QUANTUM CIRCUIS and SIMPLE QUANTUM ALGORITHMS

prof. RNDr. Jozef Gruska, DrSc.

Faculty of Informatics
Masaryk University
Brno, Czech Republic

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Part I

Quantum circuits and simple algorithms

PROLOGUE

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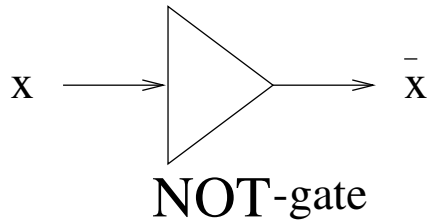
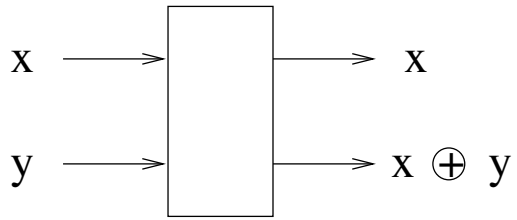
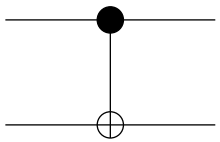
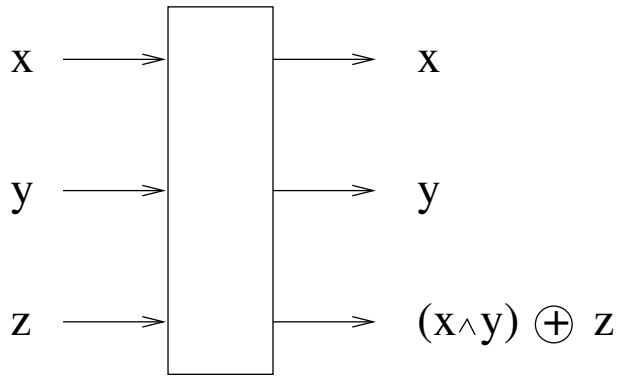


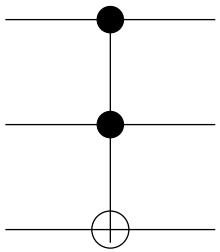
NOT-gate





CNOT = XOR gate





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In every Hilbert space there are so-called orthogonal bases all states of which are mutually orthogonal.

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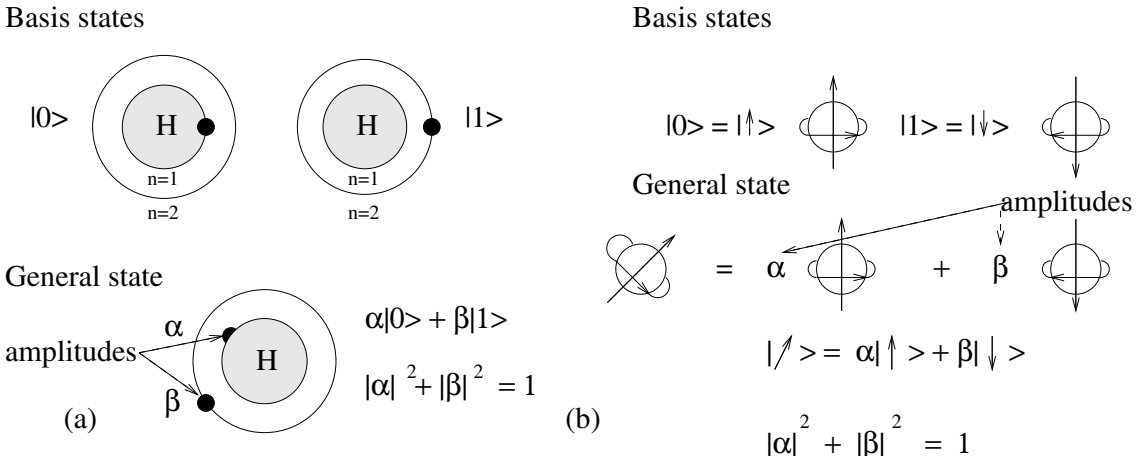
EXAMPLE: Representation of qubits by (a) an electron in a Hydrogen atom; (b) a spin-$\frac{1}{2}$ particle:



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Since vectors $|0\rangle$ and $|1\rangle$ form a basis of H_2, one of the basis of $\mathcal{H}_{2^n}$, so-called computational basis, consists of all possible n-fold tensor products where $b_i \in \{0, 1\}$ for all i.

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Example A two-qubit register has as a computational basis vectors

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

QUANTUM STATES and PROJECTION MEASUREMENTS

In case an orthonormal basis $\{\beta_i\}_{i=1}^n$ is chosen in $\mathcal{H}_n$, any state $|\phi\rangle \in \mathcal{H}_n$ can be expressed in the form

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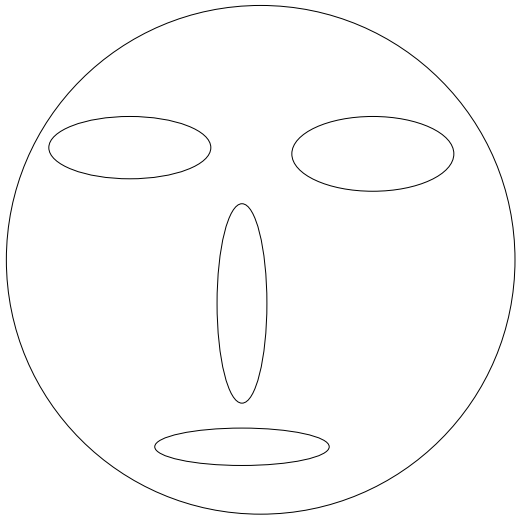
The classical "outcome" of a projection (von Neumann) measurement of the state $|\phi\rangle$ with respect to the basis $\{\beta_i\}_{i=1}^n$ is the index i of that state $|\beta_i\rangle$ into which the state $|\phi\rangle$ collapses.

QUANTUM MEASUREMENT - II

A quantum state is observed (measured) with respect to an **observable** — a decomposition of a given Hilbert space into orthogonal subspaces (such that each vector can be uniquely represented as a sum of vectors of these subspaces).



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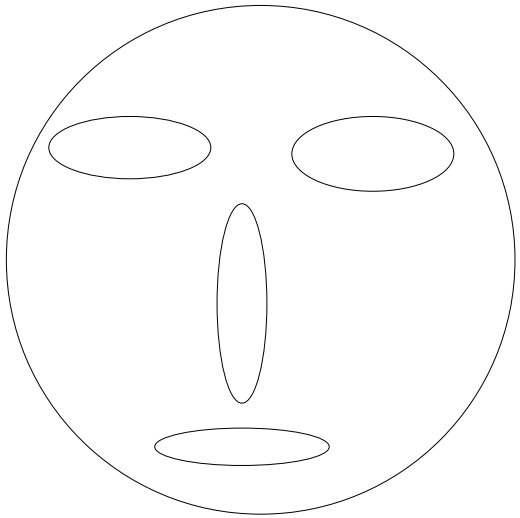

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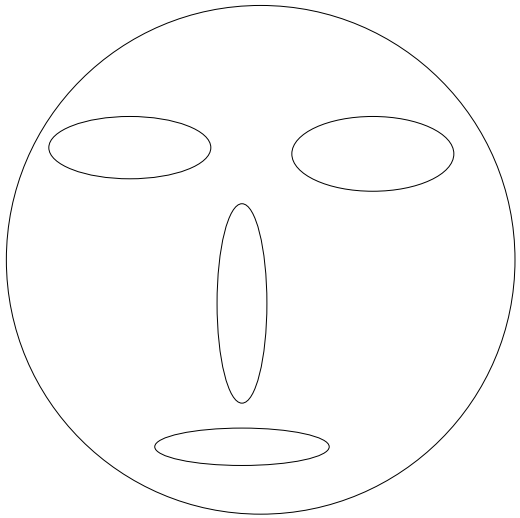

