown by the dotted line.

values of β and α. Let us choose $\beta = 3$. Using (10), we get $\alpha_c = .3742$. We therefore take $\alpha = .375$. With this the boundary condition can be solved to get $c = 0.0472$. Now the equation (13) can be re-written as

$$\int \frac{d\theta}{\sqrt{2\alpha\,\sin\theta + 2/\beta\,\cos\theta - \ \cos 2\theta/2 + 2 \times .0472}} = \pm \int d\tilde{t}. \tag{16}$$

The integral on the left hand side of (16) can be solved. However, the result is not very illuminating. We instead represent the solution graphically. This is shown in figure (4).

4. SUMMARY

To conclude, in this paper, we reviewed a toy model which captures certain qualitative behaviour of a ferromagnet as we tune its temperature in the presence of an external magnetic filed. We then constructed a time-dependent classical solution representing its relaxation from false to a true ground state after the model is appropriately quenched.

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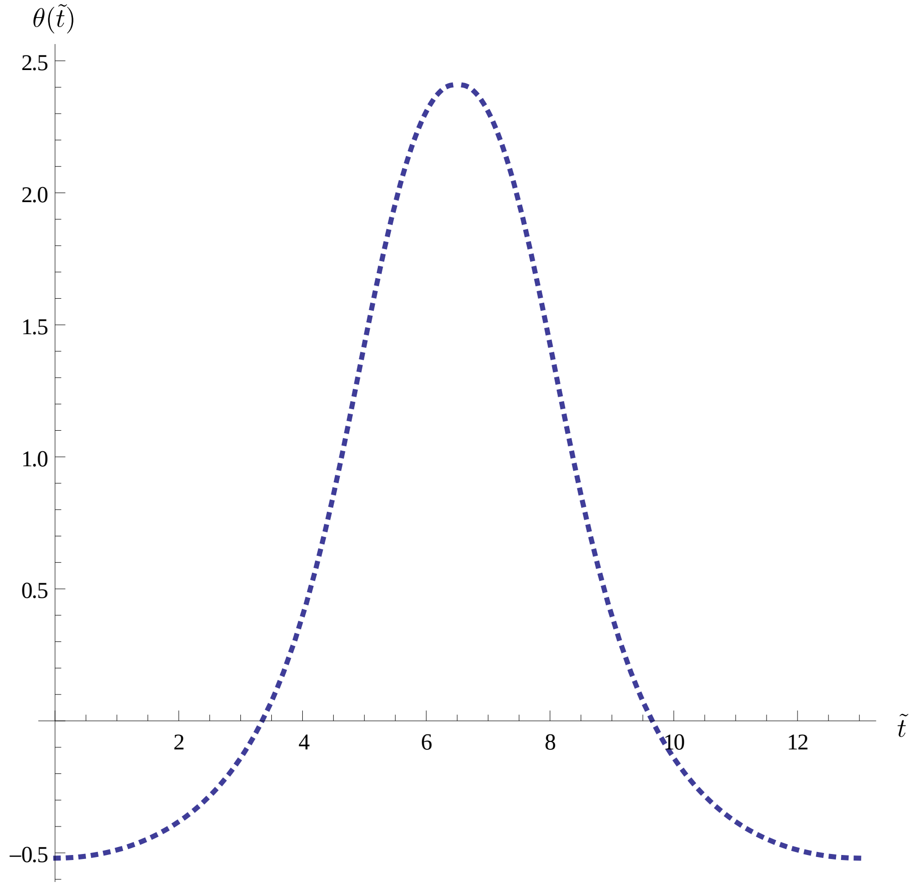

Figure 4. Behaviour of $\theta(\tilde{t})$ with $\tilde{t}$ for $\alpha = .375$ where $\alpha_c = .3742$. We have set $\beta = 3$. The figure, which would correspond to a bounced like solution in Euclidean time, is seen as an interpolating one in real time.

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