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Massive Scalar Field in Anti-deSitter Space: a Superpotential Approach

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Abstract. In this paper, we compute the ground state wave function and the ground state energy of a spin zero massive particle in Anti de Sitter space by employing supersymmetric quantum mechanics. Though the results were known earlier, our method is new.

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1. INTRODUCTION

Anti de Sitter space-time is a curved space-time with a constant negative curvature. It is a solution of Einstein equation in the presence of a negative cosmological constant. Many researchers are recently interested in this space-time due to Maldacena's conjecture of AdS/CFT correspondence (see [1]). According to this conjecture, gravitational theory in five dimensional Anti de Sitter space is dual to a gauge theory on its boundary. In this paper, we study propagation of a free spin zero massive particle in this space time. It satisfies Klein-Grodon equation. Though this problem was analyzed earlier [2], we use supersymmetric quantum mechanics to find the ground state wave function and ground state energy.

We start by reviewing the techniques of supersymmetric quantum mechanics. In section 3, we introduce d dimensional Anti de Sitter space (AdS). As mentioned before, AdS is a space-time with constant negative curvature. In section 4, we discuss how SUSY can be used to compute ground state wave function and ground state energy of a massive spin zero particle in AdS space. We hope to report on the complete spectrum and energy eigenvalues in the future.

2. HAMILTONIAN FORMULATION OF SUSY QUANTUM MECHANICS

In this section, we briefly introduce SUSY quantum mechanics. We start by considering a single particle of mass m, moving in a potential $V_1(x)$. As usually done in SUSY quantum mechanics, choose the ground state energy to be zero (by subtracting a constant). We further denote the ground state wave function as $\psi_0(x)$. The Schrodinger equation can be written as

$$H_1\psi_0(x) = E_0\psi_0(x), \tag{1}$$

and

$$H_1\psi_0(x) = \frac{-\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2} + V_1(x)\psi_0(x) = 0. \tag{2}$$

From this we get

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0''(x)}{\psi_0(x)}. \tag{3}$$

This allows us a global construction of the potential $V_1(x)$ from the knowledge of its ground state wave function. Note now that it is very simple to factorize the hamiltonian. The hamiltonian can be written as

$$H_1 = A^\dagger A, \tag{4}$$

where

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \tag{5}$$

and

$$A^\dagger = \frac{-\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x). \tag{6}$$

Then

$$H_1\psi(x) = A^\dagger A\psi(x), \tag{7}$$

or

$$H_1 = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x). \tag{8}$$

Comparing equation (2) and equation (8) we obtain

$$V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'^{(x)}. \tag{9}$$

This equation is the well known Riccati equation. The quantity $W(x)$ is generally referred to as the 'superpotential' in SUSY quantum mechanics. The solution for $W(x)$ in terms of the ground state wave function is obatained by recognising that A satisfies

$$A\psi_0(x) = 0. \tag{10}$$

This follows from the fact that the ground state energy is zero. The above equation can be written as,

$$\frac{\hbar}{\sqrt{2m}}\frac{d\psi_0(x)}{dx} + W(x)\psi_0(x) = 0. \tag{11}$$

Hence

$$W(x) = \frac{-\hbar}{\sqrt{2m}} \quad \frac{\psi_0'(x)}{\psi_0(x)}. \tag{12}$$

From which it follows that

$$\psi_0(x) = Ne^{-\int^x W(y)dy}, \tag{13}$$

where N is the normalisation constant.

After this brief account of SUSY quantum mechanics, in the next section, we introduce anti de sitter space and in section-4, we use our knowledge of SUSY quantum mechanics to construct ground state wave function and ground state energy of a massive particle in AdS space.

3. ANTI DE SITTER SPACE

A $(d+1)$-dimensional AdS space is defined as the space of constant negetive curvature which can be embedded in a $(d+2)$ dimensional Minkowski space in the following manner:

$$-(X^0)^2 - (X^{(d+1)})^2 + \sum_{i=1}^{d}(X^i)^2 = -R^2, \tag{14}$$

where R is releted to the curvature of the space. This is a space-time which satisfy Einstein equation in the presence a negative cosmological constant. The value of this constant is related to R in the above equation. The metric of $d+2$ dimensional Minkowski's space is

$$(ds)^2 = -(dX^0)^2 - \left(dX^{(d+1)}\right)^2 + \sum_{i=1}^{d}\left(dX^i\right)^2. \tag{15}$$

We parametrise the X^μ's in (15) in the following manner:

$$\begin{aligned} X^0 &= R\cosh\rho\cos\tau \\ X^{d+1} &= R\cosh\rho\sin\tau \\ X^i &= R\sinh\rho\,\Omega_i \end{aligned} \tag{16}$$

$\Omega_i{}'s$ are the spherical coordinates on a $(d-1)$ dimensional unit sphere. It satisfies

$$\sum_i \Omega_i^2 = 1. \tag{17}$$

From it, by differention, we get

$$\sum_i \Omega_i d\Omega_i = 0. \tag{18}$$

Note that with this parametrization, the relation (14) is automatically satisfied.

