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Bragg-Williams theory of phase transition: Ising model and black hole

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Abstract. In this work, we study the phase transitions of two disjoint systems within the Bragg-Williams approximation. First one is the phase transition of system from paramagnetic to ferromagnetic phase. We then use the same approximation scheme to study black hole phase transition in Anti de Sitter space. This is a first order phase transition where there is a crossover from black holes phase to Anti de Sitter phase. We argue that this transition is nicely captured by the Bragg-Williams theory.

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1. INTRODUCTION

Phase transition is often classified by the behaviour of the order parameter around the transition temperature. While for water-vapour transition the density is the order parameter, for ferromagnetic transitions, it is the magnetisation. As the name suggests, order parameter reflects the order with-in the system. For magnetic system, at high temperature, order is lost among the spins. Hence one gets expectation value of the order parameter to be zero. As we tune down the temperature, at the critical temperature, an order sets in. Consequently, below this temperature, the order parameter is non-zero. Order parameter also helps us to charecterise the nature of the phase transition. Let us denote the expectation value of a generic order parameter by $< \phi >$. While for a system undergoing second order phase transition, $< \phi >$ changes continuously around the critical temperature, for a first order phase transition, the change is rather discontinuous. It also should be mentioned that besides the order parameter, there are few other quantities that are discontinuous around the first order transition point. One of these is the entropy. This is due to the presence of the latent heat during the cross over from one phase to the other.

The purpose of this project is to study two very disjoint systems showing phase transition. One is the Ising model which has enormous impact in understanding phase transition and the other is the phase transition involving black holes. In particular, we will see how Bragg-Williams (will henceforth be called BW) theory apporach helps us bind these two together under a single umbrella. We will, in sequel, also discuss how under certain approximation, BW theory reduces to Landau's mean field theory.

Mean-field theory is an approximation where the order parameter is taken to be spatially constant. In other words, this says that we neglect the spatial fluctuation within the system. Though mean field theory often leads to answers which differ from their actual values, it has always been the first approach taken by reseachers to predict the phase diagrams. There are various formulations of mean field theory, but perhaps the best appreciated one is the Landau's mean field theory approach.

In this project, we start with Ising ferromagnet with in the BW scheme [1]. After discussing the phase diagram, we show how one reproduces Landau's mean field theory approximation from BW energy function.

Subsequently, we apply BW approach to describe first order phase transition involving black holes in Anti deSitter space (will be called AdS from now on).

In the rest of this section, we brifly list down certain thermodynamical properties of black holes and also comment about phase transition involving certain black holes [1].

General theory of relativity predicts existence of black holes. They are the sources of extremely large gravitational field. It is often said that the black holes are the objects with-in which things can fall and can not come out. Physicists argued that the end point of collapse of massive starts are black holes. J. Bakenstein and S. Hawking have helped us uncover close relations between thermodynamics and black holes. Below, we list down few general properties of black holes.

(1) Black hole has singularity inside and typically it is shielded by what is known as horizon.

(2) Black holes can be distinguished from each other only by their mass (or equivalently internal energy) M, electric or magnetic charge Q and rotation J. In this project we will consider holes with $Q = J = 0$.

(3) Black hole has surface area (A) which depends on the size of the horizon.

(4) Bakenstein showed us that entropy (S) can be associated with black holes and it is proportional to one-fourth of the horizon area.

(5) Hawking discovered that black hole can also have temperature (T). This temperature is known as Hawking temperature in the literature.

(6) Black hole obeys 0th, 1st and 2nd laws of thermodynamics.

Simplest of these black holes are the Schwarzschild black hole. It is only charectarised by mass/energy. If we take the horizon size r to be given by $r = 2M$ (We here set the Newton's constant $G = 1$. We also will set for simplicity the velocity of lingt c, Boltzman constant k and $\hbar$ also equal to 1.), then

$$A = 4\pi(2M)^2 = 16\pi M^2, \quad S = \frac{A}{4} = 4\pi M^2, \quad T = \frac{1}{8\pi M}. \tag{1}$$

It can be easily checked that the 2nd law of thermodynamics $dU = TdS$ is satisfied since

$$dU = dM \text{ and } TdS = \frac{1}{8\pi M} \times 8\pi M dM. \tag{2}$$

Here are few comments about Schwarzschild black holes. Firstly, we can think of this black holes sitting in Minkowski space. Setting $M = 0$, we have $U = 0$. This is taken to be the energy of the Minkowski's space. Secondly, we note that since themperature is inversely proportional to the mass of the black hole, as energy increases, its temperature decreases. This is typical of theormodynamic systems with negative specific heat. Such systems are generally unstable. Indeed, the Schwarzschild black hole is argued by researchers to be unstable. Because of these reasons we turn our attention to a kind of black holes which are called Schwarzschild black holes in AdS space. We will see that these black holes have positive specific heat with in certain range of parameters.

