Characterization of local quantum processes by Local Quantum Uncertainty

Indrani Chattopadhyay
Department of Applied Mathematics,
University of Calcutta

Quantum Disentanglement

- This is a **local quantum process**.
- It is defined on composite system consists of two or more subsystems, so that **resulting state is separable.**The disentangling machine(DM) is then defined as
- $$\rho^{e} \xrightarrow{DM} \rho^{d}$$
- It assumed to **preserve the local properties** of the state, by preserving the state of each subsystem.

$$\rho_A = Tr_B\left(\rho^e\right) = Tr_B\left(\rho^d\right)$$
$$\rho_B = Tr_A\left(\rho^e\right) = Tr_A\left(\rho^d\right)$$

- An exact universal disentangling process may defined in two ways.

  (i) The process may defined to convert every entangled input states to some product states.

$$\rho_{AB} \rightarrow \rho_A \otimes \rho_B$$

  (ii) Disentangling process, that transforms any entangled input state to some separable states .

$$\rho_{AB} \rightarrow \rho'_{AB} = \sum_i \omega_i \rho_A^i \otimes \rho_B^i \ ; \ Tr_A(\rho') = Tr_A(\rho), Tr_B(\rho') = Tr_B(\rho)$$

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No-Disentanglement Theorem

- Exact universal disentangling machine doesn't exist in either process.
- Hence disentangling process is defined to be either state-dependent exact disentanglement or it can be universal inexact disentanglement.

Results on state-dependent exact disentanglement

- Any set of perfectly distinguishable states can be disentangled.
- Any set of states with identical reduced density matrices can be disentangled.
- Any set of maximally entangled states can be disentangled.

Inexact Universal Disentanglement

- An inexact universal disentangling machine can disentangle any entangled state, and for which the local system are related by the reduction factors η_A and η_B as

$$Tr_B\left(\rho^d\right) = \eta_A \rho_A + \left(\frac{1-\eta_A}{2}\right) I_A$$

$$Tr_A\left(\rho^d\right) = \eta_B \rho_B + \left(\frac{1-\eta_B}{2}\right) I_B$$

- Where the reduction factors are independent of the initial entangled state and $0 \le \eta_A, \eta_B \le 1$

Universal Disentangling Machine

- Suppose two parties *A* and *B* share an entangled state of two qubit system given by

  $|\psi\rangle = \alpha\ |00\rangle + \beta\ |11\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$.

- A disentangling machine will then assumed to be some unitary operation acting on any one subsystem with some Machine state or two Local unitary operation acting of the two subsystems.

Local Cloning induces Disentanglement

- Bandyopadhyay et.al. proposed that entanglement of bipartite system $\mathcal{H}_{AB}$, can be reduced by introducing local isopropic cloner to any subsystem (Say A).

$$U|0\rangle_{A'}|b\rangle_{A''}|M\rangle = a|0\rangle_{A'}|0\rangle_{A''}|M'\rangle + b\left(|0\rangle_{A'}|1\rangle_{A''} + |1\rangle_{A'}|0\rangle_{A''}\right)|M''\rangle$$

$$U|1\rangle_{A'}|b\rangle_{A''}|M\rangle = a|1\rangle_{A'}|1\rangle_{A''}|M''\rangle + b\left(|0\rangle_{A'}|1\rangle_{A''} + |1\rangle_{A'}|0\rangle_{A''}\right)|M'\rangle$$

- The process spilt the entanglement of the joint system between two joint system($\mathcal{H}_{A'B}$ and $\mathcal{H}_{A''B}$) each having a less amount of entanglement.

Symmetric Optimal Universal Machine

- If reduction factors are chosen to be equal, i.e.

