Field theory in 18 hours

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1 Lorentz and Poincare

Lorentz transformation is linear coordinate transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1}$$

leaving

$$\eta_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2 \text{ invariant.} \tag{2}$$

Group keeping

$$(x_1^2 + x_2^2 + ...x_n^2) - (y_1^2 + y_2^2 + ...y_m^2) \tag{3}$$

is $O(m,n)$. Lorentz group is therefore $O(3,1)$.

$$\eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\rho\sigma} x^\rho x^\sigma \tag{4}$$

gives

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}. \tag{5}$$

In matrix notation

$$\Lambda^T \eta \Lambda = \eta \tag{6}$$

Taking determinant

$$(\det \Lambda)^2 = 1, \quad \det \Lambda = \pm 1. \tag{7}$$

$\det \Lambda = +1$ is called *proper* Lorentz transformation. $O(3,1)$ with $\det =\ 1$ is $SO(3,1)$.

It follows from (5) that $(\Lambda^0{}_0)^2 - \sum_{i=1}^3 (\Lambda^i{}_0)^2 = 1$. So $(\Lambda^0{}_0)^2 \geq 1$. Therefore, proper Lorentz transformation has two disconnected components $\Lambda^0{}_0 \geq 1$ and $\Lambda^0{}_0 \leq -1$. First one is called orthochronous (non for the other) transformation. We see

$$\begin{aligned} \Lambda^0{}_0 \leq -1 \quad &= \quad [\Lambda^0{}_0 \geq 1] \times [(t,x,y,z) \to (-t,x,y,z)] \\ &\text{or} \quad [\Lambda^0{}_0 \geq 1] \times [(t,x,y,z) \to (-t,-x,-y,-z)] \\ &\text{or} \quad [\Lambda^0{}_0 \geq 1] \times [(t,x,y,z) \to (-t,-x,y,z)] \end{aligned} \tag{8}$$

etc. Namely via time reversal or time reversal plus party transformation or time reversal plus spatial reflection, we go from one to the other. We will consider

$$SO(3,1) \text{ with } (\Lambda^0{}_0) \geq 1. \tag{9}$$

Infinitesimal Lorentz transformation is given by $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$. Using (5), we conclude

$$\omega_{\mu\nu} = \omega_{\nu\mu} \tag{10}$$

Anti-symmetric 4×4 matrix has six independent components:

$$\omega_{12},\ \omega_{23},\ \omega_{31} \to \text{infinitesimal rotation} \tag{11}$$

$$\omega_{01},\ \omega_{02},\ \omega_{03} \to \text{infinitesimal boost,} \tag{12}$$

Note, boost along x

$$t' = \frac{1}{\sqrt{1-v^2}}(t+vx), \quad x' = \frac{1}{\sqrt{1-v^2}}(x+vt), \quad -1 < v < 1, \tag{13}$$

which can as well be written as

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \tag{14}$$

with cosh $\phi \ = 1/\sqrt{1-v^2}$. The variable ϕ is known as rapidity.

Since $-1 \leq v \leq 1$, the Lorentz group is non-compact.

A generic element of the group is written as

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} \tag{15}$$

with $J^{\mu\nu} = -J^{\nu\mu}$ are the six generators.

Consider ϕ^is, with $i = 1, ..., n$, a basis which transform in the representation R of dimension n of the Lorentz group.

$$\phi'^i = \left(e^{-\frac{i}{2}\omega_{\mu\nu}J_R^{\mu\nu}}\right)^i{}_j \phi^j \tag{16}$$

where $(e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}})^i{}_j$ are the n dimensional matrix representation with i, j being the matrix indices. Under infinitesimal transformation:

$$\delta\phi^i = -\frac{i}{2}\omega_{\mu\nu}(J_R^{\mu\nu})^i_j\phi^j. \tag{17}$$

Physical quantities can be classified following their transformation properties under the Lorentz group.

Scalar: Remains invariant under Lorentz transformation. Rest mass of a particle is an example.

Vector: Contravariant four vector B^μ is defined through

$$B^\mu = \Lambda^\mu{}_\nu B^\nu. \tag{18}$$

Space-time coordinate x^μ with (t, $\mathbf{x}$), momentum p^μ with $(E, \mathbf{p})$ are the examples of vectors. We will learn more about it later.

Explicit form of generators $(J^{\mu\nu})^i{}_j$ as $n \times n$ matrices depend on the representation we consider. For scalar ϕ, index i has one value. So $(J^{\mu\nu})^i{}_j$ is one dimensional matrix. But since $\delta\phi = 0$, $J^{\mu\nu} = 0$. For four-vector representation, $(J^{\mu\nu})^i{}_j$ are 4×4 matrix. Here (i, j) are Lorentz indices themselves. Explicit form of the matrix is

$$(J^{\mu\nu})^\sigma{}_\rho = i(\eta^{\mu\sigma}\delta^\nu_\rho - \eta^{\nu\sigma}\delta^\mu_\rho). \tag{19}$$

To check

$$\begin{aligned}
\delta B^\sigma &= -\frac{i}{2} i(\eta^{\mu\sigma}\delta^\nu_\rho - \eta^{\nu\sigma}\delta^\mu_\rho)\omega_{\mu\nu}B^\rho \\
&= \frac{1}{2}(\omega^\sigma{}_\nu B^\nu - \omega_\nu{}^\sigma B^\nu) \\
&= \frac{1}{2}(\omega^\sigma{}_\nu B^\nu + \omega^\sigma{}_\nu B^\nu) \\
&= \omega^\sigma{}_\nu B^\nu
\end{aligned} \tag{20}$$

as expected.

Use now (19) to **show** that the Lie algebra of $SO(3,1)$ is:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \tag{21}$$

Six components of $J^{\mu\nu}$ can be re-arranged into two spatial vectors. Define

$$J^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{i0}. \tag{22}$$

Lie algebra of Lorentz group then gives (use $\epsilon^{ijk}\epsilon^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl}$)

$$\begin{aligned}
&[J^i, J^j] = i\epsilon^{ijk}J^k \\
&[J^i, K^j] = i\epsilon^{ijk}K^k \\
&[K^i, K^j] = -i\epsilon^{ijk}J^k.
\end{aligned} \tag{23}$$

Now J^i satisfy angular momentum algebra and the second equation says that the K^i transform as spatial vector.

Further, introduce $\theta^i = \frac{1}{2}\epsilon^{ijk}\omega^{jk}, \eta^i = \omega^{i0}$ to write

$$\begin{aligned}
\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} &= \omega_{12}J^{12} + \omega_{23}J^{23} + \omega_{31}J^{31} + \omega_{10}J^{10} + \omega_{20}J^{20} + \omega_{30}J^{30} \\
&= \sum_i \theta^i J^i - \sum_i \eta^i K^i
\end{aligned} \tag{24}$$

so that

$$\Lambda = e^{-i\boldsymbol{\theta}.\mathbf{J} + i\boldsymbol{\eta}.\mathbf{K}}. \tag{25}$$

Spinor representation

<u>Recalling $SU(2)$</u>

Group $SU(2)$ consists of 2×2 complex matrices U with

$$U^\dagger U = 1, \quad \det U = 1. \tag{26}$$

Writing

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{27}$$

unitarity gives $d = a^*, b = -c^*$ and $aa^* - bb^* = 1$, leaving three independent real parameters. Acts on a basis $\xi = (\xi_1, \xi_2)^T$ as

$$\xi_1' = a\xi_1 + b\xi_2, \xi_2' = -b^*\xi_1 + a^*\xi_2. \tag{28}$$

A $SU(2)$ transformation on $\xi^i, i = 1, 2 \sim SO(3)$ transformation on $x^i, i = 1, 2, 3$ if we identify

$$x = \frac{1}{2}(\xi_2^2 - \xi_1^2), \quad y = \frac{1}{2i}(\xi_2^2 + \xi_1^2), \quad z = \xi_1\xi_2. \tag{29}$$

Show using the above relations, under $SU(2)$

$$\begin{aligned} x' &= \frac{1}{2}(a^2 + a^{*2} - b^2 - b^{*2})x - \frac{i}{2}(a^2 - a^{*2} + b^2 - b^{*2})y - (ab + a^*b^*)z \\ y' &= \frac{i}{2}(a^2 - a^{*2} - b^2 + b^{*2})x + \frac{1}{2}(a^2 + a^{*2} + b^2 + b^{*2})y - i(ab - a^*b^*)z \\ z' &= (ab^* + ba^*)x + i(ba^* - ab^*)y + (aa^* - bb^*)z. \end{aligned} \tag{30}$$

Suppose we choose $a = e^{\frac{i}{2}\theta}, b = 0$, obeying $aa^* - bb^* = 1$. We get

$$x' = x \cos\theta \ + \ y \sin\theta, y' = -x \sin\theta \ + \ y \cos\theta, \ z \ = \ z. \tag{31}$$

This leads to a correspondence between $SU(2)$ and $SO(3)$ matrices

$$\begin{bmatrix} e^{\frac{i}{2}\theta} & 0 \\ c & e^{-\frac{i}{2}\theta} \end{bmatrix}, \tag{32}$$

and

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{33}$$

or in other words, between

$$e^{\frac{i}{2}\sigma^3\theta} \ \text{ and } \ e^{iJ^z\theta}. \tag{34}$$

Here σ^3 is one of the Pauli matrices. We use for Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{35}$$

Check that $\sigma^i/2$ satisfies a similar commutation relation as J^i

$$[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\epsilon^{ijk}\frac{\sigma^k}{2}. \tag{36}$$

That the $SU(2)$ and the $SO(3)$ have the same algebra means that they are indistinguishable at the level of infinitesimal transformation. However they differ globally, far away from the identity. Note while $SU(2)$ is periodic under 4π rotation, and for $SO(3)$ the period is 2π. (32) changes sign under $\theta \to \theta + 2\pi$ but (33) does not. Does this remind you the property of a particle with 1/2 integer spin?

Representations of $SU(2)$ are labeled by j values $0, 1/2, 1, 3/2....$ while for $SO(3)$ they are spaced by integers. Spin j representation has dimension $2j + 1$ where various states within it are labeled by j_z which takes values from $-j, ..., +j$.
$j = 1/2$ is called *spinorial* representation and has dimension 2. It is the fundamental representation. Rest can be constructed with tensor products of spinors.

Let us now focus on the Lorentz algebra. Define

$$\mathbf{J}^{\pm} = \frac{\mathbf{J} \pm i\mathbf{K}}{2}. \tag{37}$$

With this, (23) becomes

$$\begin{aligned} [J^{+i}, J^{+j}] &= i\epsilon^{ijk} J^{+k} \\ [J^{-i}, J^{-j}] &= i\epsilon^{ijk} J^{-k} \\ [J^{+i}, J^{-j}] &= 0. \end{aligned} \tag{38}$$

We get two copies of angular momentum algebra. We therefore find that the representation of Lorentz algebra can be labeled by two *half-integers* (j_-, j_+) with dimension of representation $(2j_- + 1)(2j_+ + 1)$.

$(\mathbf{0}, \mathbf{0})$ representation: It has dimension one. On it $\mathbf{J}^{\pm} = 0$. Therefore, $\mathbf{J}, \mathbf{K}$ are zero. Hence it is scalar representation.

$(\mathbf{1/2}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{1/2})$ representation: It has dimension 2, and spin 1/2. So they are the spinorial representation. We write $(\psi_L)_\alpha$ for $\alpha = 1, 2$ a spinor in $(1/2, 0)$ and $(\psi_R)_\alpha$ for $(0, 1/2)$. The ψ_L is called the *left-handed Weyl* spinor, and the ψ_R is called the *right-handed Weyl* spinor

For $(1/2, 0), \mathbf{J}^-$ is represented by 2×2 matrix while $J^+ = 0$. The solution in terms of 2×2 matrix of (38) is

$$\mathbf{J}^- = \frac{\boldsymbol{\sigma}}{2},\ \mathbf{J}^+ = 0;\quad \mathbf{J} = \mathbf{J}^+ + \mathbf{J}^- = \frac{\boldsymbol{\sigma}}{2},\ \mathbf{K} = -i(\mathbf{J}^+ - \mathbf{J}^-) = \frac{i\boldsymbol{\sigma}}{2}. \tag{39}$$

Now using (25)

$$\psi_L \to \Lambda_L \psi_L = e^{(-i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}} \psi_L. \tag{40}$$

For ψ_R, for $(0, 1/2)$ representation, $\mathbf{J} = \boldsymbol{\sigma}/2$ and $\mathbf{K} = -i\boldsymbol{\sigma}/2$. So,

$$\psi_R \to \Lambda_R \psi_R = e^{(-i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}} \psi_R. \tag{41}$$

Charge conjugation:

Show, using $\sigma^2 \sigma^i \sigma^2 = -\sigma^{i*}$, that $\sigma^2 {\psi_L}^*$ transforms as is a right handed Weyl spinor.

Though the physical meaning of charge conjugation will be discussed later, it is an action which transforms ψ_L to a new spinor

$$\psi_L^c = i\sigma^2 \psi_L{}^*. \tag{42}$$

Taking $*$ on both sides, and calling $\psi_L^c \sim \psi_R$ and $\psi_R^c \sim \psi_L$, we further get

$$\psi_R^c = -i\sigma^2 \psi_R{}^*. \tag{43}$$

Note that

$$(\psi_L^c)^c = (i\sigma^2\psi_L^*)^c = -i\sigma^2(i\sigma^2\psi_L^*)^* = \psi_L. \tag{44}$$

Field Representation

Field is a function of space-time with definite transformation properties under Lorentz group.

Let a generic field $\phi(x)$ transforms to $\phi'(x')$ under Lorentz transformation. The $'$ on ϕ means the change in the functional form.

Scalar field:

$$\phi'(x') = \phi(x). \tag{45}$$

Take a point P with coordinate x in one reference frame and x' in the Lorentz transformed frame. Functional form of a scalar field changes in such a way that the numerical value remains unchanged.

Consider

$$x'^\rho = x^\rho + \delta x^\rho = x^\rho + \omega^\rho{}_\sigma x^\sigma \tag{46}$$

with $(J^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$. By definition, for scalar,

$$\delta\phi = \phi'(x') - \phi(x) = 0. \tag{47}$$

But the change of the field at *fixed* coordinate is non-zero. Calling $\tilde{\delta}\phi = \phi'(x) - \phi(x)$, we get

$$\phi'(x) = \phi(x - \delta x) = \phi(x) - \delta x^\rho \partial_\rho \phi. \tag{48}$$

So

$$\begin{aligned}
\tilde{\delta}\phi &= \phi'(x) - \phi(x) = -\delta x^\rho \partial_\rho \phi \\
&= \frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho{}_\sigma x^\sigma \partial_\rho \phi \\
&= \frac{i.i}{2}\omega_{\mu\nu}(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)x^\sigma\partial_\rho\phi \\
&= -\frac{i.i}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi \\
&= -\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\phi.
\end{aligned} \tag{49}$$

Here

$$L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi. \tag{50}$$

Check that $L^{\mu\nu}$ satisfy the algebra in (21).

Recognizing $i\partial^\mu = p^\mu$, we see $L^{\mu\nu} = (x^\mu p^\nu - x^\nu p^\mu)$. Note that the spatial components $L^i = 1/2\epsilon^{ijk}L^{jk} = \epsilon^{ijk}x^i p^j$ are the components of orbital angular momentum.

Weyl field:

We saw before

$$\psi'_L(x') = \Lambda_L\psi_L(x),\ \ \psi'_R(x') = \Lambda_R\psi_R(x). \tag{51}$$

According to our definition

$$\tilde{\delta}\psi_L(x) = \psi'_L(x) - \psi_L(x). \tag{52}$$

This is equal to

$$\begin{aligned} &= \Lambda_L\psi_L(x-\delta x) - \psi_L(x) = \Lambda_L[\psi_L(x) - \delta x^\rho\partial_\rho\psi_L] - \psi_L(x) \\ &\sim \Lambda_L\psi_L(x) - \delta x^\rho\partial_\rho\psi_L - \psi_L = (\Lambda_L - 1)\psi_L - \delta x^\rho\partial_\rho\psi_L. \end{aligned} \tag{53}$$

Therefore

$$\tilde{\delta}\psi_L = (\Lambda_L - 1)\psi_L(x) - \delta x^\rho\partial_\rho\psi_L \tag{54}$$

The last part is exactly like the scalar one. We write

$$\delta x^\rho\partial_\rho\psi_L = \frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}\psi_L. \tag{55}$$

For the first part, we introduce $S^{\mu\nu}$ such that

$$(\Lambda_L - 1)\psi_L(x) = (1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} - 1)\psi_L(x). \tag{56}$$

such that

$$\tilde{\delta}\psi_L = -\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\psi_L, \tag{57}$$

where $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$. With this,

$$\Lambda_L = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}. \tag{58}$$

Comparing Λ_L with (40), we can determine $S^{\mu\nu}$.

$$\begin{aligned} \Lambda_L &= e^{(-i\boldsymbol{\theta}-\boldsymbol{\eta})\cdot\frac{\boldsymbol{\sigma}}{2}} \\ &= e^{(-i\omega^{12}\sigma^3 - i\omega^{23}\sigma^1 - i\omega^{31}\sigma^2 - \omega^{i0}\sigma^i)/2} \\ &= e^{(-i\omega_{12}\sigma^3 - i\omega_{23}\sigma^1 - i\omega_{31}\sigma^2 + \omega_{i0}\sigma^i)/2} \\ &= e^{-i\omega_{12}S^{12} - i\omega_{23}S^{23} - i\omega_{31}S^{31} - i\omega_{i0}S^{i0}}. \end{aligned} \tag{59}$$

giving (58). Here, we have written

$$S^{12} = \frac{\sigma^3}{2}, S^{23} = \frac{\sigma^1}{2}, S^{31} = \frac{\sigma^2}{2}, S^{i0} = \frac{i\sigma^i}{2}. \tag{60}$$

Like the orbital angular momentum L^i, defined previously, we write the spin angular momentum as

$$S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{\sigma^i}{2}. \tag{61}$$

Similarly, for the right handed Weyl fermion, **show** that

$$\Lambda_R = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}, \text{ with } S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{\sigma^i}{2}, \text{ but } S^{i0} = -\frac{i\sigma^i}{2}. \tag{62}$$

Dirac Field:

Under parity transformation $(t, \mathbf{x}) \to (t, -\mathbf{x})$,

$$\mathbf{K} \to -\mathbf{K}, \quad \mathbf{J} \to \mathbf{J}. \tag{63}$$

Therefore, parity transformation exchanges $\mathbf{J}^{\pm}$ and hence $j_{\pm}$. Consequently, (j_-, j_+) representation of Lorentz group is not at the same time a basis of parity transformation unless $j_- = j_+$.