We will not need to know the details of AdS space but only that it has a constant negative enery density with respect to the Minkowski space. That is why it is often called as negatively curved space time. We will in general assume that this space-time is $1 + n$ dimensional with n space-like and 1 time-like coordinate.

[1]Understanding black hole physics is beyond the scope of this project. However there are certain references which might be useful. See for example [2], [3]. We here list down some well accepted results in black hole physics and do not claim that we understand all these. We will simply use these results as a strating point to build up a model for phase transition.

Black hole with $M \neq 0, Q = J = 0$ in this space is called AdS-Schwarzschild black holes in literature. We only require the expressions of the energy density, temperature and entropy density of these black holes. They are given by:

$$T = \frac{nr^2 + (n-2)l^2}{4\pi l^2 r}, \tag{3}$$

$$S = \frac{r^{n-1}}{4}, \tag{4}$$

$$E = \frac{(n-1)(r^n l^{-2} + r^{n-2})}{16\pi}. \tag{5}$$

Here r is the horizon radius and l is associated with the energy density of the AdS space. Taking l to infinity would take us to the Minkowski space. Note that for $r \to 0, E \to 0$. $r \to 0$ can be thought of as going to AdS space. This, in turn, means that the energy density has been calculated with respect to the AdS space. The above expressions have appeared in many papers. Here we have taken it from [4]. AdS-Schwarszchild black holes also satisfy various laws of thermodynamics. It can also be seen that for large r, T increases linearly with r (or in otherword with energy). Hence, specific heat in this limit is positive and the black holes are thermodynamically stable. It has been noted earlier by Hawking and Page that AdS-Schwerzschild black hole undergo a first order phase transition from black hole phase to AdS phase as one reduces the temperature below a critical value.

Our aim in this project is to understand this transition with-in the BW approximation. But before we do so, in the next section, we introduce this apporximation via Ising model in n space dimension.

2. ISING MODEL AND BRAGG-WILLIUM THEORY

Ising model is one of the simplest model which is generally used to capture the phase diagram for ferromagnetic to paramagnetic transition. In oder to briefly introduce Ising model, let us consider classical spin variable σ_i sitting at the ith site of a lattice in n dimensional space. σ_i are allowed to take take vaules $+1$ or -1 representing spin up or spin down states respectively. The coupling is only between two nearest neihgbour spins with strength given by $+J$. With this, we write the Ising Hamiltonian as

$$H = -J \sum_i \sigma_i \sigma_{i+1}. \tag{6}$$

A useful way to chart out the phase diagram that follows from Ising model is the introduce an oder parameter $m =< \sigma >$. At high temperature, one expects $m = 0$ while at low temeprature, an order sets in, leading to $m \neq 0$. We will see in the rest of the section, how a method due to Bragg-Williams helps us to see this behaviour in a very elegant way. We follow [1] in the rest of this discussion. In the next section, we will use this procedure to understand black hole phase transition in AdS space.

Bragg-Williams theory

Let us focus our attention to the magnetic moment of the system just described. One expects the total magnetic moment is proportional to the total number of up spins (N_{up}) and down spins (N_{down}). Assuming the total number of sites as $N = N_{up} + N_{down}$, we expect