Substituting (16) into (15), we get the metric on AdS_{d+1} as

$$ds^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_i^2). \tag{19}$$

Note that τ here is periodic with period $0 \le \tau \le 2\pi$. However, we will use the universal covering space of AdS and we will take $-\infty \le \tau \le \infty$. Furthermore, $0 \le \rho$ in (19). To study the structure of AdS_{d+1}, it is convenient to introduce another coordinate θ which is related to ρ

$$\tan\theta = \sinh\rho. \tag{20}$$

Here $0 \le \theta \le \pi/2$. The metric then takes the form

$$ds^2 = \frac{R^2}{\cos^2\theta}\Big(- d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2\Big). \tag{21}$$

Hence the metric tensor $g^{\mu\nu}$ can be written as

$$g^{\mu\nu} = \begin{pmatrix} \frac{R^2}{\cos^2\theta} & 0 & 0 \\ 0 & \frac{R^2}{\cos^2\theta} & 0 \\ 0 & 0 & R^2\tan^2\theta\nu_{ij} \end{pmatrix}, \tag{22}$$

where ν_{ij} is the metric on unit $d-1$ dimensional sphere. The exact metric components of ν_{ij} can be found in books and will not be required for us. In the next section, we study a massive particle moving in AdS space.

4. MASSIVE PARTICLE IN ADS SPACE

In curved space a spin zero particle of rest mass M satisfies Klein-Gordon equation given by

$$(\Box - M^2)\psi = 0, \tag{23}$$

where, $\Box$ is laplacian in curved space and is defined as

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\partial^\mu\Big). \tag{24}$$

Here, g is the determinant of the metric $g^{\mu\nu}$ In writing down (23), we have set $\hbar = c = 1$.

For the AdS metric given in the previous section,

$$\sqrt{-g} = R^{(d+1)}\sec^2\theta\tan^{(d-1)}\theta\sqrt{det\nu_i j}. \tag{25}$$

Using (25), $\Box$ can be written as

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu)= -\frac{1}{R^2}\cos^2\theta\partial_\tau\partial_\tau + \frac{d-1}{R^2}\cot\theta\partial$$
$$+\frac{1}{R^2}\cos^2\theta\partial_\theta\partial_\theta + \frac{1}{R^2}\cot^2\theta\frac{1}{\sqrt{|\nu_{ij}|}}\partial_i(\sqrt{|\nu_{ij}|}\partial_i). \quad (26)$$

Re-writing differently, we have,

$$R^2\Box = -\cos^2\theta\partial_\tau\partial_\tau + (\mathrm{d}-1)\cot\theta\partial_\theta + \cos^2\theta\partial_\theta\partial_\theta + \cot^2\theta\nabla^2, \quad (27)$$

where we have defined

$$\frac{1}{\sqrt{|\nu_{ij}|}}\partial_i(\sqrt{|\nu_{ij}|}\partial_i) = \nabla^2_{(d-1)}. \quad (28)$$

Hence in AdS space, the Klein-Gordon equation can be written as

$$\Big(- \cos^2\theta\partial_\tau\partial_\tau+(d-1)\cot\theta\partial_\theta + \cos^2\theta\partial_\theta\partial_\theta$$
$$+\cot^2\theta\nabla^2_{(\mathrm{d}-1)}\Big)\psi - \tilde{\mathrm{M}}^2\psi = 0, \quad (29)$$

where $\tilde{M} = MR$. Let the trial wave function be

$$\psi = \exp\left(-i\omega\tau\right)Y_{l,m}(\Omega_{(d-1)})\chi(\theta). \quad (30)$$

Then the above equation becomes:

$$\Big(\omega^2\cos^2\theta + (d-1)\cot\theta\partial_\theta + \cos^2\theta\partial_\theta\partial_\theta - l(l+d-2)\cot^2\theta - \tilde{M}^2\Big)\chi(\theta) = 0. \quad (31)$$

where we have used

$$\nabla^2 Y_{l,m}(\Omega_{d-1}) = -l(l+d-2)Y_{l,m}(\Omega_{d-1}). \quad (32)$$

Let us now define $P(\theta)$ such that

$$\chi(\theta) = \tan\theta^{\frac{(1-d)}{2}}P(\theta). \quad (33)$$

Hence the above equation becomes

$$P''(\theta) + \Big((1-d-\tilde{M}^2)\sec^2\theta$$
$$-l(l+d-1)\mathrm{cosec}^2\theta + \omega^2$$
$$+\frac{(1-d)(d-3)}{4}\sec^2\theta\mathrm{cosec}^2\theta\Big)P(\theta) = 0. \quad (34)$$