To describe parity preserving interactions (electromagnetic and strong), it is often convenient to work with fields which provide representations of both Lorentz and parity transformations.

We therefore define *Dirac* field as

$$\Psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}. \tag{64}$$

Under Lorentz and parity, it transforms respectively as

$$\Psi'(x') = \begin{bmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{bmatrix} \Psi(x), \quad \text{and } \Psi'(x') = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Psi(x). \tag{65}$$

Note that Dirac field has four complex components.

Charge conjugation of Dirac field is defined through charge conjugated Weyl field as

$$\Psi^c(x) = \begin{bmatrix} -i\sigma^2\psi_R^* \\ i\sigma^2\psi_L^* \end{bmatrix} = -i \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} \Psi^* \tag{66}$$

Check that $(\Psi^c)^c = \Psi$.

Vector field:

A contravariant vector field B^μ transform under Lorentz transformation as

$$B'^{\mu}(x') = \Lambda^\mu{}_\nu B^\nu(x). \tag{67}$$

Vector field belongs to $(j_-, j_+) = (1/2, 1/2)$ representation. Since $j_- = j_+$, it is also a basis for the representation of parity transformation. An example of vector field is the gauge field A^μ in electromagnetism.

Poincare group or the *inhomogeneous Lorentz group* is formed by translation $x^\mu \to x^\mu + a^\mu$ along with the Lorentz group. Generator of translation is the momentum operator P^μ and a generic element of the translation group is written as $e^{-iP^\mu a_\mu}$. Since two translations commute

$$[P^\mu, P^\nu] = 0. \tag{68}$$

Using known relations

$$[J^i, P^j] = i\epsilon^{ijk} P^k, \quad [J^i, P^0] = 0, \tag{69}$$

show that

$$[P^\mu, J^{\rho\sigma}] = i(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho). \tag{70}$$

Lorentz algebra along with (68) and (70) defined Poincare algebra. From here it follows that $[K^i, H] = iP^i$ where $P^0 = H$ is the Hamiltonian. Since K^i does not commute with the Hamiltonian, a state can not be labeled by the eigen value of K^i.

We have already discussed the representation of Lorentz group on fields. In order to know the representation of Poincare group on fields, we need to find a way to represent the four-momentum operator P^μ. This means that we will have to specify the transformation properties of the fields under translation. *We would require all fields to be scalars under translation.* That is

$$\phi'(x') = \phi(x). \tag{71}$$

Consider infinitesimal translation $x'^\mu = x^\mu + \epsilon^\mu$. Then

$$\phi'(x) = \phi(x - \delta x) \sim \phi(x) - \epsilon^\mu \partial_\mu \phi = e^{-iP^\mu \epsilon_\mu} \phi. \tag{72}$$

So $P_\mu = i\partial_\mu$.

Using $[\partial_\mu, x_\nu] = \eta_{\mu\nu}$, **show** that indeed (70) is satisfied.

2 Classical Fields

Suppose we are working with a non-relativistic system of particles. We identify a set of coordinates $q_i(t)$. This, together with the velocities $\dot{q}_i(t)$, describe the system. First, we construct the Lagrangian L and the action S

$$S = \int_{t_1}^{t_2} dt\ L(q_i(t), \dot{q}_i(t)). \tag{73}$$

Setting the variation of the action to zero, we end up with the Euler-Lagrange equation.

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0. \tag{74}$$

Defining the momentum conjugate to q_i as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{75}$$

we construct the Hamiltonian

$$H = \sum_i p_i \dot{q}_i - L. \tag{76}$$

When we construct respective Lagrangian and the action for the fields, we simply replace

$$q_i(t) \to \phi(t, \mathbf{x}), \dot{q}_i(t) \to \partial_\mu \phi(t, \mathbf{x}). \tag{77}$$

(Thinking of an one dimensional system of particles connected by springs might be useful here. At time t, let the transverse displacement of the ith particle is given by $q_i(t)$. When the number of particles increases, we represent the displacement profile of the particles by a continuous function $\phi(t, x)$ rather than all q_is together.)

The Lagrangian has a general form

$$L = \int d^3x \ \mathcal{L}(\phi, \partial_\mu \phi), \tag{78}$$

where $\mathcal{L}$ is called the Lagrangian density. Action is analogously constructed as

$$S = \int dt \ L = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \tag{79}$$

For point particles, we consider the time integral between two values t_1 and t_2, but here we will be mostly interested in situations where the integral extends over the whole space-time. We will always assume that the fields decrease sufficiently fast so that the boundary terms can be neglected.

We define the classical dynamics again by extremizing the action. This leads to the equation of motion

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Big) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{80}$$

The conjugate four-momentum is defined by

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}, \tag{81}$$

and the Hamiltonian *density* as

$$\mathcal{H}(x) = \Pi^0 \partial_0 \phi(x) - \mathcal{L}. \tag{82}$$

Noether's theorem

A transformation $\phi(x) \to \phi'(x) = \phi(x) + \tilde{\delta}\phi(x)$ is called a symmetry transformation if it changes the Lagrangian density by a total divergence. Noether's theorem says:
If under a continuous infinitesimal transformation $\phi(x) \to \phi'(x) = \phi(x) + \tilde{\delta}\phi(x)$, the change in the Lagrangian density is found to be of the form

$$\delta \mathcal{L} = \partial^\mu W_\mu(x) \tag{83}$$

without using the equations of motion, then there exists a current density

$$j^\mu(x) = \Pi^\mu \tilde{\delta}\phi(x) - W^\mu(x) \tag{84}$$

which, for field *obeying* equations of motion, satisfies

$$\partial_\mu j^\mu = 0. \tag{85}$$

To prove, we first consider arbitrary change in the field $\tilde{\delta}\phi$ not necessarily coming from the symmetry transformation. Change in the Lagrangian density is

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\tilde{\delta}\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\tilde{\delta}(\partial_\mu\phi) \\ &= \frac{\partial\mathcal{L}}{\partial\phi}\tilde{\delta}\phi + \Pi^\mu\partial_\mu(\tilde{\delta}\phi) \\ &= \partial_\mu(\Pi^\mu\tilde{\delta}\phi) + \Big(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\Pi^\mu\Big)\tilde{\delta}\phi. \end{aligned} \tag{86}$$

Now if the above is restricted to symmetry transformation, $\delta\mathcal{L} = \partial^\mu W_\mu$. Then by construction, upon using the equations of motion (in that case, the expression inside the big brackets goes to zero),

$$\partial_\mu(\Pi^\mu\tilde{\delta}\phi - W^\mu) = 0. \tag{87}$$

Example:

Consider translation

$$x^\mu \to x' = x^\mu + \epsilon^\mu. \tag{88}$$

We have seen before

$$\tilde{\delta}\phi(x) = -\epsilon_\mu\partial^\mu\phi. \tag{89}$$

For simplicity, we will choose ϵ_μ to have only one non-vanishing component, say ϵ_α. So that $\tilde{\delta}\phi(x) = \partial^\alpha\phi$. We have dropped an overall constant as j^μ is defined only up to a constant. Further $\tilde{\delta}\partial_\mu\phi = \delta^\alpha\partial_\mu\phi$. Assuming $\mathcal{L}$ does not depend on x explicitly, we have

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\partial^\alpha\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\alpha\partial_\mu\phi \\ &= \partial^\alpha\mathcal{L} = \partial_\mu(\eta^{\mu\alpha}\mathcal{L}). \end{aligned} \tag{90}$$

Therefore

$$W^{\mu\alpha} = \eta^{\mu\alpha}\mathcal{L} \tag{91}$$

and the Noether current is

$$T^{\mu\alpha} = \Pi^\mu\partial^\alpha\phi(x) - \eta^{\mu\alpha}\mathcal{L}. \tag{92}$$

This is called the *energy-momentum tensor*. By construction, when ϕ satisfies the equation of motion

$$\partial_\mu T^{\mu\alpha} = 0. \tag{93}$$

From here, it follows that the four momentum

$$P^\alpha = \int d^3x T^{0\alpha} \tag{94}$$

is conserved. That is

$$\frac{dP^\alpha}{dt} = 0. \tag{95}$$

Real scalar field

To construct a Poincare invariant action, we start with the simplest one involving a real scalar field ϕ. In order to describe non-trivial dynamics, the action must contain $\partial_\mu \phi$. To make it Lorentz invariant we need to contract the index μ. Easy way to do this is to multiply it with $\partial^\mu \phi$. We therefore consider

$$S = \frac{1}{2}\int d^4x (\partial_\mu \phi \partial^\mu \phi - m^2\phi^2). \tag{96}$$

Here m is the rest mass of the particle as we will soon see. Euler-Lagrange equation gives

$$(\Box + m^2)\phi = 0, \quad \Box = \partial_\mu \partial^\mu. \tag{97}$$

The plane wave e^{-ipx} is a solution if

$$p^2 = m^2, \quad \text{or } (p^0)^2 - \mathbf{p}^2 = m^2, \tag{98}$$

justifying the association of m with the rest mass of a free particle. Since ϕ is real, we write the general solution as a real superposition of plane waves:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_{\mathbf{p}}e^{-ipx} + a^*_{\mathbf{p}}e^{ipx})|_{p^0=E_p}. \tag{99}$$

Here $E_p = +\sqrt{\mathbf{p}^2 + m^2}$. Note that it has both positive (e^{-iE_pt}) and negative (e^{+iE_pt}) frequency modes.

Momentum conjugate to ϕ is

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi. \tag{100}$$

Hamiltonian is therefore

$$H = \int d^3x \mathcal{H} = \frac{1}{2}\int d^3x[(\Pi^0)^2 + (\nabla\phi)^2 + m^2\phi^2]. \tag{101}$$

Complex scalar

Two real scalar fields of same mass m can be assembled into a single *complex* scalar field as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{102}$$

and the action can be written as

$$S = S_{\phi_1} + S_{\phi_2} = \int d^4x(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi) \tag{103}$$

Since we now give up the reality condition, we write the solution as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} (a_{\mathbf{p}} e^{-ipx} + b^*_{\mathbf{p}} e^{ipx})|_{p^0=E_p}, \tag{104}$$

where a_p, b_p are *independent.*

This action has a *global* $U(1)$ symmetry $\phi \to e^{i\theta}\phi, \phi^* \to e^{-i\theta}\phi^*$. The corresponding Noether current is

$$j_\mu = \Pi_\mu \delta\phi + \Pi^*_\mu \delta\phi^*, \quad \text{since } W^\mu = 0. \tag{105}$$

A simple computation gives

$$j_\mu = i(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) = i\phi^* \overleftrightarrow{\partial}_\mu \phi \tag{106}$$

and the corresponding conserved charge is

$$Q = \int d^3x j^0 = i \int d^3x \phi^* \overleftrightarrow{\partial}_0 \phi \tag{107}$$

Spinor

<u>Weyl equation</u>

First we define

$$\sigma^\mu = (1, \sigma^i), \quad \text{and } \tilde{\sigma}^\mu = (1, -\sigma^i). \tag{108}$$

Now we can show that $\psi^\dagger_L \tilde{\sigma}^\mu \psi_L$ and $\psi^\dagger_R \sigma^\mu \psi_R$ transform as vectors under Lorentz transformation. For the purpose of illustration, let us consider the second one for non-zero boost parameter η^1.

$$\psi^\dagger_R \sigma^\mu \psi_R \to \psi^\dagger_R e^{\eta^1 \sigma^{1\dagger}/2} \sigma^\mu e^{\eta^1 \sigma^1/2} \psi_R. \tag{109}$$

Therefore the time-like component transform as

$$\begin{aligned} \psi^\dagger_R \sigma^0 \psi_R \quad & \to \psi^\dagger_R (\cosh\frac{\eta^1}{2} + \sigma^1 \sinh\frac{\eta^1}{2})(\cosh\frac{\eta^1}{2} + \sigma^1 \sinh\frac{\eta^1}{2})\psi_R \\ & = \cosh\eta^1 \ \psi^\dagger_R \psi_R + \sinh\eta^1 \ \psi^\dagger_R \sigma^1 \psi_R. \end{aligned} \tag{110}$$

Similarly, it is easy to show

$$\psi^\dagger_R \sigma^1 \psi_R \to \sinh\eta^1 \ \psi^\dagger_R \psi_R + \cosh\eta^1 \ \psi^\dagger_R \sigma^1 \psi_R, \tag{111}$$

and the other components remain unchanged. This shows that $\psi^\dagger_R \sigma^\mu \psi_R$ transform as a vector under boost with η^1. For a general Lorentz transformation, the change can be easily worked out. We then construct scalar bilinears for the Lagrangian density

$$\mathcal{L}_L = i\psi^\dagger_L \tilde{\sigma}^\mu \partial_\mu \psi_L, \quad \mathcal{L}_R = i\psi^\dagger_R \sigma^\mu \partial_\mu \psi_R. \tag{112}$$

The i factors in front make the action real. Euler-Lagrange equation following from $\mathcal{L}_L$ is

$$(\partial_0 - \sigma^i \partial_i)\psi_L = 0. \tag{113}$$

Note therefore, $\partial_0^2 \psi_L = \sigma^i \partial_i \sigma^j \partial_j \psi_L = \partial_i^2 \psi_L$. Hence

$$\Box \psi_L = 0. \tag{114}$$

This is a massless Klein Gordon equation. However, the first order differential equation gives us more information about ψ_L. Consider the positive energy plane wave solution $\psi_L = u_L e^{-ipx} = u_L e^{-iEt+ip^i x^i}$ where u_L is a constant spinor. From (113), we get

$$\frac{p^i \sigma^i}{E} u_L = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E} u_L = -u_L. \tag{115}$$

Now since, from (114), $E = |\mathbf{p}|$ and the angular momentum is $\mathbf{J} = \boldsymbol{\sigma}/2$, we conclude

$$(\hat{\mathbf{p}} \cdot \mathbf{J}) u_L = -\frac{1}{2} u_L, \quad \text{with } \hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}. \tag{116}$$

The projection of spin along the direction of the momentum is known as *helicity*. We say that the left-handed massless Weyl spinor has helicity $h = -1/2$.

Show that the energy-momentum tensor is given by

$$T^{\mu\nu} = i\psi_L^\dagger \tilde{\sigma}^\mu \partial^\nu \psi_L. \tag{117}$$

Note that $\mathcal{L}_L$ does not change under global $U(1)$ transformation $\psi_L \rightarrow e^{i\theta}\psi_L$. The Noether current and charge respectively are

$$j^\mu = \psi_L^\dagger \tilde{\sigma}^\mu \psi_L, \quad Q = \int d^3x \psi_L^\dagger \psi_L. \tag{118}$$

Repeat all these computations for ψ_R. In particular, show that for ψ_R, the positive energy solution has helicity $h = +1/2$.

<u>Dirac equation</u>

First, note that since $\Lambda_L^\dagger \Lambda_R = \Lambda_R^\dagger \Lambda_L = 1$, $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$ behave as scalars under Lorentz transformation. Further the combination $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ is also invariant under parity transformation. Therefore, we can construct a Lagrangian density which is scalar under Lorentz and parity transformation as:

$$\mathcal{L}_D = i\psi_L^\dagger \tilde{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L). \tag{119}$$

Note that under parity, $\psi_L \leftrightarrow \psi_R$ and $\partial_i \rightarrow -\partial_i$. So the first two terms get interchanged.

We now define Dirac spinor as before

$$\Psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} \tag{120}$$

and introduce

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{bmatrix}, \tag{121}$$

to re-write (119) as

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi, \tag{122}$$

with $\bar{\Psi} = \Psi^\dagger \gamma^0$. It would be useful to note that the γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{123}$$

At this stage, we also define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. In matrix form

$$\gamma^5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{124}$$

Therefore $(1 \pm \gamma^5)/2$ act as projectors on the Weyl spinors.

$$\frac{1-\gamma^5}{2}\Psi = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}, \tag{125}$$

and

$$\frac{1+\gamma^5}{2}\Psi = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}, \tag{126}$$

In future discussions, in many occasions, a general four vector B_μ will appear in a combination $\gamma^\mu B_\mu$. We will write it as $\not{B}$. Therefore $\gamma^\mu \partial_\mu = \not{\partial}$. Varying $\bar{\Psi}$, we get the Dirac equation

$$(i\not{\partial} - m)\Psi = 0 \tag{127}$$

Show that from here it follows

$$(\Box + m^2)\psi_{L/R} = 0. \tag{128}$$

Therefore the Dirac equation implies massive Klein-Gordon equations for ψ_L, and ψ_R.

A general solution of Dirac equation is a superposition of plane waves

$$\Psi(x) = u(p)e^{-ipx}, \quad \text{and } \Psi(x) = v(p)e^{ipx}, \tag{129}$$

where $v(p), u(p)$ are four component spinors. Upon Substitution in Dirac equation, plane waves give

$$\begin{aligned} (\not{p} - m)u(p) &= 0 \\ (\not{p} + m)v(p) &= 0. \end{aligned} \tag{130}$$

We write $u(p)$ as

$$u(p) = \begin{bmatrix} u_L(p) \\ u_R(p) \end{bmatrix} \tag{131}$$

and first take the case $m \neq 0$. In the rest frame of the particle $p^\mu = (m, 0, 0, 0)$, the first of the (130) equations reduces to

$$(\gamma^0 - 1)u(p) = 0. \tag{132}$$

This only means $u_L = u_R$. Further from (128), we get the mass-shell condition $p^2 = m^2$. We choose $u_L = u_R = \sqrt{m}\xi$ where the two component spinor ξ satisfies $\xi^\dagger \xi = 1$.