$$m = \frac{N_{up} - N_{down}}{N}. \tag{7}$$

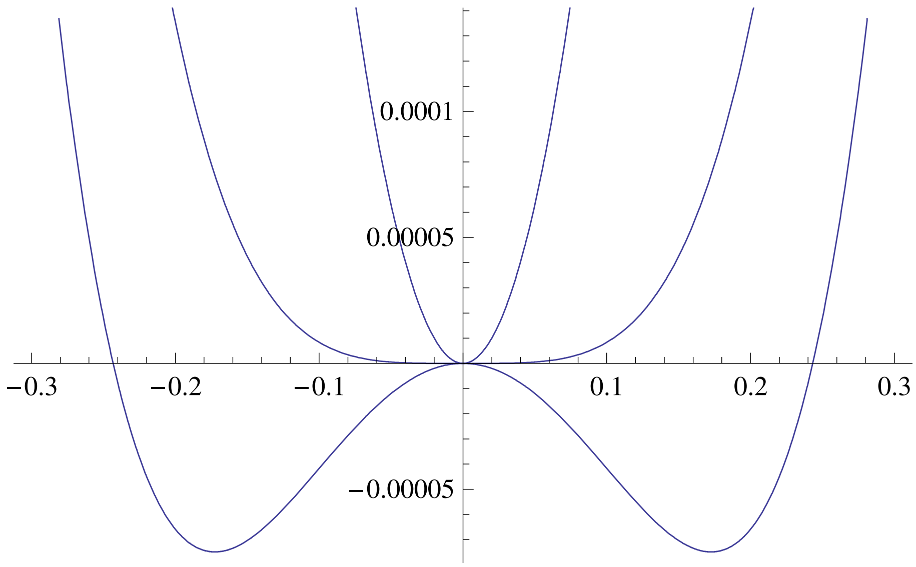

Figure 1. This is plot of $f(m,T)$ with x axis being m. The function is ploted for different values of T/Jz. The top one is for $T/Jz = 1.05$, the middle one is for $T/Jz = 1$ and the lower one is for $T/Jz = .99$.

Since entropy is defined as the logaritm of number of states, we have

$$S = \ln\,({}^N C_{N_{up}}) = \ln\,({}^N C_{N(1+m)/2}). \tag{8}$$

After simplification, entropy per unit spin can be re-expressed as

$$s = \frac{S}{N} = \ln 2 - \frac{1}{2}(1+m)\ln\,(1+m) - \frac{1}{2}(1-m)\ln\,(1-m). \tag{9}$$

Simplerly the energy per unit spin can be approximated as

$$e = \frac{E}{N} = -\frac{J\sum_i m^2}{N} = \frac{1}{2}Jzm^2. \tag{10}$$

Here z is the number of nearest neighbour sites ($z = 2n$ in n dimension). One then construct BW function

$$f(T,m) = e - Ts \tag{11}$$

Using (9) and (10) one gets,

$$f(T,m) = -\frac{1}{2}Jzm^2 + \frac{1}{2}T[(1+m)\,\ln\,(1+m) + (1-m)\,\ln\,(1-m)] - T\,\ln 2. \tag{12}$$

Behaviour of $f(m,T)$ is shown in figure 1. Note that for $T > Jz$, $f(m,T)$ has a minimum at $m = 0$. For $T < Jz$, minima are for finite non zero values of m. The symmetry in the plot $m \to -m$ is expected due to $\sigma \to -\sigma$ symmetry in Ising Hamiltonian. The shift of the minima from 0 to non-zero values start to occur at $T = T_c = Jz$. T_c is identified as the critical temperature. Since m changes continuously around T_c, we recognise this as a second order phase transition.

To conclude, we have seen in this section that BW theory captures the phase diagram of the Ising model which describes the ferromagnetic transition.

3. LANDAU'S MEAN FIELD THEORY

In the previous section, we constructed the function $f(m,T)$ which describes the phases not only close to $T = T_c$ but also away from it. This is because the magnetisation in $f(m,T)$ can take arbitrary large values. However, as it is obvious that the most interesting region in the phase diagram is around $T = T_c$. In this limit, m is close to zero. Therefore we can expand $f(m,T)$ in powers of m. This leads to

$$f(m,T) = -T \ln 2 + \frac{1}{2}(T - T_c)m^2 + \frac{T}{12}m^4 + \mathcal{O}(m^6). \tag{13}$$

Note that around $T = T_c$, the coefficient of m^2 changes sign. This leads to the change in the location of the minima as we cross T_c. Such a simple power series expansion which captures many interesting physics close to T_c is known as Landau's mean field theory approach towords phase transition. In general, it has a structure

$$f(m,T) = \sum_{i=0}^{\infty} a_i(T)m^i, \tag{14}$$

where a_is are independent of m and generally depende on T. In (13), we have

$$a_0 = T \ln 2, \;\; a_1 = 0, \;\; a_2 = \frac{1}{2}(T - T_c), \;\; a_3 = 0, \;\; a_4 = \frac{T}{12}, \;\; \text{and so on.} \tag{15}$$

In the next section, we will use BW theory to understand the phase transition involving black holes in AdS space.