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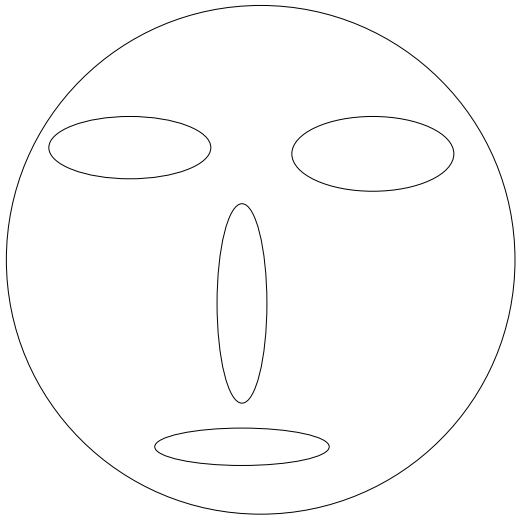

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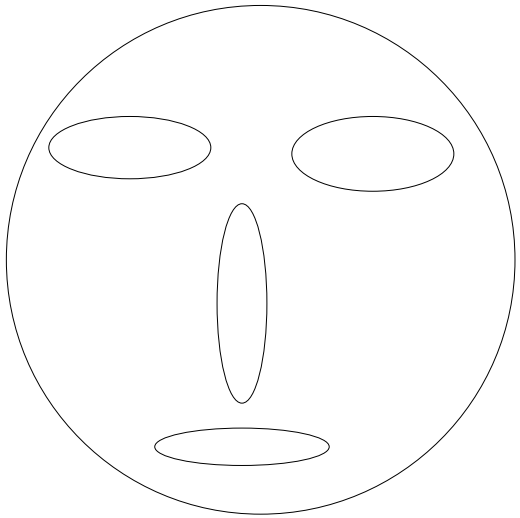

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The subspace into which projection is made is chosen **randomly** and the corresponding probability is uniquely determined by the amplitudes at the representation of $|\phi\rangle$ at the basis states of the subspace.

QUANTUM GATES and CIRCUITS

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- Quantum circuits are the simplest and best model to express quantum algorithms.

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Of importance, especially for an understanding of the power and development of methods (of programming) for very powerful real processors are also the following models:

- RAM - Random access machines
- PRAM - Parallel and shared memory random access machines

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- Finite automata (deterministic, non-deterministic, probabilistic, ultrametric,...., one-way, two-way,...; one-tape, multi-tape,....)
- Turing machines (with one or more tapes, with one or more heads, with one or more dimensional tapes - deterministic, non-deterministic, probabilistic,...) - models of universal processors
- Uniform classes of circuits - models of universal processors
- Cellular automata (one-, two-, three- and more dimensional)– models of universal processors.

Of importance, especially for an understanding of the power and development of methods (of programming) for very powerful real processors are also the following models:

- RAM - Random access machines
- PRAM - Parallel and shared memory random access machines

and also

- Interactive proof systems – to model computations by interactions.

MAIN MODELS of CLASSICAL PROCESSORS - II.

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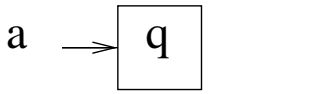
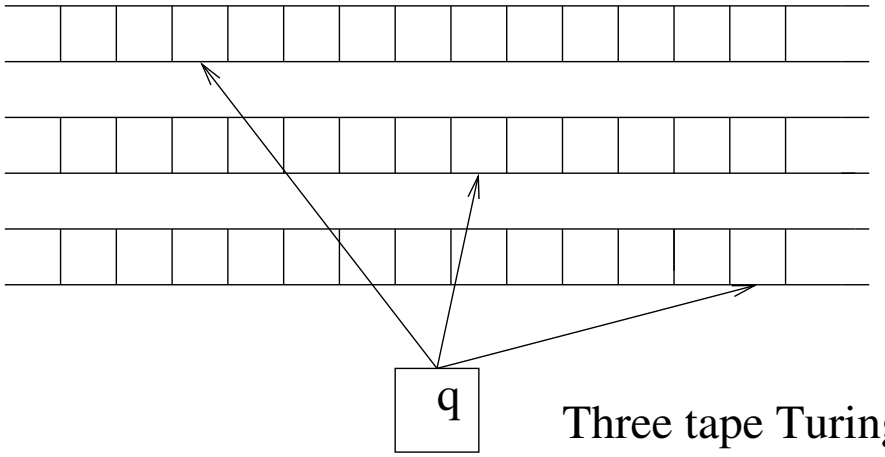
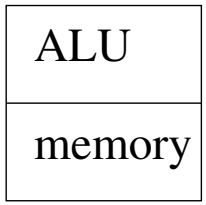


Finite automaton

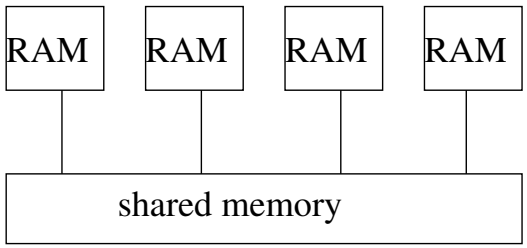


Three tape Turing machine

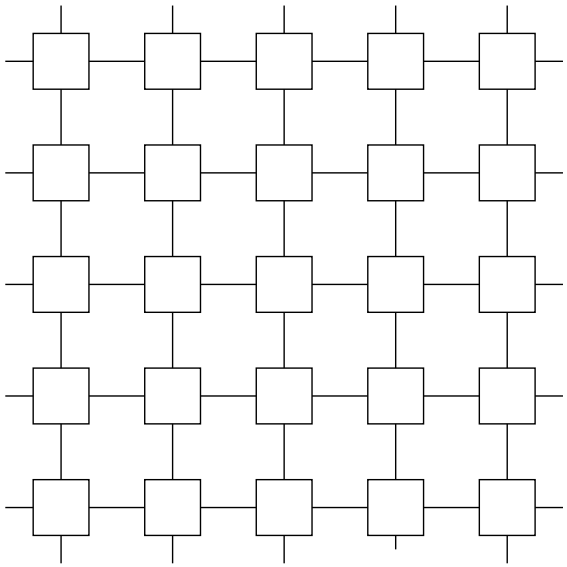
RAM



Operations: Load, Store
Add, Subtract
Jump, Jump–if



PRAM



Two–dimensional cellular automaton

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- Emerging idea: Classically controlled quantum computation (automata).

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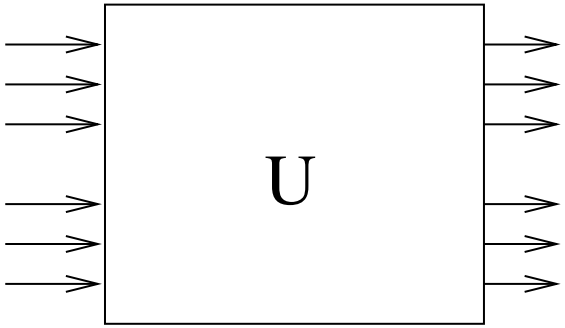

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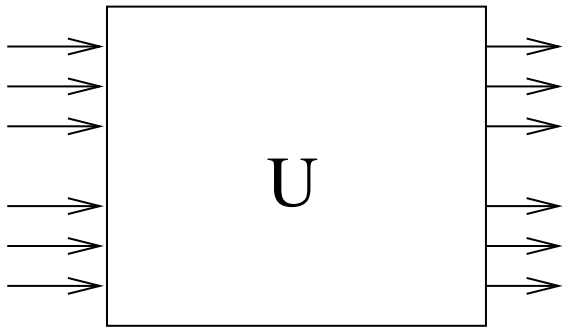

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Example: The so-called Hadamard gates are represented by matrices

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad H' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad H'' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

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Another view of unitarity (for mappings): unitary mapping U is a linear mapping that preserves the inner product, that is

$$\langle U\phi | U\psi \rangle = \langle \phi | \psi \rangle.$$

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As a generalization we have a rotation around an arbitrary real unit vector $\bar{n} = (n_x, n_y, n_z)$ defined by

$$R_{\bar{n}}(\theta) = e^{-i\theta\bar{n}\cdot\bar{\sigma}/2} = \cos\frac{\theta}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z).$$

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Of importance is that the Hadamard matrix maps the standard basis into dual one and vice verse.