For general p, we can simply boost the above solution. It goes as follows. Consider a boost along the $3rd$ direction (alternatively the z direction).

$$\begin{aligned} u'_L(p') &= e^{-\frac{\eta^3\sigma^3}{2}} u_L(p) \\ &= (\cosh\frac{\eta^3}{2} - \sigma^3 \sinh\frac{\eta^3}{2}) u_L(p). \end{aligned} \tag{133}$$

However, we note

$$\begin{aligned} \cosh\eta^3 &= \frac{1}{\sqrt{1-v^2}} = \sqrt{1-\frac{(p^3)^2}{m^2}} = \frac{E}{m} \\ \cosh\frac{\eta^3}{2} &= \sqrt{\frac{1+\cosh\eta^3}{2}} = \sqrt{\frac{E+m}{2m}} \\ \sinh\frac{\eta^3}{2} &= \sqrt{\frac{-1+\cosh\eta^3}{2}} = \sqrt{\frac{E-m}{2m}}. \end{aligned} \tag{134}$$

In the above, we have used $p^3 = mv^3/\sqrt{1-v^2}$. So

$$u'_L(p') = \Big(\sqrt{\frac{E+m}{2m}} - \sigma^3\sqrt{\frac{E-m}{2m}}\Big) u_L(p). \tag{135}$$

However, since

$$(\sqrt{E+p^3} + \sqrt{E-p^3})^2 = 2(E+m), \quad (\sqrt{E+p^3} - \sqrt{E-p^3})^2 = 2(E-m), \tag{136}$$

we can express

$$u'_L(p') = \Big(\frac{\sqrt{E+p^3}(1-\sigma^3)}{2\sqrt{m}} + \frac{\sqrt{E-p^3}(1+\sigma^3)}{2\sqrt{m}}\Big) u_L(p). \tag{137}$$

Similarly from the transformation property of u_R, we find

$$u'_R(p') = \Big(\frac{\sqrt{E+p^3}(1+\sigma^3)}{2\sqrt{m}} + \frac{\sqrt{E-p^3}(1-\sigma^3)}{2\sqrt{m}}\Big) u_R(p). \tag{138}$$

So the complete solution is (removing the $'$s for notational simplicity):

$$u^s(p) = \begin{bmatrix} \Big(\frac{\sqrt{E+p^3}(1-\sigma^3)}{2} + \frac{\sqrt{E-p^3}(1+\sigma^3)}{2}\Big)\xi^s \\ \Big(\frac{\sqrt{E+p^3}(1+\sigma^3)}{2} + \frac{\sqrt{E-p^3}(1-\sigma^3)}{2}\Big)\xi^s \end{bmatrix}. \tag{139}$$

Index s stands for the two independent solutions

$$\xi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{140}$$

For $v(p)$, we note that in rest frame $(\gamma^0 + 1)v(p) = 0$, giving $v_L = -v_R$. So the general solution is:

$$v^s(p) = \begin{bmatrix} \left(\frac{\sqrt{E+p^3}(1-\sigma^3)}{2} + \frac{\sqrt{E-p^3}(1+\sigma^3)}{2}\right)\eta^s \\ -\left(\frac{\sqrt{E+p^3}(1+\sigma^3)}{2} + \frac{\sqrt{E-p^3}(1-\sigma^3)}{2}\right)\eta^s \end{bmatrix}, \tag{141}$$

where

$$\eta^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \eta^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{142}$$

In the limit when m goes to zero, $p^\mu = (E, 0, 0, E)$. We get

$$u^1(p) = \sqrt{2E} \begin{bmatrix} 0 \\ \xi^1 \end{bmatrix}, \quad u^2(p) = \sqrt{2E} \begin{bmatrix} \xi^2 \\ 0 \end{bmatrix}, \tag{143}$$

and so on.

Taking the hermitian conjugate of (130), we also get

$$\begin{aligned} \bar{u}(p)(\not{p} - m) &= 0 \\ \bar{v}(p)(\not{p} - m) &= 0. \end{aligned} \tag{144}$$

<u>Chiral symmetry</u>

In the *absence* of the mass term, the Dirac Lagrangian has a global symmetry

$$\psi_L \to e^{i\theta_L}\psi_L, \quad \psi_R \to e^{i\theta_R}\psi_R. \tag{145}$$

This is a $U(1) \times U(1)$ symmetry. When $\theta_L = \theta_R = \alpha$, we can write the transformations in terms of the Dirac spinor

$$\Psi \to e^{i\alpha}\Psi. \tag{146}$$

The transformation with $\theta_R = -\theta_L = \beta$ can be written as

$$\Psi \to e^{i\alpha\gamma^5}\Psi. \tag{147}$$

The conserved currents for the first is called vector current as

$$j^\mu_V = \bar{\Psi}\gamma^\mu\Psi \tag{148}$$

since it transforms like a vector under Lorentz transformation. The second is the axial vector current as

$$j^\mu_A = \bar{\Psi}\gamma^\mu\gamma^5\Psi. \tag{149}$$

This transforms as an axial vector under Lorentz transformation.

Show $\bar{\Psi}\gamma^\mu\Psi$ and $\bar{\Psi}\gamma^\mu\gamma^5\Psi$ indeed behave as vector and axial vector under Lorentz transformation.

When we *turn on* the mass term in the Dirac equation, the symmetry under the transformation with $\theta_L = \theta_R$ survives while the axial $U(1)$ breaks.

Vector: The Electromagnetic field

The electromagnetic field is described by a four-vector A_μ. First we construct the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{150}$$

Components of the field strength gives the electric (E^i) and the magnetic fields (B^i), namely,

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B^k. \tag{151}$$

We now want to construct a Lagrangian such that the Maxwell equations are reproduced. The one which does that is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2). \tag{152}$$

Varying A^μ, we get the equations of motion

$$\partial_\mu F^{\mu\nu} = 0 \tag{153}$$

If we further define

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \tag{154}$$

then because of the antisymmetric nature of $\epsilon^{\mu\nu\rho\sigma}$,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{155}$$

Show that from (153) and (155), the Maxwell equations

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \partial_0 \mathbf{E} \tag{156}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} = -\partial_0 \mathbf{B} \tag{157}$$

follow.

Gauge invariance: Coulomb and Lorentz gauge

We note that the Lagrangian remains unchanged under *gauge* transformation:

$$A_\mu \to A'_\mu(x) = A_\mu - \partial_\mu \theta(x), \tag{158}$$

where θ is an arbitrary smooth function of $t, \mathbf{x}$. This means that A_μ gives a redundant description of the theory. Or, in other words, Lagrangian is telling us that there are too many of A_μs which are physically equivalent. We need to remove this redundancy or this over counting.

First consider taking

$$\theta = \int^t dt' A_0(\mathbf{x}, t'), \tag{159}$$

This makes $A'_0(x) = 0$ but of course changes A_i. Note that we can make a further gauge transformation as long as $A'_0(x) = 0$ remains unchanged. This is trivially done by choosing θ to be *independent* of time. We choose

$$\theta'(\mathbf{x}) = -\int \frac{d^3y}{4\pi|\mathbf{x}-\mathbf{y}|}\frac{\partial A'^i(\mathbf{y},t)}{\partial y^i} \tag{160}$$

and new gauge field is therefore $A''_\mu = A'_\mu - \partial_\mu\theta'$. Though we see explicit time-dependence in the right hand side, θ' is actually time independent. This follows from the equations of motion. Note that since A'_0 has been set to zero, $E^i = -\partial_0 A'^i$. Then the Maxwell equation $\partial_i E^i = 0$ says $\partial_0\partial_i A'^i = 0$. Therefore $\partial_0\theta'(\mathbf{x}) = 0$. Further, note that

$$\boldsymbol{\nabla}\cdot\mathbf{A}'' = \boldsymbol{\nabla}\cdot\mathbf{A}' - \nabla^2\theta = 0. \tag{161}$$

To derive the this result, we used the identity

$$\nabla^2_{\mathbf{x}}\Big(\frac{1}{|\mathbf{x}-\mathbf{y}|}\Big) = -4\pi\delta^3(\mathbf{x}-\mathbf{y}). \tag{162}$$

So using the gauge freedom, we have reached

$$A_0 = 0, \quad \nabla\cdot\mathbf{A} = 0. \tag{163}$$

This is called *Coulomb* or *radiation* gauge. This gauge implies *Lorentz* gauge

$$\partial_\mu A^\mu = 0. \tag{164}$$

However, in Lorentz gauge, part of gauge redundancy is still left. In this gauge,

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu = \partial_\mu\partial^\mu A^\nu. \tag{165}$$

Therefore the equations of motion reduce to

$$\Box A^\nu = 0. \tag{166}$$

What we got is a massless Klein-Gordon equation. To have a better understanding of the gauge fixing process, let us look at the plane wave solution of the above equation.

$$A_\mu(x) \sim \epsilon_\mu(k)e^{-ikx} \tag{167}$$

plus its complex conjugate. The ϵ_μ is known as the polarization vector. From (166), we get $k^2 = 0$. Further, (163) gives

$$\epsilon_0 = 0, \quad \boldsymbol{\epsilon}\cdot\mathbf{k} = 0. \tag{168}$$

We see that A_μ has two degrees of freedom given by the two polarization vectors in a plane *transverse* to the direction of the propagation.

Coupling gauge field with matter

When we add an external current term in electromagnetism, we know that the first line of the Maxwell equations in (157) gets modified. The second line, which follows from the antisymmetric property of $F^{\mu\nu}$, remains unchanged. The first pair becomes

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{169}$$

The property $\partial_\mu \partial_\nu F^{\mu\nu} = 0$ then tells us that j^ν has to be conserved,

$$\partial_\nu j^\nu = 0. \tag{170}$$

The action that produces (169) as the equations of motion is

$$S = -\int d^4x (\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu). \tag{171}$$

Under a gauge transformation,

$$S \to S - \int d^4x j^\mu \partial_\mu \theta = S + \int d^4x (\partial_\mu j^\mu) \theta. \tag{172}$$

Therefore, if we insist on the gauge invariance of the action, we get the conservation of the current as in (170). This is the first simple example of how gauge invariance works as a guiding principle in building up theory of fundamental interactions. We will now try to construct some explicit *gauge invariant* actions.

We have seen before that the Dirac action has a global $U(1)$ symmetry

$$\Psi \to e^{iq\theta} \Psi. \tag{173}$$

For the moment, we take q to be a number. Suppose we make θ a function of x. This turns the global $U(1)$ to a *local* $U(1)$ transformation. Clearly Dirac action in its present form is *not* invariant as $\partial_\mu \Psi$ does not transform to $e^{iq\theta(x)} \partial_\mu \Psi$ under local $U(1)$. However, if we modify the derivative to a *covariant* derivative as

$$\partial_\mu \Psi \to D_\mu \Psi = (\partial_\mu + iqA_\mu)\Psi, \tag{174}$$

then under the local $U(1)$

$$D_\mu \Psi \to D_\mu e^{iq\theta(x)} \Psi = e^{iq\theta} \partial_\mu \Psi + iq \partial_\mu \theta e^{iq\theta} \Psi + iqA_\mu e^{iq\theta} \Psi. \tag{175}$$

But notice, since the free electromagnetic field is invariant under $A_\mu \to A_\mu - \partial_\mu \theta$, we can absorb the piece $iq\partial_\mu \theta e^{iq\theta} \Psi$ by making a gauge transformation. Therefore, we conclude that under combined gauge and local $U(1)$ transformation, the covariant derivative transforms as

$$D_\mu \Psi \to e^{iq\theta(x)} D_\mu \Psi. \tag{176}$$

Therefore the Lagrangian

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \\ &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi - qA_\mu \bar{\Psi}\gamma^\mu \Psi \end{aligned} \tag{177}$$

is invariant under local $U(1)$ transformation. This is also known in literature as *gauging* a $U(1)$ symmetry. The conserved current under global $U(1)$ was $j^\mu_V = \bar{\Psi}\gamma^\mu \Psi$. By making $U(1)$ local, we have introduced a coupling $A_\mu j^\mu_V$.

We can repeat the same exercise for the complex scalar field. If we want to make the transformation $\phi \to e^{iq\theta}\phi$ a local symmetry of the Lagrangian, we simply replace ∂_μ by D_μ.

$$\begin{aligned}
\mathcal{L} \quad &= \partial_\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&\Rightarrow (D_\mu\phi)^* D_\mu\phi - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= \partial_\mu\phi^*\partial_\mu\phi - iqA_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) + q^2\phi^*\phi A_\mu A^\mu - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \qquad (178)
\end{aligned}$$

Again we see that by making $U(1)$ local, we have introduced a coupling between the gauge field and the global $U(1)$ current. Of course there is an extra term in the Lagrangian. But this term actually plays an important role in the Higgs mechanism.

3 Free field quantization

Scalar field

Like in quantum mechanics, we canonically quantize a theory by promoting a field to an operator and imposing nontrivial commutations relation between the field and its conjugate momentum. In Heisenberg picture, where the operators depend on time, we impose the commutation relation at *equal* time.

$$[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}). \qquad (179)$$

For notational simplicity we have written $\Pi^0 = \Pi$. Note that both the sides transform similarly under Lorentz transformation. Though ϕ is a scalar, $\Pi = \partial_0\phi$ transforms as the time component of a four vector. Therefore whole of the left hand side transforms as the time component. Now since $\int d^4x\delta^3(\mathbf{x}-\mathbf{y}) = \int dt$, and d^4x is Lorentz invariant (**Show**), $\delta^3(\mathbf{x}-\mathbf{y})$ transforms as the time-component of a four vector as well.

Promoting ϕ to an operator is straightforward. We take equation (99) and promote $a_{\mathbf{p}}$ to an operator. Then $a^*_{\mathbf{p}}$ becomes the hermitian conjugate $a^\dagger_{\mathbf{p}}$.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx})|_{p^0=E_p}. \qquad (180)$$

The relation (179), turns into a relations between $a_{\mathbf{p}}$ and its hermitian conjugate. The calculation is straightforward but somewhat tedious. The following relations will help.

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2E_p}}\int d^3x e^{ipx} \overleftrightarrow{\partial_0} \phi \qquad (181)$$

$$a^\dagger_{\mathbf{p}} = -\frac{i}{\sqrt{2E_p}}\int d^3x e^{-ipx} \overleftrightarrow{\partial_0} \phi. \qquad (182)$$

To derive this you may also need to use

$$(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}) = \int d^3x e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}. \qquad (183)$$

Further noticing $\Pi = \partial_0\phi$ one finally arrives at

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{184}$$

Rest of the commutators between operators vanish.

It is convenient sometime to put the system into a large box of size L with total volume $V = L^3$. This regularizes the divergences coming from the infinite volume and is known as *infrared* regularization. In this situation p^i takes only discrete values $(2\pi/L)n^i$ with $n^i = 0, \pm1, \pm2, ...$ Therefore the momentum integral turns into a sum.

$$\int d^3p \Rightarrow \Big(\frac{(2\pi)}{L}\Big)^3 \sum_n . \tag{185}$$

Since $\int d^3p\delta^3(\mathbf{p}-\mathbf{q}) = 1$,

$$\delta^3(\mathbf{p}-\mathbf{q}) = \Big(\frac{L}{2\pi}\Big)^3 \delta_{\mathbf{p},\mathbf{q}}. \tag{186}$$

This implies

$$\delta^3(\mathbf{p}=0) = \frac{V}{(2\pi)^3}. \tag{187}$$

Comparing with the annihilation and creation operators $a, a^\dagger$ of harmonic oscillator, we see from (184) that the real scalar field is equivalent to a collection of oscillators, one for each value of $\mathbf{p}$. The construction of the *Fock space* therefore is analogous. We first define the *vacuum* state as

$$a_{\mathbf{p}}|0\rangle = 0, \tag{188}$$

for all $\mathbf{p}$. A general state is obtained by acting creations operators on the vacuum state.

$$|\mathbf{p_1}\mathbf{p_2}...\mathbf{p_3}\rangle = (2E_{\mathbf{p_1}})^{\frac{1}{2}}(2E_{\mathbf{p_2}})^{\frac{1}{2}}...(2E_{\mathbf{p_n}})^{\frac{1}{2}} a_{\mathbf{p_1}}^\dagger a_{\mathbf{p_2}}^\dagger.....a_{\mathbf{p_n}}^\dagger|0\rangle. \tag{189}$$

We can also construct a n-particle state of momentum $\mathbf{p}$

$$|\mathbf{pp}...\mathbf{p}\rangle = (2E_{\mathbf{p}})^{\frac{n}{2}}(a_{\mathbf{p}}^\dagger)^n|0\rangle, \tag{190}$$

We find that the norm of the one-particle state is

$$\langle\mathbf{p_1}|\mathbf{p_2}\rangle = (2E_{\mathbf{p_1}})^{\frac{1}{2}}(2E_{\mathbf{p_2}})^{\frac{1}{2}}\langle 0|a_{\mathbf{p_1}}a_{\mathbf{p_2}}^\dagger|0\rangle = 2E_{\mathbf{p_1}}(2\pi)^3\delta^3(\mathbf{p_1}-\mathbf{p_2}). \tag{191}$$

The normalizations have been chosen such that combination $E_{\mathbf{p_1}}\delta^3(\mathbf{p_1}-\mathbf{p_2})$ appears on the right hand side . You are now asked to **show** that this combination is actually Lorentz invariant.