4. BLACK HOLE PHASE TRANSITION

We are now in a position to apply BW theory for black hole. In (5), we have given the thermodynamic quantities associated with the AdS black hole. Following similar method as in (12), we get

$$f(r,T) = E - TS = \frac{(n-1)(r^n l^{-2} + r^{n-2})}{16\pi} - T\frac{r^{n-1}}{4}. \tag{16}$$

In what follows, we want to treat r as the order parameter. It will play an analogus role of magnetisation m of the last section. To keep things simple, we also set $l = 1$ in the rest of discussion. Though for any $n \geq 4$ our results are going to hold, most recent works in the literarure are for $n = 4$. To understand the behaviour of the function $f(r,T)$, we find out its equilibrium points. This is given by

$$\frac{\partial f}{\partial r} = 0. \tag{17}$$

which gives

$$n - 2 + nr^2 = 4\pi r T, \tag{18}$$

leading to

$$T = \frac{n - 2 + nr^2}{4\pi r}. \tag{19}$$

This is the same expression of temperature (remember here we have set $l = 1$) given in (5). We also see writing (19) is a different way, that for a given temperature, non-zero minimum occurs at

$$r = \frac{2\pi T + \sqrt{2n - n^2 + 4\pi^2 T^2}}{n}. \tag{20}$$

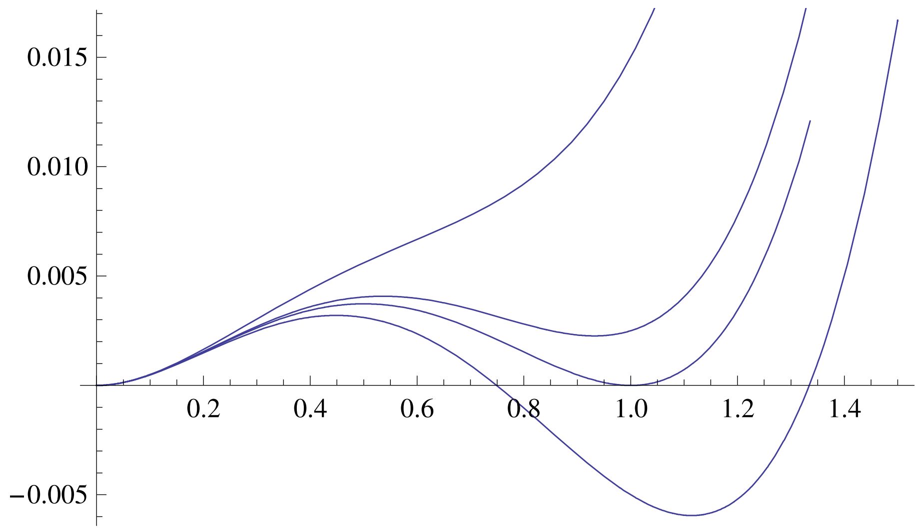

Figure 2. This is plot of $f(r,T)$ with x axis being r. The function is plotted for different values of T. The top one is for $T = .417$, the next one is for $T = .467$. The one with degenerate minima is for $T = T_c = 3/(2\pi) = .47$ and the lowest one is for $T = .497$.

This minimum to exist

$$4\pi^2 T^2 > n^2 - n \tag{21}$$

otherwise the expression in the left hand side of equation (20) will be comples. Finally, substituting the expression of (19) in (16), we get the value of $f(r,T)$ at the extrema. This is nothing but the Helmholz free energy (H). The expression is given by

$$H = \frac{r^{n-2} - r^n}{16\pi}. \tag{22}$$

We note that $H = 0$ for $r = 0$. This will be identified as *no black hole phase* as r is defined as the horizon radius. This is also called simply as the AdS phase. H is also zero for $r = 1$. Between $0 < r < 1,\ H > 0$ and for $r > 1, H < 0$. Since the thermodynamic system will try to minimise H, for $r > 1$, black hole is a stable phase and for $r < 1$, AdS is a stable phase with $r = 0$. This can also be seen if we plot $f(r,T)$ as a function of r for different T. See figure 2. For $T < T_c = 3/(2\pi)$, the AdS phase is the stable phase. At $T > T_c$, the black holes phase minimises the BW function. At $T = T_c$ both black hole and AdS phase coexist. The phase diagram is similar to what we expect for water-vapour transition. The density,there, is discontinuous around 100 degree temperature. On the other hand, here, we have discontinious change of r around $T = T_c$. As we increase T beyond T_c, r crosses over from 0 to $r \geq 1$. Consequently, entropy also jumps around T_c (we note that black hole entropy is proportional to r). From these properties, we conclude that this is a *first* order phase transition. It was discovered by Hawking and Page in 1983 in their studies of black hole in AdS space.

5. CONCLUSION

To conclude, we have seen that BW theory helps to understand phase transitions appearing in two completely different areas in physics. These are:

(1) Ising model of ferromagnetic to paramagnetic transition. This is a second order phase transition.
(2) Transition in gravity theories from ordinary phase (or Anti deSitter phase) to a black hole phase. This is an example of first order phase transition.

While in one case, magnetisation works as the order paraneter, for black hole, it is the horizon radius acts as the order parameter.

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