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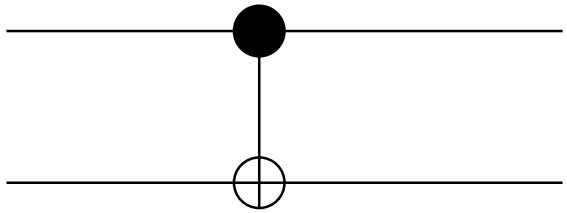
Operators σ_x, σ_z and σ'_y represent therefore a *bit error*, a *sign error* and a *bit-sign error*.

XOR (CNOT) GATE

Unitary matrix for so called XOR-gate (CNOT-gate)



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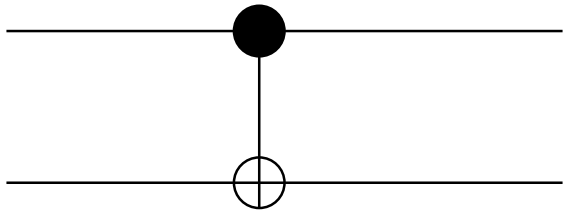

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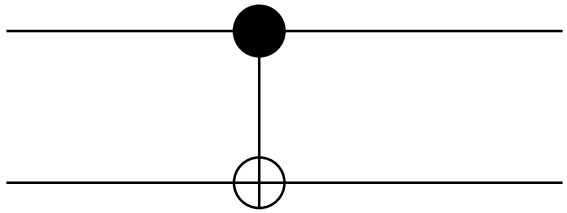

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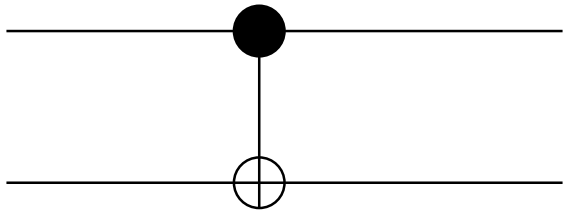

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That is, if first (control) qubit is $|0\rangle$, then neither first nor second inputs are changed; if first qubit is $|1\rangle$, then the first qubit is not changed, but second is negated.

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2. Three gates:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \sigma_z^{1/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{\pi}{4} i} \end{pmatrix}$$

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Main reason behind is that the CNOT gate maps some nonentangled states into entangled states exhibiting non-locality.

FROM GATES to UNITARY MATRICES

In general, if a quantum gate G has n inputs and outputs, then for the corresponding unitary matrix of G , with both rows and columns labeled by all n-bit strings, the entry

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is the amplitude for transition, under the mapping G, from the basis state $|x\rangle$ to the basis state $|y\rangle$.

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REPRESENTATION of GATES by MATRICES in DIFFERENT BASIS

Unitary operators have different matrix representations in different bases.

For example XOR operator has in the standard basis

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

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CNOT in BEL BASIS

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Representation of CNOT in the Bell basis $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$, where

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

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$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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Evolution in a quantum system is described by **Schrödinger linear equation**

$$i\hbar\frac{\partial\psi(t)}{\partial t} = H(t)\psi(t),$$

where $H(t)$ is a Hamiltonian (Hermitian operator representing total energy of the system), from which it follows that

$$\psi(t) = e^{-\frac{i}{\hbar}H(t)\psi(0)}$$

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In other words, if Hamiltonian is constant, then a step of evolution of a state $|\psi\rangle$ is a multiplication by a **unitary matrix** A of a vector $|\psi\rangle$, i.e.

$$A|\psi\rangle$$

HAMILTONIAN for the CNOT GATE

For the Hamiltonian

$$H = \frac{\pi\hbar}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \frac{\pi\hbar}{2} V$$

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because $V^k = 2^{k-1}V$ and therefore for $t = 1$,

$$e^{-\frac{i\pi}{2}V} = I + \frac{1}{2}(e^{-i\pi} - 1)V = I - V = CNOT.$$

QUANTUM CIRCUITS

INVERSE CNOT-GATE CIRCUIT

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An implementation of the inverse of the CNOT gate.

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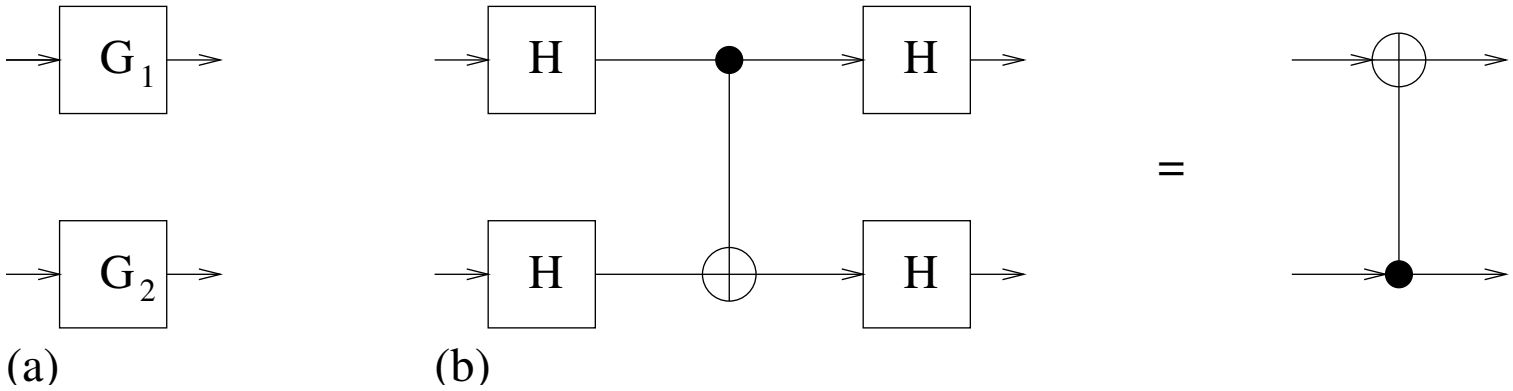

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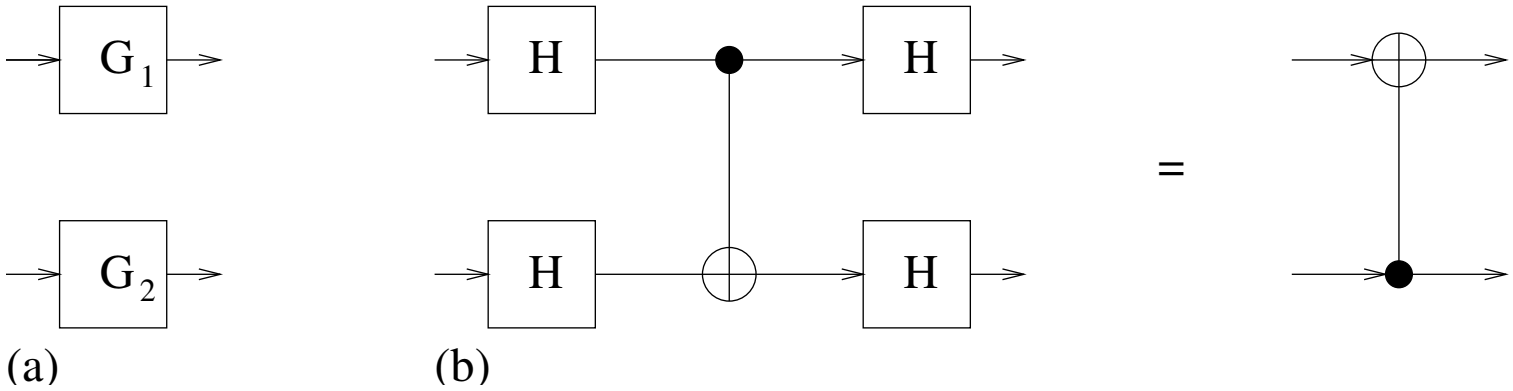