The Hamiltonian can be easily written down in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$. Using (101) and (180) and skipping the details, we get

$$H = \int \frac{d^3p}{2(2\pi)^3}E_p(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}a_{\mathbf{p}}^\dagger) = \int \frac{d^3p}{(2\pi)^3}E_p(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]). \tag{192}$$

The second term is the *zero* point energy of all the oscillators and actually gives a divergent contribution to the Hamiltonian. The integrand is $\sim (2\pi)^3\delta^3(0)$ which contribute V at finite volume. The zero point energy is therefore

$$\frac{V}{2}\int \frac{d^3p}{(2\pi)^3}E_p. \tag{193}$$

This is a divergent at large $\mathbf{p}$ since $E_p \sim |\mathbf{p}|$. However, since in the following, we mostly measure the energy *difference*, we will simply discard this infinite contribution to the energy and will take our Hamiltonian to be

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{194}$$

This is commonly known as the *normal ordering*. Given an operator O, we denote the normal ordered operator as $: O :$ by bringing all the creation operators to the left of all the annihilation operators. For example $: a_{\mathbf{p}} a^\dagger_{\mathbf{p}} := a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$. Therefore we say that (194) is obtained by normal ordering the Hamiltonian in (101). Now since $a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ is the *number* operator, the energy of a general Fock state is given by

$$H|\mathbf{p_1}...\mathbf{p_n}\rangle = (E_{\mathbf{p_1}} + ... + E_{\mathbf{p_n}})|\mathbf{p_1}...\mathbf{p_n}\rangle. \tag{195}$$

Further, from T^{0i}, which was computed earlier for the scalar field, we can write down the spatial momentum associated with this states. First

$$P^i = \int d^3x : T^{0i} := \int d^3x : \partial_0\phi\partial^i\phi : . \tag{196}$$

Using (180), we get

$$P^i = \int \frac{d^3p}{(2\pi)^3} p^i a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{197}$$

Hence, for example, $a^\dagger_{\mathbf{p}}|0\rangle$ is a state of momentum p with energy $E_p = (\mathbf{p}^2 + m^2)^{1/2}$.

We conclude this section by pointing out that a general multi-particle state is symmetric under particle exchange as $a^\dagger_{\mathbf{p}}$s commute among each other. So it obeys *Bose-Einstein* statistics. Further, since the spin operator $S^{\mu\nu}$, discussed earlier, is zero for the scalars, the quanta of the scalar field ($a^\dagger_{\mathbf{p}}$) has zero spin.

Complex scalar

Free complex scalar field has the plane wave expansion as

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_{\mathbf{p}} e^{-ipx} + b^\dagger_{\mathbf{p}} e^{ipx}) \tag{198}$$

and

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a^\dagger_{\mathbf{p}} e^{ipx} + b_{\mathbf{p}} e^{-ipx}) \tag{199}$$

The canonical commutation relation (179) gives

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = [b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}). \tag{200}$$

Rest are all zero. We define the vacuum state as

$$a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}}|0\rangle = 0. \tag{201}$$

We generate the Fock space by acting $a^\dagger_{\mathbf{p}}, b^\dagger_p$ on $|0\rangle$. Normal ordered Hamiltonian and momentum are now

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}}(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}), \quad P^i = \int \frac{d^3p}{(2\pi)^3} p^i(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{\mathbf{p}}). \tag{202}$$

The $U(1)$ charge has been constructed earlier. It is given by

$$Q = i\int \phi^\dagger \overleftrightarrow{\partial_0}\, \phi d^3x. \tag{203}$$

Show that in terms of operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$

$$Q = \int \frac{d^3p}{(2\pi)^3}(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} - b^\dagger_{\mathbf{p}} b_{\mathbf{p}}). \tag{204}$$

Note that the ordering of the operator is correct as the charge of the vacuum has to vanish. The $U(1)$ charge carried by a general Fock state is the difference between the number of quanta created by $a^\dagger_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ (integrated over all momenta).

Now we see that the state $a^\dagger_{\mathbf{p}}|0\rangle$ represents a particle with mass m, momentum $\mathbf{p}$, spin zero and charge $Q = 1$. The state $b^\dagger_{\mathbf{p}}|0\rangle$ has all same but the charge is opposite. We call this quanta an *antiparticle* (of $a^\dagger$). We see that in the expansion of ϕ, the coefficient of the *positive* energy solution e^{-ipx} becomes annihilation operator of a particle and the coefficient of the *negative* energy solution e^{ipx} becomes a creation operator of an antiparticle. For a real scalar field Q is zero, since $a_{\mathbf{p}} = b_{\mathbf{p}}$.

<u>Spinors</u>

Consider the Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(i \not{\partial} - m)\Psi. \tag{205}$$

The conjugate momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi)} = i\Psi^\dagger. \tag{206}$$

We require to quantize the half-integer spin fields by imposing *anti-commutation* relation (Otherwise, for example, energy would be unbounded from below).

$$\{\Psi_a(t,\mathbf{x}), \Psi^\dagger_b(t,\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{ab}, \quad a, b = 1, 2, 3, 4.. \tag{207}$$

The expansion of Ψ in terms of plane waves is

$$\begin{aligned} \Psi(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1,2} (a_{\mathbf{p},s} u^s(p) e^{-ipx} + b^\dagger_{\mathbf{p},s} v^s(p) e^{ipx}) \\ \bar{\Psi}(x) &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1,2} (a^\dagger_{\mathbf{p},s} \bar{u}^s(p) e^{ipx} + b_{\mathbf{p},s} \bar{v}^s(p) e^{-ipx}). \end{aligned} \tag{208}$$

The relation in (207) gives

$$\{a_{\mathbf{p},r}, a^\dagger_{\mathbf{q},s}\} = \{b_{\mathbf{p},r}, b^\dagger_{\mathbf{q},s}\} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{r,s}. \tag{209}$$

Rest of the anti-commutators are zero. The vacuum state is defined as

$$a_{\mathbf{p},r}|0\rangle = b_{\mathbf{p},r}|0\rangle = 0. \tag{210}$$

The multi-particle states are obtained by hitting the vacuum with $a^\dagger_{\mathbf{p},r}, b^\dagger_{\mathbf{p},r}$. Since the operators anti-commute among themselves, the multi-particle state is antisymmetric under exchange of two particles (obeying *Fermi-Dirac* statistics). We normalize the one-particle state as before

$$\sqrt{(2E_{\mathbf{p}})}a^\dagger_{\mathbf{p},s}|0\rangle, \sqrt{(2E_{\mathbf{p}})}b^\dagger_{\mathbf{p},s}|0\rangle. \tag{211}$$

The Hamiltonian density is

$$\mathcal{H} = \Pi\partial_0\Psi - \mathcal{L} = \bar{\Psi}(-\gamma^i\partial_i + m)\Psi. \tag{212}$$

Written in terms of annihilation and creation operators, we get the Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_{s=1,2} E_p(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}). \tag{213}$$

We skip the details of the computation here as a similar one for momentum will be done explicitly later. Few comments are now in order. Firstly, for spinors, while doing normal ordering, we put $a^\dagger_{\mathbf{p},s}$ to the left of $a_{\mathbf{p},s}$ and $b^\dagger_{\mathbf{p},s}$ to the left of $b_{\mathbf{p},s}$ adding a *minus* sign each time we exchange a position of any annihilation and creation operator, such as $: a_{\mathbf{p},s}a^\dagger_{\mathbf{p},s} := -a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s}$. Secondly, had we quantized Ψ with usual commutation relations, the $+$ sign in (213) would have been $-$, making the energy unbounded from below. Thirdly, as $\{a^\dagger_{\mathbf{p},r}, a^\dagger_{\mathbf{p},s}\} = 0$, it is impossible to have two quanta of Dirac field in the same state.

The momentum operator is obtained from Noether theorem

$$P^i = \int \frac{d^3p}{(2\pi)^3}\sum_{s=1,2} p^i(a^\dagger_{\mathbf{p},s}a_{\mathbf{p},s} + b^\dagger_{\mathbf{p},s}b_{\mathbf{p},s}). \tag{214}$$

To get used to the manipulations with spinors, let us do this calculation explicitly. First the energy-momentum tensor is (see (92))

$$\begin{aligned} T^{\mu\alpha} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\partial^\alpha\Psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi})}\partial^\alpha\bar{\Psi} - \eta^{\mu\alpha}\mathcal{L} \\ &= i\bar{\Psi}\gamma^\mu\partial^\alpha\psi - \eta^{\mu\alpha}\mathcal{L}. \end{aligned} \tag{215}$$

So

$$P^i = \int d^3x T^{0i} = -i\int d^3x \bar{\Psi}\gamma^0\partial_i\Psi = -i\int d^3x \Psi^\dagger\partial_i\Psi \tag{216}$$

since $\gamma^2 = 1$. Using explicit expansions for Ψ and $\Psi^\dagger$ we get

$$\begin{aligned} P^i \quad &= -i\int \frac{d^3xd^3pd^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\sum_{r,s}(a^\dagger_{\mathbf{q},r}u^{r\dagger}(q)e^{iqx} + b_{\mathbf{q},r}v^{r\dagger}(q)e^{-iqx}) \\ &\quad \times \partial_i(a_{\mathbf{p},s}u^s(p)e^{-ipx} + b^\dagger_{\mathbf{p},s}v^s(p)e^{ipx}) \\ &= \int \frac{d^3xd^3pd^3q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}p^i\sum_{r,s}(a^\dagger_{\mathbf{q},r}u^{r\dagger}(q)e^{iqx} + b_{\mathbf{q},r}v^{r\dagger}(q)e^{-iqx}) \\ &\quad \times (+a_{\mathbf{p},s}u^s(p)e^{-ipx} - b^\dagger_{\mathbf{p},s}v^s(p)e^{ipx}). \end{aligned} \tag{217}$$

Now, integration over x produces a δ function

$$
\begin{aligned}
= \int \frac{d^3p d^3q}{(2\pi)^3\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} p^i \sum_{r,s} \Big(& a^\dagger_{\mathbf{q},r} u^{r\dagger}(q) a_{\mathbf{p},s} u^s(p) \delta^3(\mathbf{q}-\mathbf{p}) \\
& - a^\dagger_{\mathbf{q},r} u^{r\dagger}(q) b^\dagger_{\mathbf{p},s} v^s(p) \delta^3(\mathbf{q}+\mathbf{p}) \\
& - b_{\mathbf{q},r} v^{r\dagger}(q) b^\dagger_{\mathbf{p},s} v^s(p) \delta^3(\mathbf{p}-\mathbf{q}) \\
& + b_{\mathbf{q},r} v^{r\dagger}(q) a_{\mathbf{p},s} u^s(p) \delta^3(\mathbf{p}+\mathbf{q}) \Big)
\end{aligned} \tag{218}
$$

Further, integrating over q

$$
\begin{aligned}
= \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}} p^i \sum_{r,s} \Big(& + a^\dagger_{\mathbf{p},r} u^{r\dagger}(p) a_{\mathbf{p},s} u^s(p) - a^\dagger_{\mathbf{p},r} u^{r\dagger}(-p) b^\dagger_{\mathbf{p},s} v^s(p) \\
& - b_{\mathbf{p},r} v^{r\dagger}(p) b^\dagger_{\mathbf{p},s} v^s(p) + b_{\mathbf{p},r} v^{r\dagger}(-p) a_{\mathbf{p},s} u^s(p) \Big)
\end{aligned} \tag{219}
$$

We now use the following identities (These follow easily from our previous discussion on Dirac equation. **Prove** the identities.)

$$
\begin{aligned}
& u^{r\dagger}(p) u^s(p) = 2E_{\mathbf{p}} \delta^{rs}, \quad v^{r\dagger}(p) v^s(p) = 2E_{\mathbf{p}} \delta^{rs} \\
& v^{r\dagger}(p) u^s(-p) = 0, \quad u^{r\dagger}(p) v^s(-p) = 0
\end{aligned} \tag{220}
$$

to get

$$
P^i = \int \frac{d^3p}{(2\pi)^3} \sum_r p^i (a^\dagger_{\mathbf{p},r} a_{\mathbf{p},r} - b_{\mathbf{p},r} b^\dagger_{\mathbf{p},r}). \tag{221}
$$

The normal ordered P^i is then,

$$
P^i = \int \frac{d^3p}{(2\pi)^3} \sum_r p^i (a^\dagger_{\mathbf{p},r} a_{\mathbf{p},r} + b^\dagger_{\mathbf{p},r} b_{\mathbf{p},r}). \tag{222}
$$

Angular momentum is a Noether charge and has contributions from orbital and spin part. **Show** that the spin part is

$$
S^i = \frac{1}{2} \int d^3x \Psi^\dagger \Sigma^i \Psi \tag{223}
$$

where

$$
\Sigma^i = \begin{bmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix} \tag{224}
$$

Further use the mode expansions to check, in the rest frame,

$$
\begin{aligned}
S^z a^\dagger_{\mathbf{p},1}|0\rangle &= \frac{1}{2} a^\dagger_{\mathbf{p},1}|0\rangle, \quad S^z a^\dagger_{\mathbf{p},2}|0\rangle = -\frac{1}{2} a^\dagger_{\mathbf{p},2}|0\rangle. \\
S^z b^\dagger_{\mathbf{p},1}|0\rangle &= -\frac{1}{2} b^\dagger_{\mathbf{p},1}|0\rangle, \quad S^z b^\dagger_{\mathbf{p},2}|0\rangle = \frac{1}{2} b^\dagger_{0,2}|0\rangle
\end{aligned} \tag{225}
$$

Finally the conserved charge following from the symmetry $\Psi \to e^{i\alpha}\Psi$ is

$$
Q = \int \frac{d^3p}{(2\pi)^3} \sum_r (a^\dagger_{\mathbf{p},r} a_{\mathbf{p},r} - b^\dagger_{\mathbf{p},r} b_{\mathbf{p},r}). \tag{226}
$$

So

$$
\begin{aligned}
Qa^\dagger_{\mathbf{p},1}|0\rangle &= +a^\dagger_{\mathbf{p},1}|0\rangle, \quad Qa^\dagger_{\mathbf{p},2}|0\rangle = +a^\dagger_{\mathbf{p},2}|0\rangle. \\
Qb^\dagger_{\mathbf{p},1}|0\rangle &= -b^\dagger_{\mathbf{p},1}|0\rangle, \quad Qb^\dagger_{\mathbf{p},2}|0\rangle = -b^\dagger_{\mathbf{p},2}|0\rangle.
\end{aligned} \tag{227}
$$

The state created by $a^\dagger_{\mathbf{p},s}$ are called particle and the one created by $b^\dagger_{\mathbf{p},s}$ are called an antiparticle state. In fact, in electrodynamics, we identify $a^\dagger_{\mathbf{p},s}|0\rangle$ as an electron and $b^\dagger_{\mathbf{p},s}|0\rangle$ as a positron.

<u>Electromagnetic field</u>: Lorentz (or covariant) gauge

Consider the Lagrangian

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{228}
$$

The momentum conjugate to A_i is

$$
\Pi^i = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_i)}. \tag{229}
$$

In order to quantize the theory, we would like to impose an equal time commutation relation

$$
[A^i(\mathbf{x},t), \Pi^j(\mathbf{y},t)] = -\delta^{ij}\delta^3(\mathbf{x}-\mathbf{y}). \tag{230}
$$

Note that the momentum conjugate to $A^i = \Pi^i = -\Pi_i$. In a Lorentz covariant theory, one would like to make the above commutation relation covariant

$$
[A^\mu(\mathbf{x},t), \Pi^\nu(\mathbf{y},t)] = i\eta^{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y}). \tag{231}
$$

However, owing to the antisymmetry of $F_{\mu\nu}$, it is easy to check that $\Pi^0 = 0$. Therefore, it can not have non-trivial commutation relation with A^0.

To overcome this problem, we modify the Lagrangian as follows:

$$
\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{232}
$$

Note that, we have now broken the gauge invariance by adding the extra piece. Conjugate momenta is defined as

$$
\Pi^\mu = \frac{\partial L}{\partial(\partial_0 A_\mu)}, \tag{233}
$$

so that $\Pi^i = -F^{0i} = E^i$ and $\Pi^0 = -\partial_\mu A^\mu$. We can therefore implement the canonical commutation relation. The equations of motion that follow from $\mathcal{L}'$ are

$$
\Box A^\mu = 0. \tag{234}
$$

Hence A^μ has a plane wave expansion

$$
A_\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{\lambda=0}^{3} \left(\epsilon_\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}e^{-ipx} + \epsilon^*_\mu(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}e^{ipx}\right) \tag{235}
$$

with $E_p = \mathbf{p}$ and $p^2 = 0$. The $\epsilon_\mu(\mathbf{p}, \lambda)$ are the four independent polarization vectors. The Lorentz transformation properties of A_μ is carried by these vectors. We will choose a frame where $p^\mu = (p, 0, 0, p)$ and take these linearly independent vectors as

$$\begin{aligned}\epsilon^\mu(\mathbf{p}, 0) &= (1, 0, 0, 0)\\ \epsilon^\mu(\mathbf{p}, 1) &= (0, 1, 0, 0)\\ \epsilon^\mu(\mathbf{p}, 2) &= (0, 0, 1, 0)\\ \epsilon^\mu(\mathbf{p}, 3) &= (0, 0, 0, 1).\end{aligned} \tag{236}$$

Note that $\epsilon^\mu(\mathbf{p}, 1), \epsilon^\mu(\mathbf{p}, 2)$ are transverse to the direction of propagation. In a general frame ϵ^μ can be found by a Lorentz transformation.

From (231), it follows that

$$[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = -(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q})\eta_{\lambda\lambda'} \tag{237}$$

and the rest are zero.