This equation can be written in the form of Schrodinger equation

$$-\frac{d^2P(\theta)}{d\theta^2} + V_1P(\theta) = 0, \quad (35)$$

where

$$V_1 = \Big(-(1-d-\tilde{M}^2) - \frac{(1-d)(d-3)}{4}\Big)\mathrm{sec}^2\theta + \Big(l(l+d-2) - \frac{(1-d)(d-3)}{4}\Big)\mathrm{cosec}^2\theta + \omega^2. \quad (36)$$

To this end we use SUSY quantum mechanics and use an ansatz for the superpotential W

$$W = A\tan\theta + B\cot\theta. \quad (37)$$

We first would like to see if there exist A and B such that the superpotential reproduces back (36). Note that with this choice of W,

$$\begin{aligned} W^2 &= A^2\tan^2\theta + B^2\cot^2\theta + 2AB \\ &= A^2\sec^2\theta - A^2 + B^2\mathrm{cosec}^2\theta - B^2 + 2AB, \end{aligned} \quad (38)$$

and

$$W' = A\sec^2\theta - B^2\mathrm{cosec}^2\theta. \quad (39)$$

Hence the partner potential is

$$\begin{aligned} V_1 &= W^2 - W' \\ &= (A^2 - A)\sec^2\theta + (\mathrm{B}^2 + \mathrm{B})\mathrm{cosec}^2\theta - (\mathrm{A} - \mathrm{B})^2. \end{aligned} \quad (40)$$

But since from (36)

$$V_1 = \Big(-(1-d-\tilde{M}^2) + \frac{(d-1)(d-3)}{4}\Big)\mathrm{sec}^2\theta + \Big(l(l+d-2) + \frac{(d-1)(d-3)}{4}\Big)\mathrm{cosec}^2\theta + \omega^2, \quad (41)$$

we get by comparing (40) and (41),

$$A^2 - A = \frac{(d^2 - 1 + 4\tilde{M}^2)}{4}. \quad (42)$$

From where it follows,

$$A = \frac{1 \pm \sqrt{d^2 + 4\tilde{M}^2}}{2}. \quad (43)$$

From (40) and (41), we also get,

$$\omega^2 = -(A-B)^2. \quad (44)$$

Hence we have

$$\omega = \mp(A - B). \tag{45}$$

Hence there are two cases to be considered. One is $\omega = A - B$ and $\omega = -A + B$. The value of B can therefore be calculated as:

$$B = \omega + A \text{ or } B = A - \omega. \tag{46}$$

All together now, there are four possibilites. In the following, we pair those possibilites as (A_1, B_1), (A_2, B_2), (A_3, B_3) and (A_4, B_4). In the following, we give the expressions of all these and then determine the relevent pair by considering boundary conditions on the wave function. Note now, for $B = A - \omega$

$$B = \frac{1 \pm \sqrt{d^2 + 4\tilde{M}^2}}{2} - \omega = \frac{1}{2}\Big((1 - 2\omega) \pm \sqrt{d^2 + 4\tilde{M}^2}\Big). \tag{47}$$

$$B_1 = \frac{1}{2}\Big((1 - 2\omega) + \sqrt{d^2 + 4\tilde{M}^2}\Big), A_1 = \frac{1 + \sqrt{d^2 + 4\tilde{M}^2}}{2}. \tag{48}$$

The other pair is

$$B_2 = \frac{1}{2}\Big((1 - 2\omega) - \sqrt{d^2 + 4\tilde{M}^2}\Big), A_2 = \frac{1 - \sqrt{d^2 + 4\tilde{M}^2}}{2}. \tag{49}$$

Rest of the pairs can be found out noticing the possibility $B = A + \omega$. So

$$B = \frac{1 \pm \sqrt{d^2 + 4\tilde{M}^2}}{2} + \omega = \frac{1}{2}\Big((1 + 2\omega) \pm \sqrt{d^2 + 4\tilde{M}^2}\Big) \tag{50}$$

$$B_3 = \frac{1}{2}\Big((1 + 2\omega) + \sqrt{d^2 + 4\tilde{M}^2}\Big), A_3 = \frac{1 + \sqrt{d^2 + 4\tilde{M}^2}}{2}. \tag{51}$$

$$B_4 = \frac{1}{2}\Big((1 - 2\omega) + \sqrt{d^2 + 4\tilde{M}^2}\Big), A_4 = \frac{1 - \sqrt{d^2 + 4\tilde{M}^2}}{2}. \tag{52}$$