The processing in the network on the left side of the identity in Figure 33b for the input $|0\rangle|1\rangle$ can be depicted as shown on next slide:

$$|0\rangle|1\rangle \quad \overset{H-\text{gates}}{\longrightarrow} \quad |0'\rangle|1'\rangle$$

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$$\overset{\text{CNOT gate}}{\longrightarrow}$$

$$
\begin{aligned}
|0\rangle|1\rangle \quad & \stackrel{H-\text{gates}}{\longrightarrow} \quad |0'\rangle|1'\rangle \\
& = \quad \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
& = \quad \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|0\rangle - |0\rangle|1\rangle - |1\rangle|1\rangle) \\
& \stackrel{\text{CNOT gate}}{\longrightarrow} \quad \frac{1}{2}(|0\rangle|0\rangle + |1\rangle|1\rangle - |0\rangle|1\rangle - |1\rangle|0\rangle)
\end{aligned}
$$

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|0\rangle|1\rangle \quad & \stackrel{H-\text{gates}}{\longrightarrow} \quad |0'\rangle|1'\rangle \\
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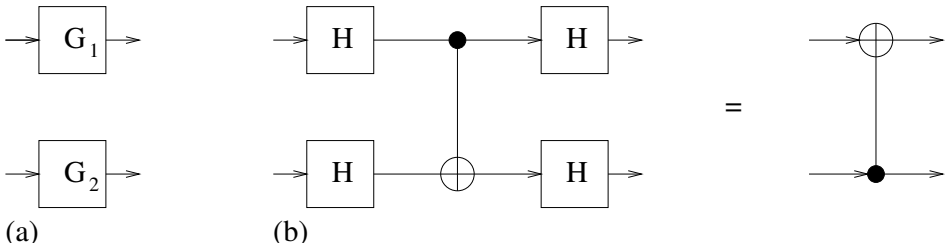
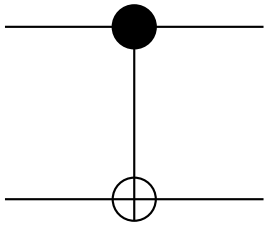
$$

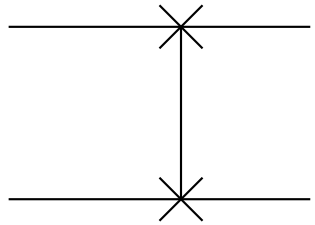


GRAPHICAL REPRESENTATION of BASIC GATES

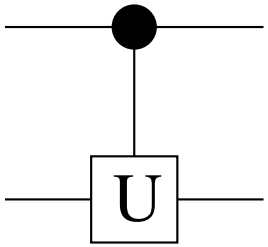
CNOT–gate



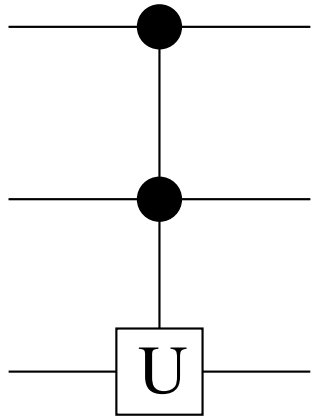
SWAP–gate



Λ(U)–gate

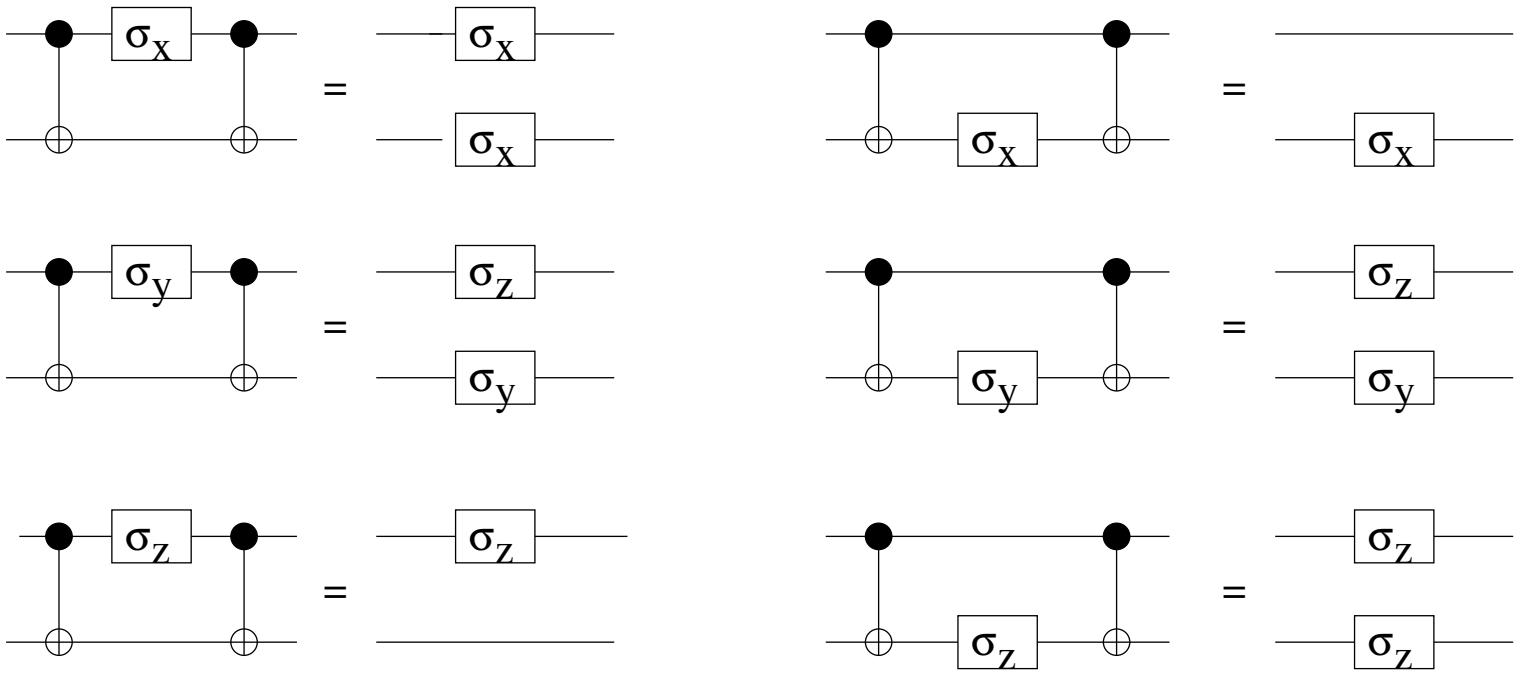


Λ_2 (U)–gate



SOME USEFUL IDENTITIES - I.

Several simple identities between elementary gates are surprisingly useful.