Notice that the norm of a single particle state is given by

$$\langle \mathbf{p}, \lambda | \mathbf{p}, \lambda\rangle = (2E_\mathbf{p})\langle 0|a_{\mathbf{p},\lambda}a^\dagger_{\mathbf{p},\lambda}|0\rangle = -\eta_{\lambda\lambda}2E_\mathbf{p}V, \tag{238}$$

where V is the volume cutoff. We now see $\lambda = 0$ state has negative norm. This puts the probabilistic interpretation of the state in trouble. In order to get rid of such states, we impose the Lorentz gauge condition. We say that for any two physical states $|P\rangle$ and $|P'\rangle$, the following relation holds

$$\langle P'|\partial_\mu A^\mu|P\rangle = 0. \tag{239}$$

Note that we can break

$$\partial_\mu A^\mu = (\partial_\mu A^\mu)^+ + (\partial_\mu A^\mu)^- \tag{240}$$

where the $+$ part contains $a_{\mathbf{p},\lambda}$ and the $-$ part contains $a^\dagger_{\mathbf{p},\lambda}$. Since $(\partial_\mu A^\mu)^- = (\partial_\mu A^\mu)^{+\dagger}$, (239) is equivalent to

$$(\partial_\mu A^\mu)^+|P\rangle = 0. \tag{241}$$

Since the above is a Lorentz invariant constraint, we can explore its meaning in any frame. Let us choose a frame where $p^\mu = (E_p, 0, 0, E_p)$. Then

$$\begin{aligned}(\partial_\mu A^\mu)^+ &= -i\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_\lambda p_\mu \epsilon^\mu(\mathbf{p}, \lambda)a_{\mathbf{p},\lambda}e^{-ipx}\\ &= -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}(a_{\mathbf{p},0} - a_{\mathbf{p},3})e^{-ipx}\end{aligned} \tag{242}$$

Now if (241) has to hold, then $(a_{\mathbf{p},0} - a_{\mathbf{p},3})|P\rangle = 0$. To go from the first to the second line, we used the fact that in the chosen frame $p_\mu\epsilon^\mu(\mathbf{p}, \lambda) = 0$ for $\lambda = 1, 2$.

Let us now consider the first excited state $(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle$.

$$\begin{aligned}a_{\mathbf{k},0}(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle &= -(2\pi)^3\delta^3(\mathbf{p} - \mathbf{k})\\ a_{\mathbf{k},3}(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle &= -(2\pi)^3\delta^3(\mathbf{p} - \mathbf{k}).\end{aligned} \tag{243}$$

Therefore, $(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle$ satisfies the physical state condition. However, $(a^\dagger_{\mathbf{p},0} + a^\dagger_{\mathbf{p},3})|0\rangle$ does not. But note that scalar products of $(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle$ with states created by $a^\dagger_{\mathbf{p},1}, a^\dagger_{\mathbf{p},2}$ are zero by virtue of their commutation relations. It is easy to check that this state is also orthogonal to itself. Therefore, $(a^\dagger_{\mathbf{p},0} - a^\dagger_{\mathbf{p},3})|0\rangle$ is a *null* state and decouples. The physical states are only formed by the transverse excitations $a^\dagger_{\mathbf{p},1}, a^\dagger_{\mathbf{p},2}$!

4 Perturbations

Particless are not free. They interact and we need a framework to describe their interaction. The types of interactions depend on the kind of fields we are considering. Fro example, in the case of real scalar fields, interactions could be of the form term

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4, \tag{244}$$

with the corresponding Hamiltonian $\mathcal{H}_{\text{int}} = \lambda/4!\phi^4$,. Here, λ is known as the coupling constant. We can also consider a term

$$-\,qA_\mu\bar{\Psi}\gamma^\mu\Psi \tag{245}$$

as in (177) as the interaction term in quantum electrodynamics. If λ or q is small, we may treat the respective interaction terms small, and, hope to compute their effects by doing a perturbation expansion around the free field configurations. This is what we will learn in this and next few sections.

What we expect to happen as we turn on interaction? A hint can be obtained from (244). Expanding ϕ^4 in terms of creation and annihilation operators of the free theory, we get terms like $a^\dagger a^\dagger a^\dagger a^\dagger$, $a^\dagger a^\dagger a^\dagger a$ and others. This tells us that as we switch on interactions, there are creations and annihilation of particles, changing the particle number.

Evolution operator

In presence of interactions, the field $\phi(x)$ satisfies a complicated equation of motion* and an exact form of solution is hard to obtain. We need to set up a perturbation expansion. To proceed, first we would like to relate ϕ to a field ϕ_I whose evolution is determined by the *free* part of the Hamiltonian. For ϕ_I, that means, if we know $\phi_I(t,\mathbf{x})$ at some fixed time t_0, at a later time it is given by

$$\phi_I(t,\mathbf{x}) = e^{iH_0(t-t_0)}\phi_I(t_0,\mathbf{x})e^{-iH_0(t-t_0)}. \tag{246}$$

Since $\phi_I(t,\mathbf{x})$ evolves as a free field, it has standard plane wave expansion

$$\phi_I(t,\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}(a_{\mathbf{p}}e^{-ipx} + a^\dagger_{\mathbf{p}}e^{ipx}). \tag{247}$$

We can now relate ϕ with ϕ_I as follows. Let $\phi(t_0,\mathbf{x}) = f(\mathbf{x})$. We *choose* $\phi_I(t,\mathbf{x})$ such that at $t = t_0$ it is given by $f(\mathbf{x})$. Then

$$\phi(t,\mathbf{x}) \;=\; e^{iH(t-t_0)}\phi(t_0,\mathbf{x})e^{-iH(t-t_0)}$$

*We are going to concentrate on scalar field here. But the construction goes through for the other fields as well

$$
\begin{aligned}
&= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\phi(t_0,\mathbf{x})e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\
&= U^\dagger(t,t_0)\phi_I(t,\mathbf{x})U(t,t_0). \qquad (248)
\end{aligned}
$$

Here, we have defined

$$
U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}, \qquad (249)
$$

which is commonly known as the *evolution* operator. Note that it is not same as $e^{-iH_{\rm int}(t-t_0)}$ as H_0 and $H_{\rm int}$ generally do not commute. It is easy to check the following relation however holds:

$$
\begin{aligned}
i\frac{\partial U}{\partial t} &= e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\
&= e^{iH_0(t-t_0)}H_{\rm int}\, e^{-iH_0(t-t_0)}U(t,t_0) \\
&= H_I U(t,t_0). \qquad (250)
\end{aligned}
$$

The last line *defines* H_I for us. With the boundary condition $U(t_0,t_0)=1$, the solution is

$$
U(t,t_0) = T\Big\{e^{-i\int_{t_0}^t dt' H_I(t')}\Big\}. \qquad (251)
$$

Here $T\{..\}$ means the following. If we Taylor expand the exponential, the terms in the Taylor series are *time ordered*. Time ordering among two operators $\phi(x)$ and $\phi(y)$ is given by

$$
\begin{aligned}
T\Big\{\phi(x)\phi(y)\Big\} &= \phi(x)\phi(y) \qquad \text{if } \ x^0 > y^0 \\
&= \phi(y)\phi(x) \qquad \text{if } \ y^0 > x^0. \qquad (252)
\end{aligned}
$$

That the form (251) is a solution can be directly checked. To see this, we first notice that the equation (250) has a solution

$$
U(t,t_0) = 1 - i\int_{t_0}^t dt_1 H_I(t_1)U(t_1,t_0), \qquad (253)
$$

This expression is not useful because of the $U(t_1,t_0)$ term on the right hand side. We can get a solution iteratively if we replace $U(t_1,t_0)$ again by (253)

$$
U(t,t_0) = 1 - i\int_{t_0}^t H_I(t_1)\Big(1 - i\int_{t_0}^{t_1} H_I(t_2)U(t_2,t_0)dt_2\Big)dt_1. \qquad (254)
$$

Carrying this out repeatedly, we obtain

$$
U(t,t_0) = 1 + \sum_{n=1}^{\infty}(-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n H_I(t_1)H_I(t_2)...H_I(t_n). \qquad (255)
$$

The above solution can be re-written using the time-ordering defined before. Take, for example, $n=2$ term in (251) Note that

$$
\frac{1}{2}\int_{t_0}^t dt_1 \int_{t_0}^t dt_2 T\Big\{H_I(t_1)H_I(t_2)\Big\} \qquad (256)
$$

$$
= \frac{1}{2}\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) \qquad (257)
$$

$$
+ \frac{1}{2}\int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 H_I(t_2)H_I(t_1) \qquad (258)
$$

The first term is obtained for $t_1 > t_2$ and the second for $t_2 > t_1$. Now we interchange the dummy variables t_1 and t_2 in the second integral. Adding two contributions, we reach at

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2). \tag{259}$$

This is the $n = 2$ term in (255). One can repeat this exercise for other values of n as well.

We now want to compute the n-point *Green's function*. The reason for that will be discussed later. The n-point Green's function is defined as

$$\langle 0|\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle \tag{260}$$

when x_is are time ordered, meaning $x_1^0 > x_2^0 > ...x_n^0$. We can re-write this as

$$\langle 0|U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_0)U^\dagger(t_2,t_0)\phi_I(x_2)U(t_2,t_0)...U^\dagger(t_n,t_0)\phi_I(x_n)U(t_n,t_0)|0\rangle. \tag{261}$$

However since $U^\dagger(t_2,t_0) = U(t_0,t_2)$ and $U(t_1,t_0)U(t_0,t_2) = U(t_1,t_2)$ (**Show**), we get

$$\langle 0|U^\dagger(t_1,t_0)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)U(t_2,t_3)...U(t_{n-1},t_n)\phi_I(x_n)U(t_n,t_0)|0\rangle. \tag{262}$$

We now introduce a new variable t such that $t \gg t_1 > t_2 > ... > t_n \gg -t$ and use $U(t_n,t_0) = U(t_n,-t)U(-t,t_0)$ to re-write the above as

$$\langle 0|U^\dagger(t,t_0)[U(t,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)U(t_2,t_3)...U(t_{n-1},t_n)\phi_I(x_n)U(t_n,-t)]U(-t,t_0)|0\rangle. \tag{263}$$

Now notice that the terms inside the square brackets are already time ordered. So we can write

$$\begin{aligned}
[...] &= T\Big\{\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)U(t,t_1)U(t_1,t_2)...U(t_n,-t)\Big\}. \\
&= T\Big\{\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)U(t,-t)\Big\} \\
&= T\Big\{\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)e^{-i\int_{-t}^{t} H_I(t')dt'}\Big\}.
\end{aligned} \tag{264}$$

In the last line we used the property $T\{A(x_1)A(x_2)...T\{B(x_n)B(x_{n+1}\}\} = T\{A(x_1)Ax_2)...B(x_n)B(x_{n+1})\}$. Since this is true for arbitrary t_0, we can choose $t_0 = -t$ and then take $t \to \infty$ to reduce (263) to

$$\langle 0|U^\dagger(\infty,-\infty)T\Big\{\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)e^{-i\int_{-\infty}^{t} H_I(\infty)dt'}\Big\}|0\rangle. \tag{265}$$

The term $\langle 0|U^\dagger(\infty,-\infty) = (U(\infty,-\infty)|0\rangle)^\dagger$ and $U(\infty,-\infty)$ simply evolves a vacuum state from infinite past to infinite future. Assuming that the vacuum state is stable, applying $U(\infty,-\infty)$ we get again a vacuum state but it may differ by a phase factor (in quantum mechanics, two state vectors differing by a phase factor are physically same.). So generally, we have

$$U(\infty,-\infty)|0\rangle = e^{i\alpha}|0\rangle. \tag{266}$$

So

$$\langle 0|U(\infty,-\infty)|0\rangle = \langle 0|e^{i\alpha}|0\rangle = e^{i\alpha} \tag{267}$$

giving

$$e^{i\alpha} = \langle 0|T\Big\{e^{-i\int_{-\infty}^{\infty} dt' H_I(t')}\Big\}|0\rangle. \tag{268}$$

With these manipulations, we finally arrive at an important formula:

$$\langle 0|T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|0\rangle = \frac{\langle 0|T\{\phi_I(x_1)...\phi_I(x_n)e^{-i\int_{-\infty}^{\infty} dt' H_I(t')}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-\infty}^{\infty} dt' H_I(t')}\}|0\rangle}. \tag{269}$$

Scattering matrix and the LSZ reduction formula.

In Schrodinger picture, a state vector, at an initial time T_i, is an eigen vector of a set of commuting operators with eigen values written collectively as a. We denote the state by $|a(T_i)\rangle$. Similarly, at final time T_f, we denote eigen state as $|b(T_f)\rangle$. We know $|a(T_i)\rangle$ evolves as $e^{-iH(t-T_i)}|a(T_i)\rangle$. So the amplitude for a process for which the initial state $|a(T_i)\rangle$ goes to the final state $|b(T_f)\rangle$ is given by

$$\langle b(T_f)|e^{-iH(T_f-T_i)}|a(T_i)\rangle. \tag{270}$$

In the limit when $T_f - T_i \to \infty$, $e^{-iH(T_f-T_i)}$ is called the S-matrix. We can easily check that $S^\dagger S = 1$. If $|a\rangle$ is the initial state normalized to 1 and $|n\rangle$ is a complete set of states, then $|a\rangle$ going to $|n\rangle$ summed over all n must be one. That is

$$\sum_n |\langle n|S|a\rangle|^2 = 1. \tag{271}$$

This, in turn, means

$$\sum_n \langle a|S^\dagger|n\rangle\langle n|S|a\rangle = \langle a|S^\dagger S|a\rangle, \tag{272}$$

where we have used the completeness condition on n. Comparing the last two equations, we conclude that $S^\dagger S = 1$. Since in field theory operators like ϕ and others explicitly depend on time, it is convenient to work in the Heisenberg representation. Operators in the two representations are related by $A_H(t) = e^{iHt}Ae^{-iHt}$, while the state vector in Heisenberg representation is time independent and related to the one in the Schrodinger representation as $|a, T_i\rangle_H = e^{iHt}|a(t)\rangle$. T_i here is just a label. Now connecting this with our previous discussion, we see

$$|a, T_i\rangle_H = e^{iHT_i}|a\rangle, \quad |b, T_f\rangle_H = e^{iHT_f}|b\rangle. \tag{273}$$

Therefore,

$$\langle b|S|a\rangle = \langle b, T_f|a, T_i\rangle_H. \tag{274}$$

For notational simplicity we will omit the subscript H henceforth.

Let us consider a general S-matrix element written in Heisenberg representation

$$\langle \mathbf{p_1}, \mathbf{p_2}, T_f|\mathbf{k_1}, \mathbf{k_2}, T_i\rangle. \tag{275}$$

We are considering here a single species of real scalar particle, so that the states are labeled by only their momenta. We will, at the end, be interested in the limits $T_f \to \infty, T_i \to -\infty$. Our task will be to relate this matrix element with a 4-point Green's function. Further, we will consider the case when $\mathbf{k_1}, \mathbf{k_2} \neq \mathbf{p_1}, \mathbf{p_2}$.

First notice that for *free* fields, as in (182), we can express a and $a^\dagger$ in terms of fields and its derivatives. This is not true when the interactions are turned on. However, in the limit

$t \to -\infty$, we expect the theory to reduce to a free theory. Incoming particles are infinitely far away from each other and interaction decreases fast with the distance (not in QCD though). So, for $t \to -\infty$, $\phi \to \phi_{in}$. Similarly, when $t \to \infty$, $\phi \to \phi_{out}$. We now consider (182), with ϕ_{in} playing the role of free field ϕ. Though the integrands in (182) are time dependent but the results after the integrations are *not*. So we can perform the integrals at $t \to -\infty$ as well. Therefore,

$$\begin{aligned}\sqrt{2E_{\mathbf{k}}}a_{\mathbf{k}}^{\dagger(in)} &= -i\int_{t\to-\infty} d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi_{in} \\ &= -i \text{ Lt}_{t\to-\infty}\int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi. \end{aligned} \tag{276}$$

Similarly, for $t \to \infty$,

$$\sqrt{2E_{\mathbf{k}}}a_{\mathbf{k}}^{\dagger(out)} = -i \text{ Lt}_{t\to\infty}\int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi_{out}. \tag{277}$$

We can use the above relations to write

$$\begin{aligned}&\langle \mathbf{p_1}, \mathbf{p_2}, T_f|\mathbf{k_1}, \mathbf{k_2}, T_i\rangle \\ &= \sqrt{2E_{\mathbf{k_1}}}\langle \mathbf{p_1}, \mathbf{p_2}, T_f|a_{\mathbf{k_1}}^{\dagger(in)}|\mathbf{k_2}, T_i\rangle \\ &= -i \text{ Lt}_{t\to-\infty}\int d^3x e^{-ik_1x}\langle \mathbf{p_1}, \mathbf{p_2}, T_f| \overleftrightarrow{\partial}_0 \phi|\mathbf{k_2}, T_i\rangle. \end{aligned} \tag{278}$$

Further, since for an integrable function $f(x)$,

$$\text{Lt}_{t\to\infty}\int d^3x f(x) - \text{Lt}_{t\to-\infty}\int d^3x f(x) = \int_{-\infty}^{\infty} dt\frac{\partial}{\partial t}\int d^3x f(x), \tag{279}$$

we can write

$$\begin{aligned}\sqrt{2E_{\mathbf{k}}}(a_{\mathbf{k}}^{\dagger(in)} - a_{\mathbf{k}}^{\dagger(out)}) &= i\int d^4x\partial_0(e^{-ikx} \overleftrightarrow{\partial}_0 \phi) \\ &= i\int d^4x\ e^{-ikx}(\Box + m^2)\phi(x). \end{aligned} \tag{280}$$

Therefore,

$$\sqrt{2E_{\mathbf{k_1}}}\langle \mathbf{p_1}, \mathbf{p_2}, T_f|a_{\mathbf{k_1}}^{\dagger(in)} - a_{\mathbf{k_1}}^{\dagger(out)}|\mathbf{k_2}, T_i\rangle = i\int d^3x\ e^{-ik_1x}(\Box + m^2)\langle \mathbf{p_1}, \mathbf{p_2}, T_f|\phi(x)|\mathbf{k_2}, T_i\rangle. \tag{281}$$

But when $\mathbf{k_1} \neq \mathbf{p_1}$, or $\mathbf{p_2}$, $a_{\mathbf{k_1}}^{\dagger(out)}$ acting on the out state $\langle \mathbf{p_1}, \mathbf{p_2}, T_f|$ will give zero. So

$$\langle \mathbf{p_1}, \mathbf{p_2}, T_f|\mathbf{k_1}\mathbf{k_2}, T_i\rangle = i\int d^3x\ e^{-ik_1x}(\Box + m^2)\langle \mathbf{p_1}, \mathbf{p_2}, T_f|\phi(x)|\mathbf{k_2}, T_i\rangle. \tag{282}$$

We can repeat this exercise now to remove $\mathbf{p_1}$ from the final state.