All these four values of B satisfies the equation

$$B^2 + B = l(l + d - 2) + \frac{(d-1)(d-3)}{4}. \tag{53}$$

Now we proceed to find the ground state energy eigenvalues ω in terms of physical parameters d, $\tilde{M}$ and l. To this end let us consider the first pair (A_1, B_1) for which

$$\begin{aligned} B_1^2 + B_1 =& \frac{1}{4}\Big(1 + d^2 + 4\tilde{M}^2 \\ &+2\sqrt{d^2 + 4\tilde{M}^2}\Big) + \omega^2 - \omega - \omega\sqrt{d^2 + 4\tilde{M}^2} \\ &+\frac{1}{2}1 + \sqrt{d^2 + 4\tilde{M}^2} - \omega \\ &= l(l + d - 2) + \frac{(d-1)(d-3)}{4}. \end{aligned} \tag{54}$$

Solving this equation we get

$$\omega = \omega_1 = \frac{1+\sqrt{d^2+4\tilde{M}^2}}{2} \pm l + \frac{d-2}{2}. \tag{55}$$

Similarly subtituting the other three values of B, we get

$$\begin{aligned}
\omega_2 &= \frac{1-\sqrt{d^2+4\tilde{M}^2}}{2} \pm l + \frac{d-2}{2} \\
\omega_3 &= \frac{-1-\sqrt{d^2+4\tilde{M}^2}}{2} \pm l + \frac{d-2}{2} \\
\omega_4 &= \frac{-1+\sqrt{d^2+4\tilde{M}^2}}{2} \pm l + \frac{d-2}{2}.
\end{aligned} \tag{56}$$

Using the superpotetial we can construct the ground state wave function.

$$P(\theta) = N\Big(\frac{\cos^A\theta}{\sin^B\theta}\Big), \tag{57}$$

where N is a normalisation constant. Hence

$$\chi_0(\theta) = N\frac{\cos\theta^{\frac{d-1+2A}{2}}}{\sin\theta^{\frac{d-1+2B}{2}}}. \tag{58}$$

Since we want χ_0 to be non-singular between $0 \leq \theta \leq \pi/2$, we need to put restrictions on the solutions. $\chi_0(\theta)$ is non-singular at $\theta = 0$ when

$$\frac{(d-1+2B)}{2} < 0. \tag{59}$$

This is satisfied by B_1 iff

$$\frac{(d-2\omega+\sqrt{d^2+4\tilde{M}^2}}{2} > 0 \tag{60}$$

Furthermore, we note $\chi(\theta)$ is non-singular at $\theta = \pi/2$

$$\frac{(d-1+2A)}{2} > 0. \tag{61}$$

This is satisfied if we choose A_1 from all possible A's. From this it follows that the physically interesting wave function is:

$$\chi_0 = (\mathrm{cos}\theta)^{(\frac{\mathrm{d}+\sqrt{\mathrm{d}^2+4\tilde{\mathrm{M}}^2}}{2})}(\mathrm{sin}\theta)^{\mathrm{l}}. \tag{62}$$

The ground state wave function is

$$\begin{aligned}
\psi_0 &= \exp(-i\omega\tau)Y_{0,0}(\Omega_{d-1})\chi_0(\theta) \\
&= \exp(-i\omega\tau)Y_{0,0}(\Omega_{d-1})\mathrm{cos}\theta^{\frac{\mathrm{d}+\sqrt{\mathrm{d}^2+\tilde{\mathrm{M}}^2}}{2}}.
\end{aligned} \tag{63}$$

The energy is

$$\begin{aligned}\omega &= 1+\frac{\sqrt{d^2+4\tilde{M}^2}}{2}+l+\frac{(d-2)}{2}\\ &= 1+\frac{\sqrt{d^2+4\tilde{M}^2}}{2}+l+\frac{d}{2}-1\\ &= \frac{1}{2}\Big(d+\sqrt{d^2+4\tilde{M}^2}\Big)+l.\end{aligned} \tag{64}$$

The ground state energy is obtained by putting l=0.

$$\omega = \frac{1}{2}\Big(d+\sqrt{d^2+4\tilde{M}^2}\Big). \tag{65}$$

5. CONCLUSION

Using supersymmetric quantum mechanics, we have computed the ground state wave function and the ground state energy of a massive particle in Anti de Sitter space. Though these results were known earlier, we believe that the use of SUSY quantum mechanics in this problem is new. We expect to report on a more complete analysis of this problem elsewhere.

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