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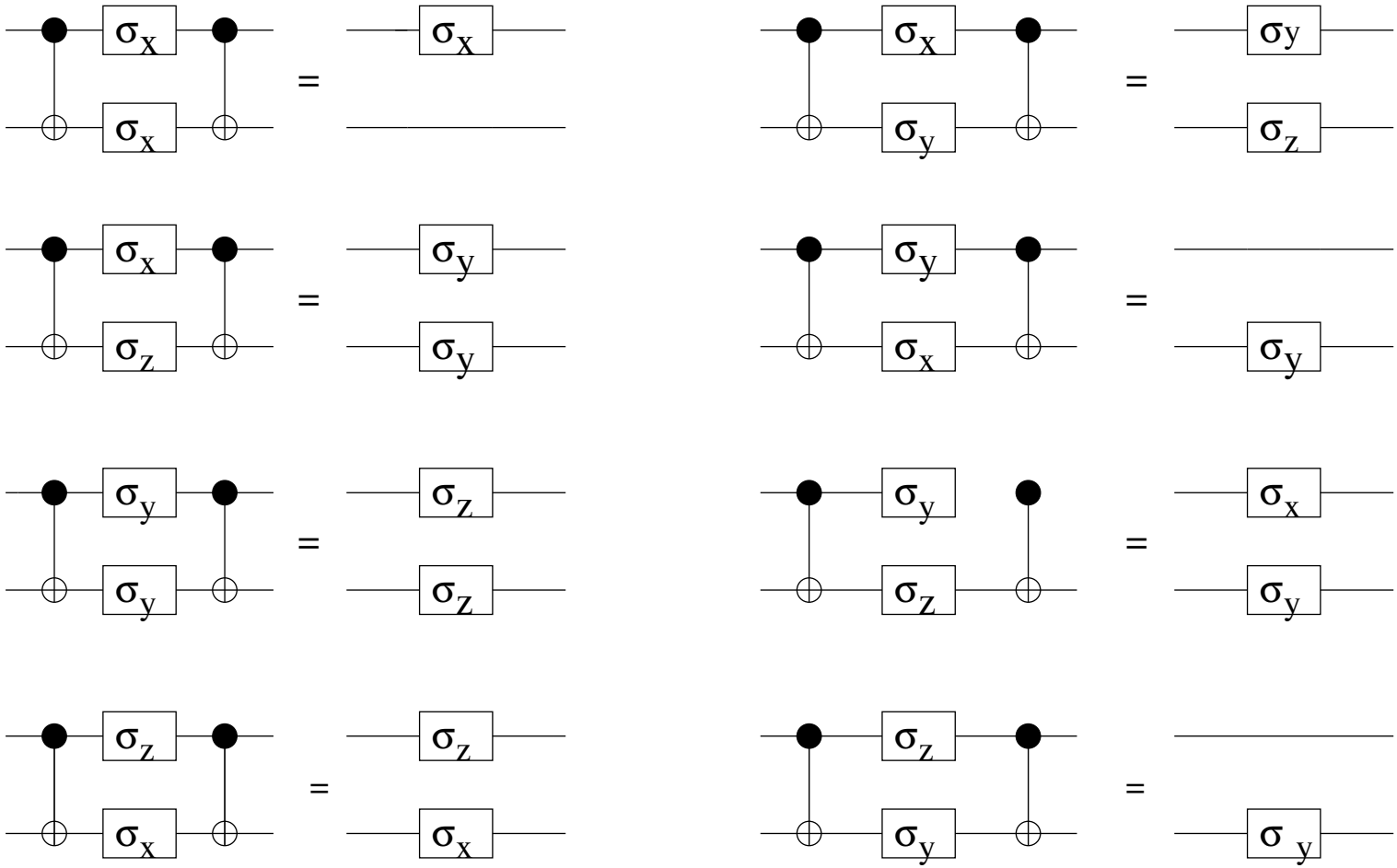

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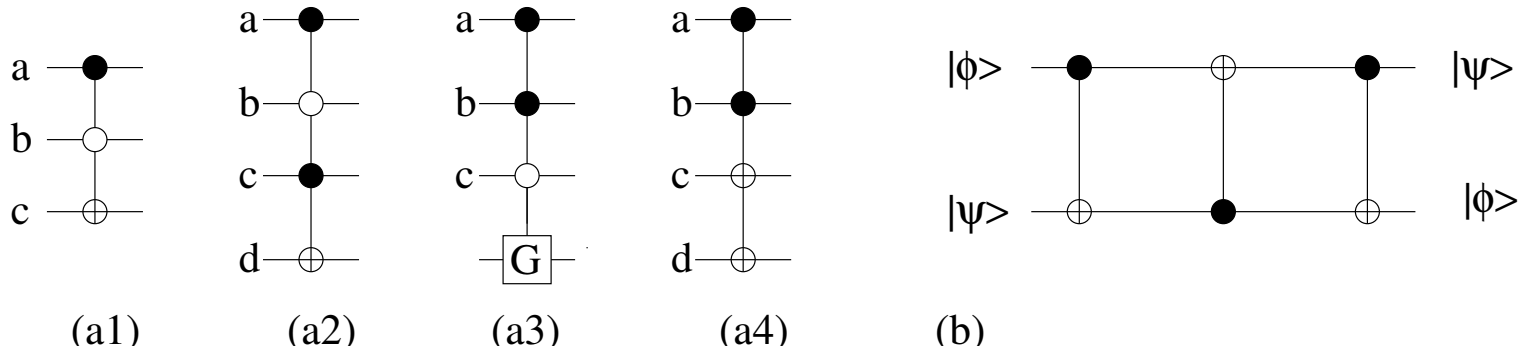

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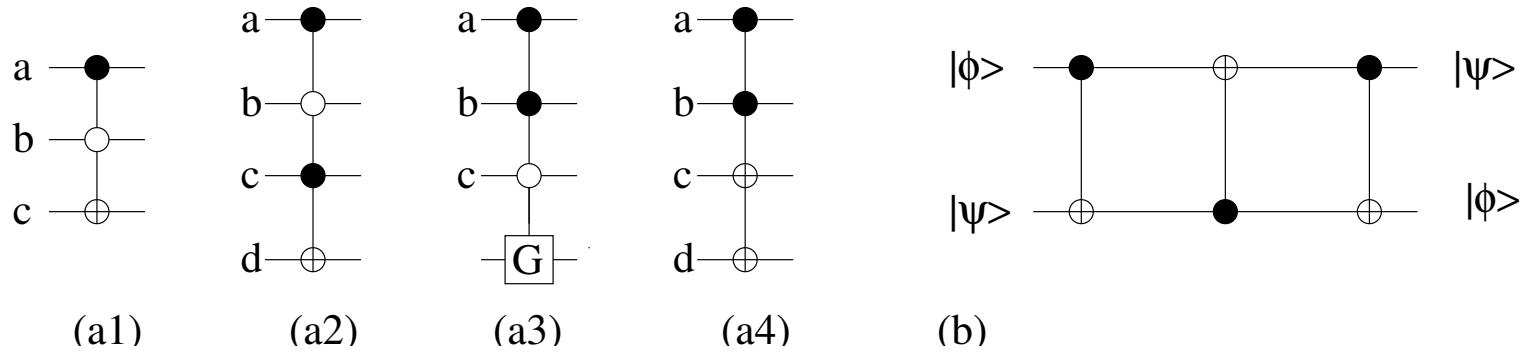

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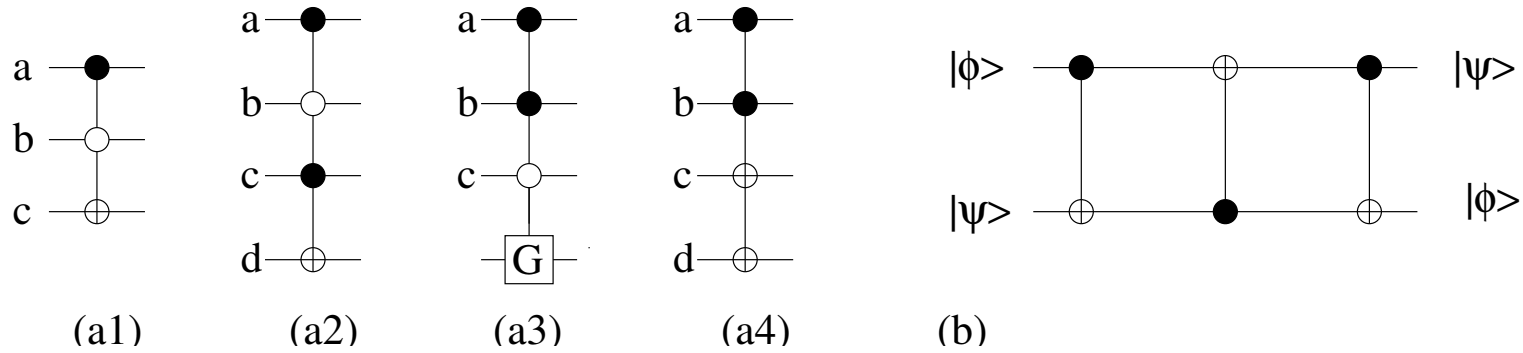