$$\langle \mathbf{p_1}, \mathbf{p_2}, T_f|\phi(x)|\mathbf{k_2}, T_i\rangle = \sqrt{2E_{\mathbf{p_1}}}\langle \mathbf{p_2}, T_f|a_{\mathbf{p_1}}^{(out)}\phi(x)|\mathbf{k_2}, T_i\rangle. \tag{283}$$

However,

$$\sqrt{2E_{\mathbf{p_1}}}\langle \mathbf{p_2}, T_f|a^{(out)}\phi(x)|\mathbf{k_2} = \sqrt{2E_{p_1}}\langle \mathbf{p_2}, T_f|T\{a_{\mathbf{p_1}}^{(out)}\phi(x) - a_{\mathbf{p_1}}^{(in)}\phi(x)\}|\mathbf{k_2}, T_i\rangle. \tag{284}$$

To see that, we simply take the hermitian conjugate of (276).

$$\sqrt{2E_{p_1}}a_{\mathbf{p_1}}^{(in)} == i\ \mathrm{Lt}_{y^0\to-\infty}\int d^3y e^{ip_1y}\overleftrightarrow{\partial_0}\ \phi(y). \tag{285}$$

Therefore, due to the time ordering in (284), $a_{\mathbf{p_1}}^{(in)}$ will sit after $\phi(x)$. It will then hit the *in* state to produce zero. Consequently, we get back (283). Hence,

$$\sqrt{2E_{\mathbf{p_1}}}(a_{\mathbf{p_1}}^{(out)} - a_{\mathbf{p_1}}^{(in)}) = i\int d^4y\ e^{ip_1y}(\Box_y + m^2)\phi(y). \tag{286}$$

So

$$\langle\mathbf{p_1},\mathbf{p_2},T_f|\phi(x)|\mathbf{k_2},T_i\rangle = i\int d^4y e^{ip_1y}(\Box_y + m^2)\langle\mathbf{p_2},T_f|T\{\phi(y)\phi(x)\}|\mathbf{K_2},T_i\rangle, \tag{287}$$

and

$$\langle\mathbf{p_1},\mathbf{p_2},T_f|\mathbf{k_1}\mathbf{k_2},T_i\rangle = (i)^2\int d^4xe^{-ik_1x}(\Box_x+m^2)\int d^4ye^{ip_1x}(\Box_y+m^2)\langle\mathbf{p_2},T_f|T\{\phi(y)\phi(x)\}|\mathbf{K_2},T_i\rangle. \tag{288}$$

We can iterate this procedure further to finally arrive at

$$\begin{aligned}&\langle\mathbf{p_1},\mathbf{p_2},T_f|\mathbf{k_1},\mathbf{k_2},T_i\rangle =\\ &\quad(i)^4\int\prod_{i=1,2}d^4x_i\prod_{j=1,2}d^4y_je^{ip_1y_1+ip_2y_2-ik_1x_1-ik_2x_2}(\Box_{y_1}+m^2)(\Box_{y_2}+m^2)\\ &\quad\times(\Box_{x_1}+m^2)(\Box_{x_2}+m^2)\langle 0|T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|0\rangle.\end{aligned} \tag{289}$$

Exactly similar analysis relates $\langle\mathbf{p_1},...\mathbf{p_n},T_f|\mathbf{k_1},..\mathbf{k_n},T_i\rangle$ to a n-point Green's function. This formula is known as the LSZ reduction formula, named after three German physicists Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann. This important formula connects the theory with the experiment. While the left hand side is what we determine through experiments, we derive the right hand side starting from a Lagrangian!

In Schrodinger picture

$$\langle\mathbf{p_1},\mathbf{p_2},T_f|\mathbf{k_1},\mathbf{k_2},T_i\rangle = \langle\mathbf{p_1},\mathbf{p_2}|S|\mathbf{k_1},\mathbf{k_2}\rangle. \tag{290}$$

It is convenient to write the S-matrix as $S = 1 + iT$. Then,

$$\langle\mathbf{p_1},\mathbf{p_2}|S|\mathbf{k_1},\mathbf{k_2}\rangle = \langle\mathbf{p_1},\mathbf{p_2}|iT|\mathbf{k_1},\mathbf{k_2}\rangle. \tag{291}$$

We have restricted to the case where no initial and final momenta are same. So the matrix element of the identity operator vanishes. The non-trivial part of the S-matrix is in the iT. So we write,

$$\begin{aligned}&\langle\mathbf{p_1},\mathbf{p_2}|iT|\mathbf{k_1},\mathbf{k_2}\rangle =\\ &\quad(i)^4\int\prod_{i=1,2}d^4x_i\prod_{j=1,2}d^4y_je^{ip_1y_1+ip_2y_2-ik_1x_1-ik_2x_2}(\Box_{y_1}+m^2)(\Box_{y_2}+m^2)\\ &\quad\times(\Box_{x_1}+m^2)(\Box_{x_2}+m^2)\langle 0|T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|0\rangle.\end{aligned} \tag{292}$$

Let us now use a notation for the Green's function. The N-point Green's function will be written as

$$G(x_1, ..., x_n) = \langle 0|T\{\phi(x_1)....\phi(x_n)\}|0\rangle \tag{293}$$

It's Fourier transform $\tilde{G}$ is

$$G(x_1, ..., x_n) = \int \prod_{i=1}^{n} \Big(\frac{d^4 k_i}{(2\pi)^4}\Big) e^{-i\sum_{i=1}^{n} x_i k_i}\ \tilde{G}(k_1, ..., k_n). \tag{294}$$

Now note that

$$(\Box_{x_1} + m^2) G(x_1, .., x_4) = -\int \prod_{i=1}^{4} \Big(\frac{d^4 k_i}{(2\pi)^4}\Big)(k_1^2 - m^2) e^{-i\sum_{i=1}^{4} x_i k_i} \tilde{G}(k_1, .., k_4). \tag{295}$$

So, we can re-write (292) as (**Show**)

$$\begin{aligned}\Big(\prod_{i=1}^{2} \frac{i}{k_i^2 - m^2}\Big)\Big(\prod_{j=1}^{2} \frac{i}{p_j^2 - m^2}\Big)\langle \mathbf{p_1}, \mathbf{p_2}|iT|\mathbf{k_1}, \mathbf{k_2}\rangle = \\ \int \prod_{i=1,2} d^4 x_i \prod_{j=1,2} d^4 y_j\ e^{ip_1 y_1 + ip_2 y_2 - ik_1 x_1 - ik_2 x_2} G(x_1, x_2, x_3, x_4).\end{aligned} \tag{296}$$

Few comments are in order. The denominator of the left hand side contains factors like $p_i^2 - m^2$. For a physical particle, it is zero. The meaning of such factors is that we must first compute the right hand side *without* making use of the relation between $(p^0)^2$ and $\mathbf{p}^2$. After computing, once we use the relation between $(p^0)^2$ and $\mathbf{p}^2$, the expression develops poles of the form $1/(p_i^2 - m^2)$. These factors then cancel similar poles arising from the left hand side. It will be more clear in later sections where we do some explicit calculations.

Propagator

In order to evaluate the right hand side of (269), we first need to compute $\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle$, known as the Feynman propagator. To avoid cluttering of indices, from now on, we will leave the subscript I. It should be understood that we are working with fields which evolve in time with the free Hamiltonian in time.

In order to compute the propagator, it is convenient to separate out ϕ in two parts:

$$\phi(x) = \phi^+(x) + \phi^-(x). \tag{297}$$

Here the ϕ^+ contains the annihilation operator $a_{\mathbf{p}}$, while ϕ^- contains the creation operator $a_{\mathbf{p}}^\dagger$. In particular, $\phi^+(x)|0\rangle = \langle 0|\phi^-(x) = 0$.

When $x^0 > y^0$, we see that

$$\begin{aligned} T\{\phi(x)\phi(y)\} &= \phi(x)\phi(y) \\ &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + [\phi^+(x), \phi^-(y)] \\ &= :\phi(x)\phi(y): + [\phi^+(x), \phi^-(y)] \end{aligned} \tag{298}$$

For $y^0 > x^0$, we get

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + [\phi^+(y), \phi^-(x)]. \tag{299}$$

Combining last two equations, we write

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + D(x-y), \tag{300}$$

where

$$D(x-y) = \theta(x^0-y^0)[\phi^+(x), \phi^-(y)] + \theta(y^0-x^0)[\phi^+(y), \phi^-(x)]. \tag{301}$$

The theta function is 1 when the argument is positive. Otherwise it is zero. Note that since $D(x-y)$ is a commutator between ϕ^+ and ϕ^-, it is a number, not an operator. Further, since vacuum expectation value of a normal ordered product vanishes, $D(x-y)$ is just the Feynman propagator. We write

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = D(x-y). \tag{302}$$

The commutator can be computed using $[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$ with the result

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}}\Big(\theta(x^0-y^0)e^{-ip(x-y)} + \theta(y^0-x^0)e^{ip(x-y)}\Big) \tag{303}$$

It is convenient to re-write $D(x-y)$ as follows:

$$D(x-y) = \ \mathrm{Lt}_{\epsilon\to 0^+} \ \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}. \tag{304}$$

To see that the last two expressions are indeed same, we start from (304). We re-write

$$\begin{aligned} D(x-y) \quad &= \quad \mathrm{Lt}_{\epsilon\to 0^+} \ \int \frac{d^3p}{(2\pi)^3}e^{-\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\frac{i}{(p^0)^2-E^2_{\mathbf{p}}+i\epsilon}e^{-ip^0(x^0-y^0)} \\ &= \quad \mathrm{Lt}_{\epsilon\to 0^+} \ \int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int_{-\infty}^{\infty}\frac{dp^0}{4\pi E_{\mathbf{p}}} \\ &\quad \times\Big(\frac{i}{p^0-(E_{\mathbf{p}}-i\epsilon)} - \frac{i}{p^0+(E_{\mathbf{p}}-i\epsilon)}\Big)e^{-ip^0(x^0-y^0)}. \end{aligned} \tag{305}$$

To evaluate the p^0 integrals, we use a result from the complex analysis

$$\mathrm{Lt}_{\epsilon\to 0^+}\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\frac{e^{-i\xi(x^0-y^0)}}{\xi+i\epsilon} = -i\theta(x^0-y^0). \tag{306}$$

Taking $\tilde{p}^0 = p^0 - E_{\mathbf{p}}$, we get

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\frac{i}{p^0-(E_{\mathbf{p}}-i\epsilon)}e^{-ip^0(x^0-y^0)} \\ &= \frac{i}{2\pi}\int d\tilde{p}^0\frac{e^{-i(\tilde{p}^0+E_p)(x^0-y^0)}}{\tilde{p}^0+i\epsilon} \\ &= \theta(x^0-y^0)e^{-iE_{\mathbf{p}}(x^0-y^0)}, \end{aligned} \tag{307}$$

and taking $\tilde{p}^0 = -p^0 - E_{\mathbf{p}}$, we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{i}{p^0 + (E_{\mathbf{p}} - i\epsilon)} e^{-ip^0(x^0-y^0)} \\
&= \frac{i}{2\pi} \int_{\infty}^{-\infty} d\tilde{p}^0 \frac{e^{i(\tilde{p}^0+E_p)(x^0-y^0)}}{\tilde{p}^0 + i\epsilon} \\
&= -\theta(y^0 - x^0) e^{iE_{\mathbf{p}}(x^0-y^0)}. \qquad (308)
\end{aligned}$$

First, substituting these back to (305), and then changing the integration variable $\mathbf{p} \to -\mathbf{p}$ in the second expression, we get (303).

Before we end this subsection, here are a few comments. The Feynman propagator in momentum space can be read off from (303).

$$\tilde{D}(p) = \frac{i}{p^2 - m^2 + i\epsilon}. \qquad (309)$$

Further

$$(\Box_x + m^2) D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} (-p^2 + m^2) e^{-ip(x-y)} = -i\delta^4(x-y). \qquad (310)$$

Wick's theorem

Wick's theorem allows us to re-express time ordered product of operators in terms of their normal ordered products. We have already seen an example of that while discussing the Feynman propagator:

$$T\{\phi(x)\phi(y)\} =: \phi(x)\phi(y) : + D(x-y). \qquad (311)$$

The general statement of the theorem is:

$$\begin{aligned}
&T\{\phi(x_1)\phi(x_2)...\phi(x_n)\} \\
&\quad =: \phi(x_1)\phi(x_2)...\phi(x_n) : \\
&\quad + D(x_1 - x_2) : \phi(x_3)...\phi(x_n) : + \text{ permutations} \\
&\quad + D(x_1 - x_2) D(x_3 - x_4) : \phi(x_5)...\phi(x_n) : + \text{ permutations} \\
&\quad + ... \\
&\quad + \Big(D(x_1 - x_2)....D(x_{n-1} - x_n) + \text{ permutations} \quad (n \text{ even}) \\
&\quad \text{or } \ D(x_1 - x_2)....D(x_{n-2} - x_{n-1}) : \phi(x_n) : + \text{ permutations} \quad (n \text{ odd}) \Big) \qquad (312)
\end{aligned}$$

This result can be proved by the method of induction, but we will not try to do so here. It is straightforward to check by an explicit computation however that the following holds:

$$\begin{aligned}
&T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\} = \\
&\quad : \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : \\
&\quad + D(x_1 - x_2) : \phi(x_3)\phi(x_4) : + D(x_1 - x_3) : \phi(x_2)\phi(x_4) : + D(x_1 - x_4) : \phi(x_2)\phi(x_3) : \\
&\quad + D(x_2 - x_3) : \phi(x_1)\phi(x_4) : + D(x_2 - x_4) : \phi(x_1)\phi(x_3) : + D(x_3 - x_4) : \phi(x_1)\phi(x_2) : \\
&\quad + D(x_1 - x_2) D(x_3 - x_4) + D(x_1 - x_3) D(x_2 - x_4) + D(x_1 - x_4) D(x_2 - x_3). \qquad (313)
\end{aligned}$$

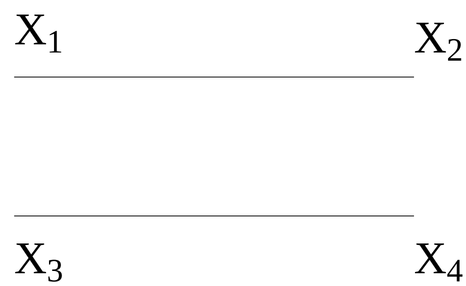

Figure 1: Diagram for $D(x_1 - x_2)D(x_3 - x_4)$

When we take the vacuum expectation values on both sides, from the right, all the normal ordered terms drop out. We get a very simple result:

$$\begin{aligned}&\langle 0|\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle\\ &\quad = D(x_1 - x_2)D(x_3 - x_4) + D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3). \quad (314)\end{aligned}$$

We can associate a physical meaning to this expression. We interpret $D(x_1 - x_2)$ as the amplitude for propagation of a particle from space-time points x_1 to x_2. Then $D(x_1 - x_2)D(x_3 - x_4)$ is the amplitude for a process where one particle goes from x_1 to x_2 and x_3 to x_4 without *interacting*. We draw a diagram to represent the process as shown in fig (1). This is a Feynman diagram in position space. The graph is called a *disconnected* graph as it factorizes into two disconnected parts. The graphs which are not disconnected are called the connected graphs.

Some computations with the scalars

Suppose we want to compute the right hand side of (296) via perturbation in λ. Since we have appropriately normalized the vacuum state, to leading order, the denominator contributes one. The numerator gives

$$\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p_1x_1+p_2x_2-k_1x_3-k_2x_4)}\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle. \quad (315)$$

But, due to the Wick's theorem, this is equal to

$$\begin{aligned}&\int d^4x_1 d^4x_1 d^4x_3 d^4x_4 e^{i(p_1x_1+p_2x_2-k_1x_3-k_2x_4)}(D(x_1 - x_2)D(x_3 - x_4)\\ &\quad + D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3)). \quad (316)\end{aligned}$$

Introducing new variables $X = (x_1 + x_2)/2$ and $x = x_1 - x_2, \tilde{X} = (x_3 + x_4)/2$ and $\tilde{x} = x_3 - x_4$, we see that the first term (and analogously the other terms) can be re-written as

$$\begin{aligned}&\Big(\int d^4x d^4X e^{i(p_1+p_2)X+i(p_1-p_2)x/2}D(x)\Big)\Big(\int d^4\tilde{x} d^4\tilde{X} e^{-i(k_1+k_2)\tilde{X}-i(k_1-k_2)\tilde{x}/2}D(x)\Big)\\ &\quad = (2\pi)^4\delta^4(p_1 + p_2)(2\pi)^4\delta^4(k_1 + k_2)\frac{i}{p_1^2 - m^2}\frac{i}{k_1^2 - m^2}. \quad (317)\end{aligned}$$

Few comments are in order. First, one should check that the Jacobian due to the change of variables is unity. Secondly, in the last line, to write the propagators in momentum space, we

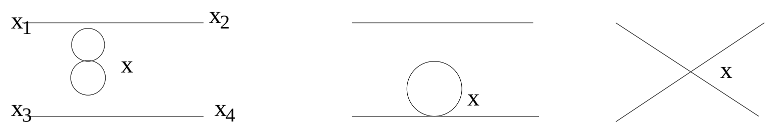

Figure 2: The left one is (1), middle one is (2) and the last one is (3)

have used the property of the δ functions (appearing in the front). Now from (296), we see

$$\langle \mathbf{p_1}, \mathbf{p_2} | iT | \mathbf{k_1}, \mathbf{k_2} \rangle = -(2\pi)^8 (p_2^2 - m^2)(k_2^2 - m^2) \delta^4(p_1 + p_2) \delta^4(k_1 + k_2). \tag{318}$$

However, the expression on the right is actually zero when we use the *on-shell* conditions on $p_2^2 - m^2$ and $k_2^2 - m^2$. So to leading order in λ, there is no contribution to the scattering matrix. This is expected as at the lowest order, the particles are free!

The fact that the disconnected graphs did not contribute is a generic feature. This is because these graphs do not produce enough pole factors to cancel the ones coming from the LSZ formula.

Next we move to the graphs that contributes at the first order in λ. Let us first focus on the numerator. The right hand side of (296) gives

$$\begin{aligned} &- \int d^4x_1 d^4x_1 d^4x_3 d^4x_4 e^{i(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)} \\ &\quad \times \Big(- \frac{i\lambda}{4!} \Big) \int d^4x \langle 0 | T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\} | 0 \rangle \end{aligned} \tag{319}$$

Wick's theorem on $\langle 0 | T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(x)\} | 0 \rangle$ produces three distinct kind of terms:

$$\begin{aligned} &(1) \quad D(x_1 - x_2) D(x_3 - x_4) D(x - x) D(x - x) \\ &(2) \quad D(x_1 - x_2) D(x_3 - x) D(x_4 - x) D(x - x) \\ &(3) \quad D(x_1 - x) D(x_2 - x) D(x_3 - x) D(x_4 - x) \end{aligned} \tag{320}$$

The diagrams are shown in fig (2). We note that first two are the disconnected diagrams and hence, following our earlier arguments, they will not contribute. The third one is a *connected* diagram and we need to focus on this. Actually, it is easy to see that there are 4! such terms (through Wick's theorem). Therefore, the right hand side of (319) gives

$$\begin{aligned} &\int d^4x_1 d^4x_1 d^4x_3 d^4x_4 e^{i(p_1 x_1 + p_2 x_2 - k_1 x_3 - k_2 x_4)} \\ &\quad \times (4!) \times \Big(- i \frac{\lambda}{4!} \Big) \int d^4x D(x_1 - x) D(x_2 - x) D(x_3 - x) D(x_4 - x) \end{aligned} \tag{321}$$

Now we first perform integrations over x_1, x_2, x_3, x_4. Consider the term

$$\int d^4x_1 e^{ip_1 x_1} D(x_1 - x). \tag{322}$$

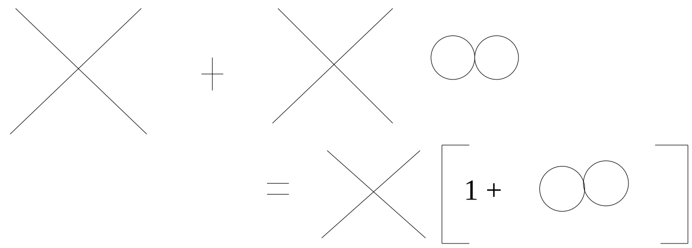

Figure 3: A connected diagram is getting dressed by vacuum-to-vacuum graph of order λ

We define $y_1 = x_1 - x$ and re-write the above expression as

$$\int d^4 y_1 e^{ip_1 y_1 + ip_1 x} D(y_1) = e^{ip_1 x} \tilde{D}(p_1) = \frac{ie^{ip_1 x}}{p_1^2 - m^2}. \tag{323}$$

Therefore (321) reduces to

$$(-i\lambda)\tilde{D}(p_1)\tilde{D}(p_2)\tilde{D}(k_1)\tilde{D}(k_2) \int d^4 x e^{i(p_1+p_2-k_1-k_2)x} \tag{324}$$

$$= \quad (-i\lambda)(2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2) \frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \frac{i}{k_1^2 - m^2} \frac{i}{k_2^2 - m^2}. \tag{325}$$

So finally we have, to first order in λ, the contribution to $\langle \mathbf{p_1}, \mathbf{p_2} | iT | \mathbf{k_1}, \mathbf{k_2} \rangle$ from the numerator is

$$= (-i\lambda)(2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2). \tag{326}$$

We note that the integration over the internal point x has contributed a momentum conserving δ function.

Now we turn our attention to the denominator. To first order in λ, this produces a contribution

$$1 + 6 \times \frac{(-i\lambda)}{4!} \int d^4 x D^2(x - x). \tag{327}$$

These are called vacuum-to-vacuum graph as they do not have external legs. However, notice that each contribution coming from the numerator can be dressed with vacuum-to-vacuum graphs. This is best represented by the first line of the fig (3). The second line of the figure shows as to how this factors out into two pieces. The piece inside the brackets cancels the denominator (327) to first order. So, as long as we neglect disconnected graphs coming from the numerator in which a disconnected component is a vacuum diagram, we can safely set the denominator to *one*.

Consider an interaction term

$$\mathcal{L}_{int} = -\frac{\lambda}{3!}\phi^3. \tag{328}$$

Find out, up to the second order in perturbation, the Feynman diagrams that contributes to the the four point correlation functions.

It is now easy to see how to compute a general amplitude. We summarize the techniques by a set of *Feynman rules* in momentum space.

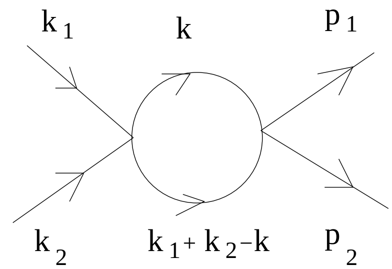

Figure 4: A second order Feynman diagram in momentum space

1. Draw all the connected graphs corresponding to a given initial state and a final state. The number of lines meet at each vertex is determined by the interaction term (4 for a ϕ^4 for example).

2. To each external leg, we associate a factor which cancels the pole factor in the LSZ formula. Therefore we can omit these factors from the graph. We then obtain directly the matrix element of iT. This is called *amputating the external legs* in a graph†

3. There is an overall energy-momentum conserving δ function. In order to avoid writing this δ function every time, we define matrix element $\mathcal{M}_{fi}$ as

$$\langle \mathbf{p_1}, ...\mathbf{p_n}|iT|\mathbf{k_1}, ...\mathbf{k_m}\rangle = (2\pi)^4\delta^4(\sum_i p_i - \sum_j k_j)i\mathcal{M}_{fi}, \tag{329}$$

where i, f refer to the initial and final states. For the process in (326), $\mathcal{M}_{fi} = -\lambda$.

4. Energy-momentum conservation must be imposed at each vertex separately. For example, in the loop diagram, it is shown explicitly in fig (4). Two external momenta k_1 and k_2 flowing into the left vertex, and the momenta associated with the internal lines flowing out of this vertex is $k_1 + k_2$. The *virtual particles* associated with the internal lines produce the final states with momenta p_1 and p_2. Over all δ function assures $p_1 + p_2 = k_1 + k_2$. So momentum is conserved also at the right vertex.

5. Associate, to each vertex, a factor $-i$ times the coupling constant.

6. To each internal line associate a propagator, with the value of the momentum given by the energy-momentum conservation.

7. Multiply with the combinatorial factor which combines the equivalent diagrams, $1/n!$ from the expansion of the exponential at order n and the numerical factors that come from the definition of the coupling constant. To first order in ϕ^4 it is $1/4!$, in second order it is $(1/4!)^2$ etc.

Having come this far, let's once more go back to have a fresh look at (324). We started with an initial state of two scalars with momenta $\mathbf{k_1}$ and $\mathbf{k_2}$ $|\mathbf{k_1}, \mathbf{k_2}\rangle$. Final state is also two scalars

†Amputating a line with a (tadpole like) loop sitting on it is a bit tricky and requires the notion of mass renormalization. This is beyond the scope of this course.

with momenta $\mathbf{p_1}$ and $\mathbf{p_2}$ $\langle \mathbf{p_1}, \mathbf{p_2}|$. The $\phi^4(x)$ interaction term has facilitated this transition. In fact, we can think it as an overlap term:

$$\langle \mathbf{p_1}, \mathbf{p_2}|T\{\int d^4x\phi^4(x)\}|\mathbf{k_1}, \mathbf{k_2}\rangle \sim \langle 0|a_{\mathbf{p_1}}a_{\mathbf{p_2}}T\{\int d^4x\phi^4(x)\}a^\dagger_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}|0\rangle \tag{330}$$

Following Wick's theorem

$$T\{\phi^4(x)\} =: \phi^4(x) : +6D(x-x) : \phi^2(x) : +3D(x-x)^2. \tag{331}$$

Consider the first term

$$\int d^4x\langle 0|a_{\mathbf{p_1}}a_{\mathbf{p_2}} : \phi^4(x) : a^\dagger_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}|0\rangle. \tag{332}$$

The only term that contributes in this matrix element is $\langle 0|a_{\mathbf{p_1}}a_{\mathbf{p_2}}(\phi^-)^2(\phi^+)^2a^\dagger_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}|0\rangle$ where ϕ^+ has $a_{\mathbf{p}}$ and ϕ^- has $a^\dagger_{\mathbf{p}}$. Note that:

$$\begin{aligned}
\phi^+(x)|\mathbf{k_1}\rangle &= \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}e^{-iqx}a_{\mathbf{q}}\sqrt{2E_{\mathbf{k_1}}}a^\dagger_{\mathbf{k_1}}|0\rangle && (333)\\
&= \int \frac{d^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}e^{-iqx}\sqrt{2E_{\mathbf{k_1}}}(2\pi)^3\delta^3(\mathbf{q}-\mathbf{k_1})|0\rangle. && (334)\\
&= e^{-ik_1x}|0\rangle. && (335)
\end{aligned}$$

Similarly $\langle \mathbf{p_1}|\phi^-(x) = \langle 0|e^{ip_1x}$. Now, in $: \phi^4 :$ there are six terms of the form $(\phi^-)^2(\phi^+)^2$. So we have

$$6 \times \Big(-\frac{i\lambda}{4!}\Big)\int d^4x\langle 0|a_{\mathbf{p_1}}a_{\mathbf{p_2}}(\phi^-)^2(\phi^+)^2a^\dagger_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}|0\rangle. \tag{336}$$

Since (**Check**)

$$(\phi^+)^2a^\dagger_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}|0\rangle = 2e^{-ik_1x-ik_2x}|0\rangle \tag{337}$$

(336) reduces to

$$-i\lambda\int d^4xe^{i(p_1+p_2-k_1-k_2)x} = (-i\lambda)(2\pi)^4\delta^4(p_1+p_2-k_1-k_2). \tag{338}$$

This is the term that appears on the right hand side of (326).

5 Feynman rules for fermions and the gauge bosons

Fermion propagator

Since fermions anti-commute, we first need to generalize the time-ordering:

$$\begin{aligned}
T\{\Psi(x)\bar{\Psi}(y)\} \quad &= \Psi(x)\bar{\Psi}(y) \quad \text{for} \;\; x^0 > y^0\\
&= -\bar{\Psi}(y)\Psi(x) \quad \text{for} \;\; y^0 > x^0.
\end{aligned} \tag{339}$$

So, for example,

$$T\{\Psi(x_1)\Psi(x_2)\Psi(x_3)\Psi(x_4)\} = (-1)^3\Psi(x_3)\Psi(x_1)\Psi(x_4)\Psi(x_2) \quad \text{for} \;\; x^0_3 > x^0_1 > x^0_4 > x^0_2. \tag{340}$$

For normal ordering, as we noted previously, we use a negative sign every time we interchange $a, a^\dagger$.

$$: a_{\mathbf{p}} a_{\mathbf{q}} a^\dagger_{\mathbf{r}} := (-1)^2 a^\dagger_{\mathbf{r}} a_{\mathbf{p}} a_{\mathbf{q}} = (-1)^3 a^\dagger_{\mathbf{r}} a_{\mathbf{q}} a_{\mathbf{p}}. \tag{341}$$

As before, we write

$$T\{\Psi(x)\bar{\Psi}(y)\} =: \Psi(x)\bar{\Psi}(y) : + \, S(x-y) \tag{342}$$

where $S(x-y)$ is the propagator for the Dirac field. One needs to be a bit careful while using Wick's theorem as minus signs appear during fermion exchange. For example:

$$: \Psi(x_1)\Psi(x_2)\bar{\Psi}(x_3)\bar{\Psi}(x_4) := -S(x_1 - x_3) : \Psi(x_2)\bar{\Psi}(x_4) : . \tag{343}$$

Feynman propagator for the Dirac field is a 4×4 matrix in the Dirac indices. It's explicit form can be easily obtained if we note that the scalar propagator satisfies equation (310), namely,

$$(\Box_x + m^2) D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} (-p^2 + m^2) e^{-ip(x-y)} = -i\delta^4(x-y), \tag{344}$$

with

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}. \tag{345}$$

We expect the fermion propagator to similarly satisfy

$$(i \not{\partial} - m)_{ab} (S(x-y))_{bc} = i\delta_{ac}\delta^4(x-y). \tag{346}$$

Now

$$\begin{aligned} (i \not{\partial} + m)(i \not{\partial} - m) S(x-y) &= -(\partial_\mu \partial^\mu + m^2) S(x-y) \\ &= i(i \not{\partial} + m)\delta^4(x-y). \end{aligned} \tag{347}$$

This equation is solved by

$$(S(x-y))_{ab} = (i \not{\partial} + m)_{ab} D(x-y). \tag{348}$$

Therefore

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{349}$$

In momentum space, the propagator is then

$$\tilde{S}(p) = \frac{i(\not{p} + m)}{p^2 - m^2} = \frac{i}{\not{p} - m}. \tag{350}$$

Before we list down the Feynman rules for the fermions, it will useful to consider the following process

$$\text{fermion } (k_1) \ + \ \text{fermion } (k_2) \ \rightarrow \ \text{fermion } (p_1) + \ \text{fermion } (p_2), \tag{351}$$

through a massive scalar exchange. The Hamiltonian describing such a process has an interaction part $\int d^3xg\bar{\Psi}\Psi\phi$ where g is the coupling constant. The full Hamiltoninan is therefore given by (the *Yukawa* theory)

$$H = H_{Dirac} + H_{Klein-Gordon} + \int d^3xg\bar{\Psi}\Psi\phi. \tag{352}$$

Since we have two fermions in the initial state and two in the final, the leading contribution to the S-matrix comes from the second order term

$$\langle \mathbf{p_2}, \mathbf{p_1}|T\{\frac{1}{2!}(-ig)^2 \int d^4x\bar{\Psi}_a(x)\Psi_a(x)\phi(x) \int d^4y\bar{\Psi}_b(y)\Psi_b(y)\phi(y)\}|\mathbf{k_2}, \mathbf{k_1}\rangle. \tag{353}$$

Note that here we are working with fermions. Therefore, there are spin quantum numbers associated with the states.

As before, Wick's theorem turns the T ordering to N ordering. Further note that

$$\begin{aligned}\Psi_a(x)^+|\mathbf{k_1}, s\rangle &= \int \frac{d^3q}{(2\pi)^3\sqrt{2E_\mathbf{q}}} \sum_r a_{\mathbf{q},\mathbf{r}} u^r_a(q)e^{-iqx}\sqrt{2E_{\mathbf{k_1}}}a^\dagger_{\mathbf{k_1},\mathbf{s}}|0\rangle \\ &= e^{-ik_1x}u^s_a(k_1)|0\rangle.\end{aligned} \tag{354}$$

Similarly,

$$\langle \mathbf{p_1}, s|\bar{\Psi}^-_a(x) = \langle 0|e^{ip_1x}\bar{u}^s_a(p_1). \tag{355}$$

So the contribution to the S-matrix element in (353) is (up to an overall sign)

$$\frac{1}{2!}(-ig)^2 D(x-y)\langle \mathbf{p_2}, \mathbf{p_1}|\bar{\Psi}^-_a(x)\bar{\Psi}^-_b(y)\Psi^+{}_a(x)\Psi^+{}_b(y)|\mathbf{k_2}, \mathbf{k_1}\rangle. \tag{356}$$

It is easy to check that there is only one such term. However, interchanging x and y, we generate a similar piece which adds up to cancel $1/2!$. Therefore, we get

$$\begin{aligned}&(-ig)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2_\phi} \int d^4yd^4xe^{-i(k_1x+k_2y)+i(p_1x+p_2y)-iq(x-y)}\bar{u}_a(p_1)u_a(k_1)\bar{u}_b(p_2)u_b(k_2) \\ &= (-ig)^2 \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2_\phi}(2\pi)^4\delta^4(p_1-k_1-q)(2\pi)^4\delta^4(p_2-k_2+q) \\ \times \quad & \bar{u}(p_1)u(k_1)\bar{u}(p_2)u(k_2),\end{aligned} \tag{357}$$

where we have suppressed some of the indices. Now integrating over q, we arrive at

$$= \frac{i}{(p_1-k_1)^2-m^2_\phi}(-ig)^2(2\pi)^4\delta^4(p_2+p_1-k_2-k_1)\bar{u}(p_1)u(k_1)\bar{u}(p_2)u(k_2). \tag{358}$$

This gives a contribution to the scattering matrix:

$$\mathcal{M}_{fi} = \frac{-ig^2}{(p_1-k_1)^2-m^2_\phi}\bar{u}(p_1)u(k_1)\bar{u}(p_2)u(k_2). \tag{359}$$

There is another diagram that contributes to the S-matrix to this order. It would be good to **compute** that. The Feynman diagram for the process is shown in (5). The scalar particle is

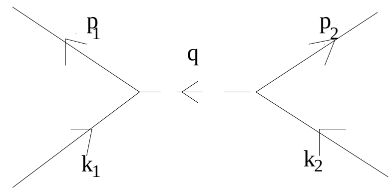

Figure 5: Feynman diagram for the Yukawa interaction in momentum space

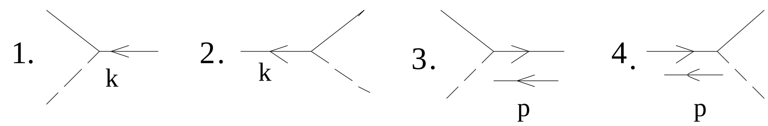

Figure 6: external legs for fermions and anti-fermions

denoted by a dashed line and fermions by solid lines. For external lines, we have the following possibilities.

$$1.\ \Psi|\mathbf{k},s\rangle = u^s(k),\ 2.\ \langle\mathbf{k},s|\bar{\Psi} = \bar{u}^s(k),\ 3.\ \bar{\Psi}|\mathbf{p},s\rangle = \bar{v}^s(p)\ 4.\ \langle\mathbf{p},s|\Psi = v^s(p). \tag{360}$$

Diagrammatic representations are shown in (6). Note that each vertex gives a factor of $(-ig)$ here.