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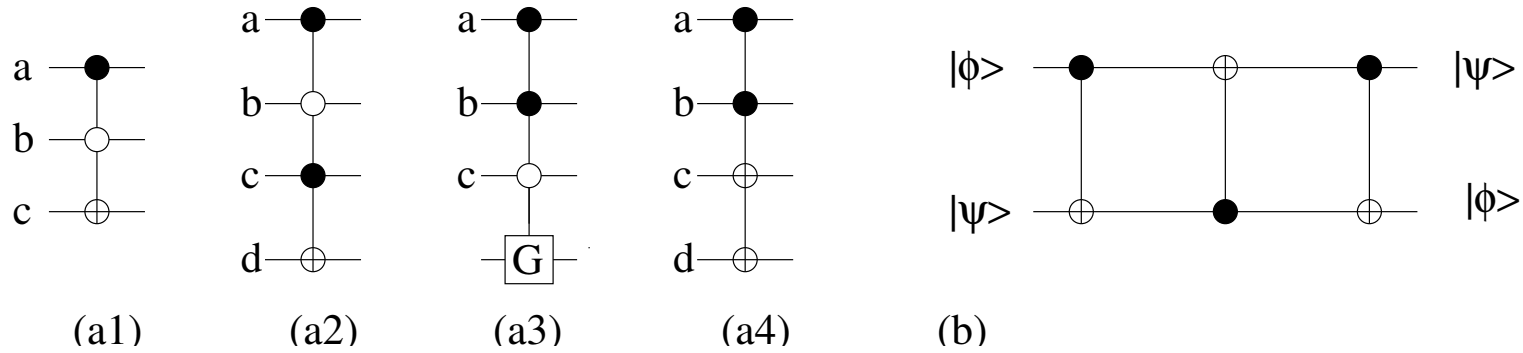

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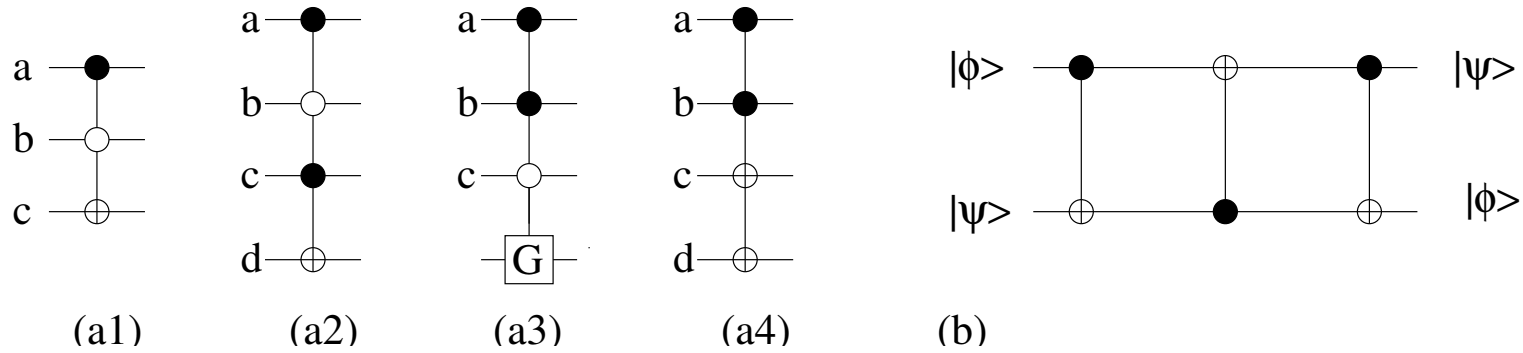

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$$I_{34} I_{24} I_{34} \begin{pmatrix} \alpha\alpha' \\ \alpha\beta' \\ \beta\alpha' \\ \beta\beta' \end{pmatrix} = I_{34} I_{24} \begin{pmatrix} \alpha\alpha' \\ \alpha\beta' \\ \beta\beta' \\ \beta\alpha' \end{pmatrix} = I_{34} \begin{pmatrix} \alpha\alpha' \\ \beta\alpha' \\ \beta\beta' \\ \alpha\beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' \\ \beta\alpha' \\ \alpha\beta' \\ \beta\beta' \end{pmatrix} = \begin{pmatrix} \alpha'\alpha \\ \alpha'\beta \\ \beta'\alpha \\ \beta'\beta \end{pmatrix}$$

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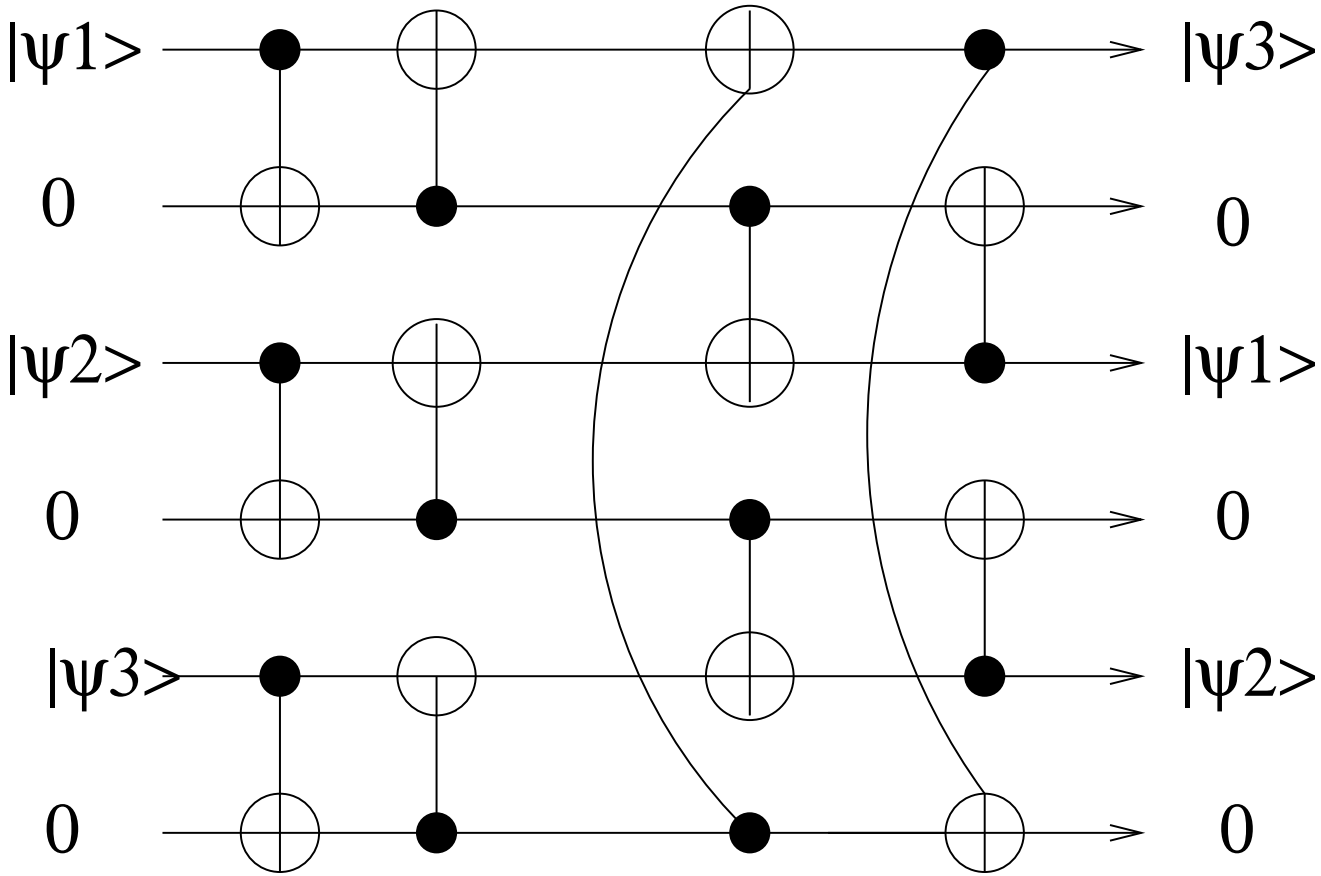

HADAMARD GATE

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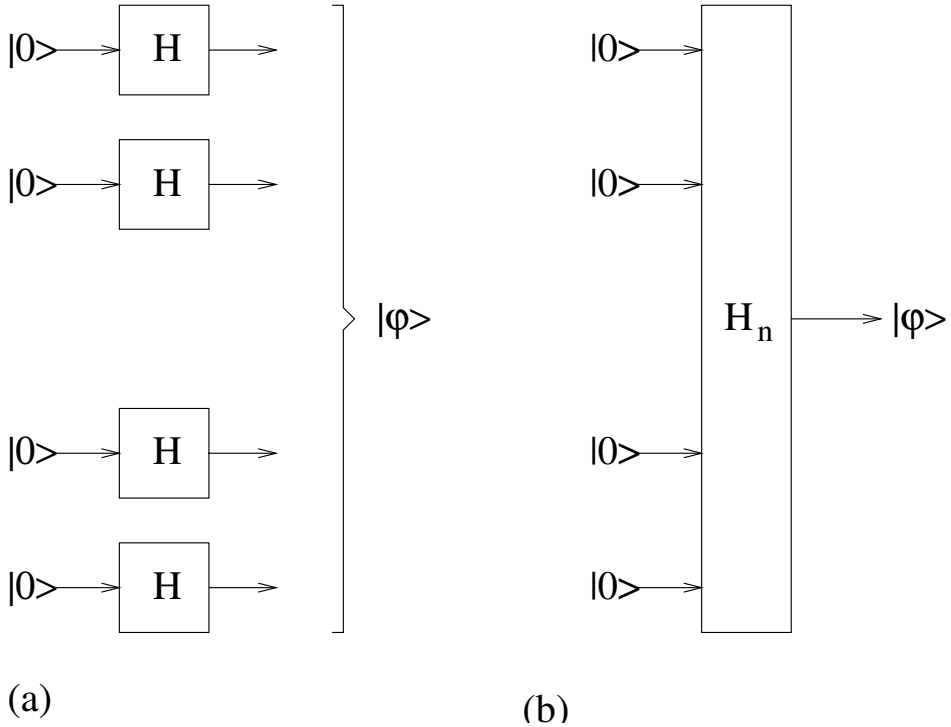
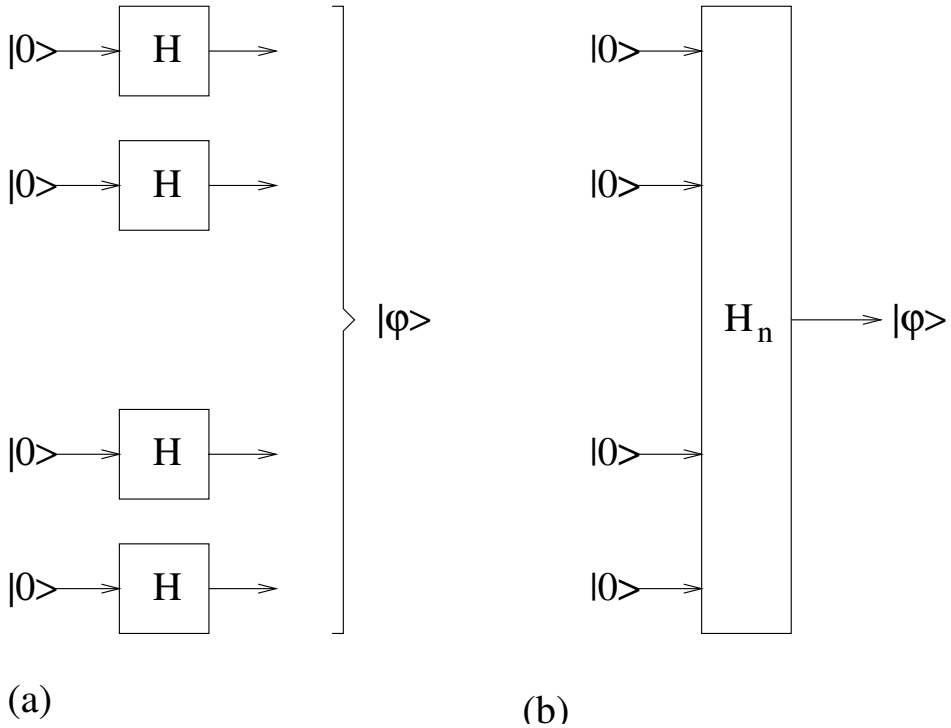


HADAMARD GATE

The Hadamard transform H_n is implemented by the circuit in Figure a, and Figure b contains the usual notation for the circuit for H_n.



The Hadamard circuit/gate H_n when applied to the state $|0^{(n)}\rangle$ provides as the outcome the state

$$|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle.$$

HADAMARD GATE APPLIED TO BASIS STATES

For an $x \in \{0,1\}^n$ it holds

$$H_n(|x\rangle) = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle$$

where $x \cdot y$ stands for the inner product of strings x and y, that is their bit-wise xor-multiplications and then the addition modulo 2.

SIMPLE QUANTUM ALGORITHMS

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$$f : \{0, 1, \ldots, 2^n - 1\} \Longrightarrow \{0, 1, \ldots, 2^n - 1\}$$

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With a **single application** of the mapping U_f we get

$$U_f|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle|f(i)\rangle$$

Hence, IN A SINGLE COMPUTATIONAL STEP 2^n VALUES OF f ARE "COMPUTED"! - in some sense.

IMPACTS of PROJECTIVE MEASUREMENTS

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This fact we usually interpret that y is the (classical) result of the measurement of the second register of the state $|\phi\rangle$ with respect to the standard basis.

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Another useful operator related to functions

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$$\begin{aligned} U_f|x, \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\rangle &= \frac{1}{\sqrt{2}}(|x, 0 \oplus f(x)\rangle - |x, 1 \oplus f(x)\rangle \\ &= (-1)^{f(x)}|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned}$$

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Given a function $f : \{0,1\} \to \{0,1\}$, as a **black box**, the task is to determine whether f is constant or balanced.