We need to impose energy-momentum conservation at each vertex and integrate over the loop momentum. For fermions, one needs to be also careful with the overall sign of the diagrams (fermions anti-commute). The direction of the momentum on a fermion line is important. On external lines, like boson, the direction of the momentum is always ingoing for initial particles and outgoing for the final state particles. On internal lines, momentum must be assigned in the direction of particle-number flow (in electron, it is the direction of the negative charge flow). Arrow on a fermion line indicates this direction. For external anti-particle, momentum flows opposite to the arrow. This is shown in (6) by an arrow on the extra line next to it.

<u>Gauge field propagator</u>

We will work here in covariant gauge. We have seen that A^μ satisfies mass-less Klein-Gordon equation as in (234) and we have four of these equations corresponding to four values of μ. Further, ϕ and A_μ satisfy:

$$\begin{aligned}
[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})] &= i\delta^3(\mathbf{x}-\mathbf{y}),\\
[A^\mu(t,\mathbf{x}),\Pi^\nu(t,\mathbf{y})] &= i\eta^{\mu\nu}\delta^3(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{361}$$

Note that the time component of the gauge field has an opposite sign. We can then write

$$D_{\mu\nu}(x-y) = \langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = -\eta_{\mu\nu}D(x-y)|_{m=0}. \tag{362}$$

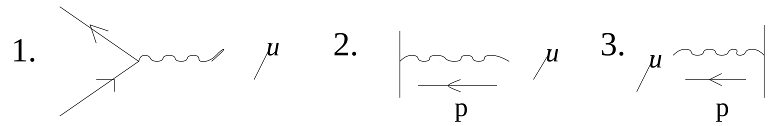

Figure 7: The vertex, the external photon lines contribution $\epsilon_\mu(p)$ and $\epsilon^*_\mu(p)$ are shown as 1, 2 and 3 respectively

This gives

$$\tilde{D}_{\mu\nu}(p) = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon} \tag{363}$$

In other words, the spatial components have same propagators as scalars but the time component has an opposite sign.

We will now set the Feynman rules for quantum electrodynamics closely following the Yukawa interaction. We first replace the scalar ϕ by A_μ and the interaction term by $\int d^3xe\bar{\Psi}\gamma^\mu\Psi A_\mu$.. The full Hamiltonian is therefore given by

$$H = H_{Dirac} + H_{gaugefields} + \int d^3xe\bar{\Psi}\gamma^\mu\Psi A_\mu. \tag{364}$$

Comparing with the Yukawa interactions, we can list down the changes in the Feynman diagrams

1. The new vertex now gives $-ie\gamma^\mu$.

2. Photon propagator is drawn by a wavy line and is given by $-i\eta_{\mu\nu}/(q^2 + i\epsilon)$

External photon lines give

$$A_\mu|\mathbf{p}\rangle = \epsilon_\mu(p), \quad \langle\mathbf{p}|A_\mu = \epsilon^*_\mu(p). \tag{365}$$

These are shown in the fig (7). We will now use the interaction Hamiltonian to study the process:

$$e^- \; e^+ \to \mu^- \; \mu^+. \tag{366}$$

The term that contributes in the leading order to this scattering process comes from a second order in perturbation:

$$\begin{aligned} &2 \times \frac{1}{2!}(-ie)^2 \int \frac{d^4q}{(2\pi)^4} d^4xd^4y \langle\mathbf{p_1p_2}|\bar{\Psi}^-_a(x)\gamma^\mu_{ab}\Psi^-_b(x)\bar{\Psi}^+_c(y)\gamma^\nu_{cd}\Psi^+_d(y)|\mathbf{k_1k_2}\rangle \\ &\times \Big(\frac{-i\eta_{\mu\nu}e^{-iq(x-y)}}{q^2 + i\epsilon}\Big). \end{aligned} \tag{367}$$

Suppressing some of the indices, this gives

$$\mathcal{M}_{fi} = \frac{ie^2}{(k_1 + k_2)^2}\bar{v}(k_1)\gamma^\mu u(k_2)\bar{u}(p_2)\gamma_\mu v(p_1) \tag{368}$$

The Feynman diagram for the process is shown in (8). What we will calculate is the quantity

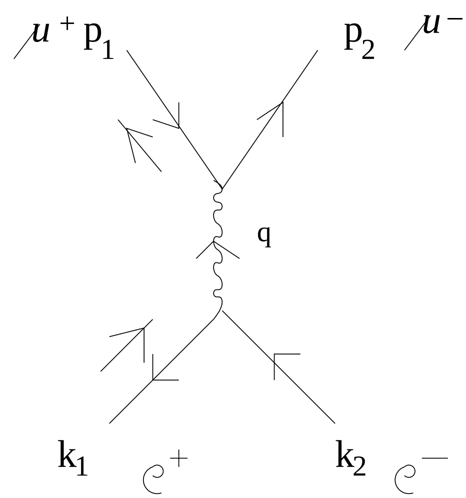

Figure 8: The process $e^+ \; e^- \to \mu^+ \; \mu^-$

$|\mathcal{M}_{fi}|^2$. The reason will be clear a bit later. To do that let us first notice $\bar{u}(p_2)\gamma^\mu v(p_1)$ is a number. So

$$(\bar{u}\gamma^\mu v)^* = (\bar{u}\gamma^\mu v)^\dagger = v^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^0\gamma^0 u = \bar{v}\gamma^\mu u. \tag{369}$$

Hence

$$\mathcal{M}^*_{fi} = \frac{-ie^2}{(k_1+k_2)^2}\bar{v}(p_1)\gamma_\mu u(p_2)\bar{u}(k_2)\gamma^\mu v(k_1). \tag{370}$$

We write

$$|\mathcal{M}_{fi}|^2 = \frac{e^4}{(k_1+k_2)^4}L^{\mu\mu}\tilde{L}_{\mu\nu}, \tag{371}$$

with

$$L^{\mu\nu} = \bar{v}^r(k_1)\gamma^\mu u^s(k_2)\bar{u}^s(k_2)\gamma^\nu v^r(k_1), \quad \tilde{L}_{\mu\nu} = \bar{u}^{r'}(p_2)\gamma_\mu v^{s'}(p_1)\bar{v}^{s'}(p_1)\gamma_\nu u^{r'}(p_2) \tag{372}$$

In the last expression, we have put the spin indices explicitly.

In most of the experiments, the electron and the positron beams are unpolarized. So the measured cross section is an average over the electron and the positron spins r and s. The detectors on the other hand are generally blind to the final spins of muons. So we sum over the final spins r' and s'. Therefore, the expression that we are interested in is

$$\frac{1}{2}\sum_r \frac{1}{2}\sum_s \sum_{r'} \sum_{s'} |\mathcal{M}_{fi}|^2. \tag{373}$$

To proceed, let us sum the spin indices of $L^{\mu\nu}$ and $\tilde{L}_{\mu\nu}$. To do that the following relations help (**show**):

$$\sum_s u^s(k)\bar{u}^s(k) = \not{k} + m, \quad \sum_r v^r(k)\bar{v}^r(k) = \not{k} - m. \tag{374}$$

So

$$\sum_{s,r} L^{\alpha\beta} = \sum_{s,r} \bar{v}^r_a(k_1)\gamma^\alpha_{ab} u^s_b(k_2)\bar{u}^s_c(k_2)\gamma^\beta_{cd} v^r_d(k_1) \tag{375}$$

$$= \gamma^\alpha_{ab}(\not{k}_2 + m_e)_{bc}\gamma^\beta_{cd}(\not{k}_1 - m_e)_{da} \tag{376}$$

$$= Tr[\gamma^\alpha(\not{k}_2 + m_e)\gamma^\beta(\not{k}_1 - m_e)]. \tag{377}$$

Now to evaluate the trace, we use the following identities (**show**):

$$Tr[\gamma^\alpha\gamma^\sigma\gamma^\beta\gamma^\rho] = 4(\eta^{\alpha\sigma}\eta^{\beta\rho} - \eta^{\alpha\beta}\eta^{\sigma\rho} + \eta^{\alpha\rho}\eta^{\sigma\beta}) \tag{378}$$

$$Tr[\gamma^\alpha\gamma^\sigma] = 4\eta^{\alpha\sigma}, \tag{379}$$

to get

$$\sum_{s,r} L^{\alpha\beta} = 4(k_1^\alpha k_2^\beta + k_2^\alpha k_1^\beta - \eta^{\alpha\beta} k_2^\sigma k_{1\sigma} - \eta^{\alpha\beta} m_e^2). \tag{380}$$

Similarly

$$\sum_{s',r'} L_{\alpha\beta} = 4(p_{1\alpha}p_{2\beta} + p_{2\alpha}p_{1\beta} - \eta_{\alpha\beta} p_2^\sigma p_{1\sigma} - \eta_{\alpha\beta} m_\mu^2). \tag{381}$$

Therefore, multiplying the last two expressions and doing a bit of simplification, we finally arrive at

$$\frac{1}{4}\sum |\mathcal{M}_{fi}|^2 = \frac{8e^4}{(k_1+k_2)^4}(k_1.p_1\ k_2.p_2 + k_1.p_2\ k_2.p_1 + m_e^2\ p_1.p_2 + m_\mu^2\ k_1.k_2 + 2m_e^2 m_\mu^2). \tag{382}$$

We define now the *Mandlestam* variables defined as:

$$s = (k_1+k_2)^2 = (p_1+p_2)^2, \quad t = (k_1-p_1)^2 = (k_2-p_2)^2,$$

$$u = (k_2-p_1)^2 = (k_1-p_2)^2 \tag{383}$$

$$\text{such that } s+t+u = 2m_\mu^2 + 2m_e^2. \tag{384}$$

In terms of these variables, (382) simplifies to

$$\frac{1}{4}\sum |\mathcal{M}_{fi}|^2 = \frac{2e^4}{s^2}[t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^4 + 4m_e^2 m_\mu^2 + m_\mu^4)] \tag{385}$$

If we set the electron mass to zero, the amplitude is

$$\frac{1}{4}\sum |\mathcal{M}_{fi}|^2 = \frac{8e^4}{(k_1+k_2)^4}(k_1.p_1\ k_2.p_2 + k_1.p_2\ k_2.p_1 + m_\mu^2 k_1.k_2). \tag{386}$$

To write it in a more explicit form, we go to the center-of-mass frame as shown in fig (9). Initial four-momenta of the incident electrons (taken in z direction) are then $k_2 = (E, E\hat{z})$ and $k_1 = (E, -E\hat{z})$. This is because the electrons are mass-less here. On the other hand, owing to the presence of $\delta^4(k_1 + k_2 - p_1 - p_2)$, we have $\mathbf{p_1} = -\mathbf{p_2}$ since $\mathbf{k_1} = -\mathbf{k_2}$. However, because $E^2_{\mathbf{p_1}} - \mathbf{p_1}^2 = E^2_{\mathbf{p_2}} - \mathbf{p_2}^2$, $E_{\mathbf{p_1}} = E_{\mathbf{p_2}}$. This along with the δ function constraint, gives $p_1 = (E, -\mathbf{p_1})$ and $p_2 = (E, +\mathbf{p_2})$. Therefore, the following relations hold:

$$\begin{aligned} s &= (k_1+k_2)^2 = (p_1+p_2)^2 = 4E^2, \\ k_1.p_1 &= E^2 - E|\mathbf{p_2}|\cos\theta \\ k_2.p_2 &= E^2 - E|\mathbf{p_2}|\cos\theta \\ k_1.p_2 &= E^2 + E|\mathbf{p_2}|\cos\theta \\ k_2.p_1 &= E^2 + E|\mathbf{p_2}|\cos\theta \\ k_1.k_2 &= 2E^2. \end{aligned} \tag{387}$$

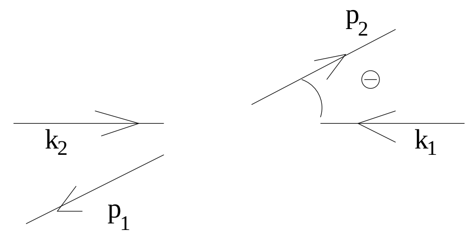

Figure 9: The center-of-mass frame

Substituting these in (386), we get the final formula as

$$\frac{1}{4}\sum|\mathcal{M}_{fi}|^2 = e^4\Big[\Big(1+\frac{m_\mu^2}{E^2}\Big)+\Big(1-\frac{m_\mu^2}{E^2}\Big)\cos^2\theta\Big]. \tag{388}$$

If the scattering takes place at a very high energy such that $m_\mu/E \ll 1$, we can even neglect the muon masses and can further approximate the result as

$$\frac{1}{4}\sum|\mathcal{M}_{fi}|^2 = e^4(1+\cos^2\theta). \tag{389}$$

<u>Cross section</u>

Suppose a hard sphere of radius a is located within a total area A. Assume another sphere approaching the static one. The likelihood of the incoming sphere to scatter is clearly proportional to the ratio $\pi a^2/A$. If we call this the probability of scattering P, then,

$$P \sim \frac{\pi a^2}{A}, \quad \pi a^2 = PA. \tag{390}$$

This equation relates the cross sectional area of the sphere in terms of beam area. Now, even if all the objects are not hard spheres, we can still define an effective cross section σ as

$$\sigma = PA. \tag{391}$$

Consider now a beam with particle density ρ approaching the target with velocity v. In time t, this beam fills up a volume ρvtA, where A is the area normal to the beam which fully contains it. Choosing t such that the volume contains only one particle, we write

$$\rho vtA = 1, \quad \text{or } A \; = \; \frac{1}{\rho vt}. \tag{392}$$

Therefore,

$$\sigma = \frac{P}{t}.\frac{1}{\rho v}. \tag{393}$$

The factor P/t is called the transition rate (probability of scattering per unit time). The other factor ρv is simply the flux of the incoming particles. In other words, the cross section is defined as the transition rate per unit incident flux.

This transition rate is related to the mod square of S_{fi} introduced earlier. Remember

$$S_{fi} = \delta_{fi} + (2\pi)^4\delta^4(\sum_i p_i - \sum_j p_j)i\mathcal{M}_{fi}. \tag{394}$$

The first part does not contribute to the scattering. The square of the second part gives

$$\begin{aligned} |S_{fi}|^2 &= (2\pi)^8\delta^4(\sum_i p_i - \sum_j p_j)\delta^4(0)|\mathcal{M}_{fi}|^2 \\ &= (2\pi)^8\delta^4(\sum_i p_i - \sum_j p_j)\frac{VT}{(2\pi)^4}|\mathcal{M}_{fi}|^2. \end{aligned} \tag{395}$$

In the first line, we used $\delta^4(x)f(x) = \delta^4(x)f(0)$ with $f(x) = \delta^4(x)$. In the second line, we assumed that the scattering process was taking place in a volume V and in time T (similar to (187)). Therefore, the transition rate per unit time is

$$\frac{|S_{fi}|^2}{T} = (2\pi)^4\delta^4(\sum_i p_i - \sum_j p_j)V|\mathcal{M}_{fi}|^2. \tag{396}$$

This quantity should be identified with P/t if the final state has specified momenta. For a range of momenta, we need to integrate this expression over momenta. We will skip the details here (details can be found in various books, see [4, 1, 3] for example). The final result for a process $a + b \to c + d$ is

$$\sigma = \Big(\frac{1}{(4(E_aE_b - \mathbf{p}_a\mathbf{p}_b)^2 - m_a^2m_b^2)^{1/2}}\Big)\int\Big(\frac{d^3p_c}{2E_c(2\pi)^3}\frac{d^3p_d}{2E_d(2\pi)^3}\Big)(2\pi)^4\delta^4(p_a + p_b - p_c - p_d)|\mathcal{M}_{fi}|^2. \tag{397}$$

Here the terms inside the first big brackets come from appropriately writing the flux. The terms within the second big brackets come because we are integrating over final state configurations. Then there is the momentum conserving delta function.

Now let us apply the above formula for $e^+e^- \to \mu^+\mu^-$. We will assume that the electrons are massless. Further, we will use the frame shown in fig (9), where $\mathbf{p}_a = -\mathbf{p}_b$. We first do the integration over p_d. The $\delta^3(\mathbf{p}_a + \mathbf{p}_b - \mathbf{p}_c - \mathbf{p}_d)$ produces $\mathbf{p}_c = -\mathbf{p}_d$. This, after the $\mathbf{p}_d$ integration, sets $E_c = (\mathbf{p}_c^2 - m^2)^{1/2} = E_d$. Hence

$$\Big(\frac{d^3p_c}{2E_c(2\pi)^3}\frac{d^3p_d}{2E_d(2\pi)^3}\Big)(2\pi)^4\delta^4(p_a + p_b - p_c - p_d) = \int\frac{d\Omega dp_c p_c^2}{16\pi^2E_cE_d}\delta(E_{cm} - E_c - E_d). \tag{398}$$

To further integrate the p_c part, let $x = E_c + E_d - E_{cm}$. Then

$$\frac{dx}{dp_c} = \frac{d}{dp_c}(E_c + E_d - E_{cm}) = \frac{p_cE_{cm}}{E_cE_d}. \tag{399}$$

With this, the integration over p_c gives

$$\frac{d\Omega}{16\pi^2}\frac{p_c}{E_{cm}} = \frac{d\Omega}{16\pi^2}\frac{E_{cm}/2}{E_{cm}}. \tag{400}$$

On the other hand, it is easy to check that the terms in the first big brackets in (397) give $1/(2E_{cm}^2)$. The rest part has already been computed in (388). Putting all the pieces together, we get the differential cross section $d\sigma/d\Omega$. In particular in the high energy limit, when the masses of muons can be neglected, we get a simple result

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{cm}^2}(1+\cos^2\theta), \quad \alpha = \frac{e^2}{4\pi}. \tag{401}$$

The total cross-section is therefore

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d(\cos\,\theta) d\phi \frac{d\sigma}{d\Omega} = \frac{4\pi\alpha^2}{3E_{cm}^2}. \tag{402}$$

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