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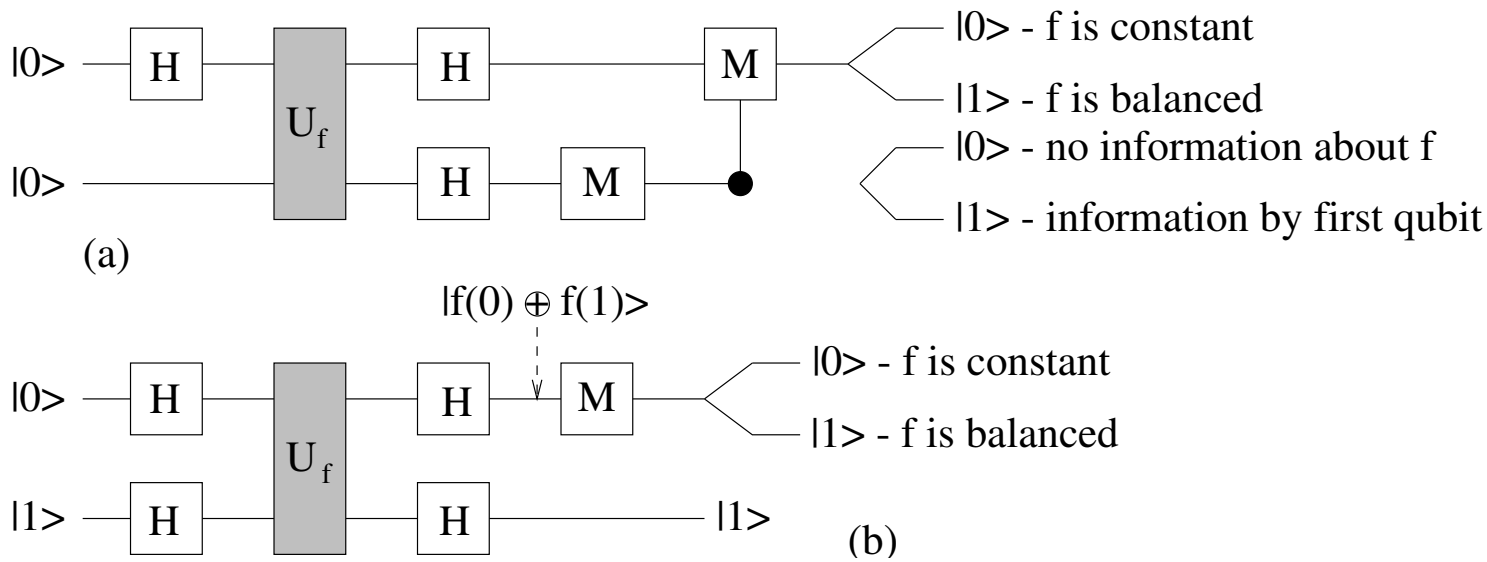
The corresponding circuit is shown in the following Figure (a).



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&= \frac{1}{2}(\sum_{x=0}^{1}(-1)^{f(x)}|x\rangle)(|0\rangle-|1\rangle) \\
&= \frac{1}{2}(-1)^{f(0)}(|0\rangle+(-1)^{f(0)\oplus f(1)}|1\rangle)(|0\rangle-|1\rangle). \qquad (1)
\end{aligned}$$

From the right side in (1), the two possibilities for f to be constant lead to the left sides in (2) and (3) and two possibilities for f to be balanced lead to the left sides in (4) and (5):

$$\frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) = |0'\rangle|1'\rangle \text{ if } f(0) = 0; \quad (2)$$

$$\frac{1}{2}(|0\rangle + |1\rangle)(|1\rangle - |0\rangle) = -|0'\rangle|1'\rangle \text{ if } f(0) = 1; \quad (3)$$

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By measuring the first bit, with respect to the dual basis, we can immediately see whether f is constant or balanced.

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Unfortunately, there is no projection measurement that can faithfully distinguish such non-orthogonal states.

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What has been achieved by these operations? The values of f were transferred to the amplitudes!!!

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$$|\phi_1\rangle = \alpha|\psi_a\rangle + \beta|\psi_b\rangle \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1,$$

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A single measurement of $|\phi_1\rangle$, with respect to $\mathcal{D}$, therefore provides the solution of the problem with probability 1.

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One measurement therefore shows whether f is balanced or not.

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The probability of error can be reduced to less than $\frac{1}{2^k}$ with only $k + 1$ queries.

Therefore, in spite of the fact that there is an exponential gap between deterministic classical and exact quantum query complexity, the gap between randomized classical complexity and quantum query complexity is in this case constant in the case of constant error.

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The task is to determine which of the above conditions holds for f and, in the second case, to determine also s.

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$n-1$ independent applications of the scheme *Hadamard-twice* therefore produce $n-1$ pairs $(y_1, f(x_1)), \ldots, (y_{n-1}, f(x_{n-1}))$, distributed uniformly and independently over all pairs $(y, f(x))$.

To solve the problem two registers are used, both with n qubits, and the initial states $|0^{(n)}\rangle$, and (expected) $\mathcal{O}(n)$ repetitions of the following version of the so-called Hadamard-twice scheme:

1. *Apply the Hadamard transformation on the first register, with the initial value* $|0^{(n)}\rangle$*, to produce the superposition* $\frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x, 0^{(n)}\rangle$.
2. *Apply* U_f *to compute* $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x, f(x)\rangle$.
3. *Apply Hadamard transformation on the first register to get*

$$\frac{1}{2^n} \sum_{x,y\in\{0,1\}^n} (-1)^{x\cdot y} |y, f(x)\rangle.$$

4. *Observe the resulting state to get a pair* $(y, f(x))$.

Case 1: f is one-to-one. After performing the first three steps of the above procedure all possible states $|y, f(x)\rangle$ in the superposition are distinct and the absolute value of their amplitudes is the same, namely 2^{-n}.

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$$(y_1, f(x_1)), \ldots, (y_{n-1}, f(x_{n-1})) \text{ such that } y_i \cdot s \equiv 0 \pmod 2,$$

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If these vectors are linearly independent, then the system of $n - 1$ linear equations in $\mathbf{Z}_2$,

$$y_i \cdot s \equiv 0 \pmod n$$

can be solved to obtain s.

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If $f(0) \neq f(s)$, then f is one-to-one. If the vectors obtained by the scheme

Hadamard-twice are not linearly independent, then the whole process has to be repeated.

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Hence at least in half of the cases any classical algorithm needs to perform $\Omega(\sqrt{2^n})$ f-queries.

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The basic trick is to compute the sum in (6) but only for x such that $hw(x) \leq k$,

Let F_k be such a function that for $x \in \{0,1\}^n$,

$$F_k(x) = \begin{cases} x \cdot w_f \text{ if } hw(x) \leq k \\ 0; \text{ otherwise} \end{cases}$$

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Therefore if V_{F_k} is applied to the (initial) state

$$|\psi_k\rangle = \frac{1}{\sqrt{M_k}} \sum_{x \in \{0,1\}^n}^{hw(x) \leq k} |x\rangle,$$

where $M_k = \sum_{i=0}^{k} \binom{n}{i}$,

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In order to compute $|\psi'_k\rangle$, at most k calls of f are needed. Let us now measure all n qubits of $|\psi''_k\rangle = H_n|\psi'_k\rangle$.

The probability that this way we get w_f is

$$Pr(|\psi''_k\rangle \text{ yields at measurement } w_f) = |\langle w_f|\psi''_k\rangle|^2 = \frac{M_k}{2^n} = \frac{1}{2^n}\sum_{i=1}^{k}\binom{n}{i}$$

and, as one can easily calculate, this probability is more than 0.95 if $k = \frac{n}{2} + \sqrt{n}$.

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For a given $f : \{0,1\} \to \{0,1\}$ we define an oraculum mapping

$$C_f(a+bi) = (-1)^{0 \oplus f(0)} a + (-1)^{1 \oplus f(1)} bi$$

For the four possible functions f we get the following four functions C_f:

$$\begin{array}{lll} C_{00}(x) = x^* & \text{if} & f(0)=0, f(1)=0 \\ C_{01}(x) = x & \text{if} & f(0)=0, f(1)=1 \\ C_{10}(x) = -x & \text{if} & f(0)=1, f(1)=0 \\ C_{11}(x) = -x^* & \text{if} & f(0)=1, f(1)=1 \end{array}$$

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The Deutsch problem can now be formulated as follows: A function is chosen secretly from the set of functions $\{C_{00}, C_{01}, C_{10}, C_{11}\}$ and the task is to determine, with a single query, which type of the function it is - balanced or constant.

Algorithm Given f, calculate $(i-1)C_f(1+i)$. If the outcome is real, then the function chosen is balanced; otherwise it is constant.

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Correctness:

$$\begin{aligned}
(i-1)C_{00}(1+i) &= (i-1)(1-i) = 2i \\
(i-1)C_{01}(1+i) &= (i-1)(1+i) = -2 \\
(i-1)C_{10}(1+i) &= (i-1)(-1-i) = 2 \\
(i-1)C_{11}(1+i) &= (i-1)(1-i) = -2i
\end{aligned}$$