$\eta_A = \eta_B = \eta$, the final state remain entangled for all initial state, if $\eta > 1/3$. Fidelity of cloner is related with the reduction factor by $F = \frac{1}{2}(1+\eta)$

- It is possible to disentangle arbitrary pure two-qubit entangled state, by applying universal isotropic cloner whose Fidelity $F \leq 2/3$ in one subsystem.
- If isotropic local cloning machine is applied on both of the subsystem, then the any pure bipartite state entangled states can be disentangled, if the common reduction factor

$$\eta < 1/\sqrt{3} \left(\text{i.e., } F < \frac{1+\sqrt{3}}{2\sqrt{3}} \right)$$

Asymmetric Optimal Universal Machine

- When the disentangling machine is allowed to operate locally on both the subsystems, and reduction factors are not bound to be equal(asymmetric case) then it is shown that for optimal disentanglement process the reduction factors η_A and η_B satisfy the following relation:

$$\eta_A \eta_B \leq 1/3$$

Disentanglement resulting from Decoherence Process in Open System Dynamics

- Decoherence process is the destruction of quantum interference.
- Disentanglement and Decoherence phenomena are shown to be connected by *Dodd et.al.* for open quantum dynamics. All possible initial state of the two particle system become separable after a finite time, under the evolution process that produce decoherence of both particles.

Disentangling Capacity of a Joint Unitary

In evolution of pure bipartite system, the Entangling $E^{\uparrow}(U_{AB})$ and Disentangling ($E^{\downarrow}(U_{AB})$) Capacity of a joint unitary U_{AB} are

$$E^{\uparrow}(U) = \sup_{\Psi}\left[E\left(U'|\Psi\rangle\right) - E\left(|\Psi\rangle\right)\right]$$

$$E^{\downarrow}(U) = \sup_{\Psi}\left[E\left(|\Psi\rangle\right) - E\left(U'|\Psi\rangle\right)\right]$$

U' is the extension of U in the extended Hilbert space($H'_{AB}=H'_{AaBb}$) of bipartite system, introducing ancillary spaces to both subsystem, $|\psi\rangle$ be an arbitrary state of H'.

For any **2×2** unitary **$E^{\uparrow}$(U) =$E^{\downarrow}$(U)** whereas from **2×3** dimension the two capacities are not always equal.

Discord

- A measure of non-classicality of bipartite correlation.
- Consider a composite quantum system

$$H_{AB} = H_A \otimes H_B\,,\ \dim(H_A) = d_A\,,\ \dim(H_B) = d_B$$

- The total correlation of a density matrix ρ_{AB} of the composite system is characterized by the quantum mutual information
- $I(\rho) = H(\rho_A) + H(\rho_B) - H(\rho)\,,$ (1)
- where H(.) is the von Neumann entropy function.
- ρ_A *and* ρ_B are local subsystems of parties A and B respectively. where H(.) is the von Neumann entropy function.
- ρ_A *and* ρ_B are local subsystems of parties A and B respectively.

- A generalization of the classical conditional entropy is *$H(\rho_{B|A})$, where $\rho_{B|A}$ is the state of the subsystem B* given a measurement on subsystem *A. By optimizing over all possible* measurements in *A, we get an alternative version of* mutual information as
-
- $$Q_A(\rho) = H(\rho_B) - \min_{\{E\kappa\}} \sum_k \rho_\kappa H(\rho_{B|\kappa}) \quad , \quad (2)$$
- Where $\rho_{B|k} = \frac{Tr_A(E_k \otimes I_B \rho)}{Tr(E_k \otimes I_B \rho)}$ is the state of B
- conditioned on outcome k of the measurement performed on subsystem A and $\{E_k\}$ represents the set of positive operator valued measure(POVM) elements.

- Then the discrepancy between the two measures of information defined above in equations (1) and (2) will be termed as Quantum Discord :

$$D_A(\rho) = I(\rho) - Q_A(\rho) \qquad (3)$$

- States having highly mixed in this sense, though not have much entanglement, but may used as resource for performing some information theoretic tasks exponentially faster than any classical algorithm.
- Even separable states having this resource are shown to be powerful than classical system.

- The discord is always non-negative.
- The value of this measure reaches zero for classically correlated states.
- Discord is not a symmetric quantity *$D_A(\rho)$ and $D_B(\rho)$ denotes the left* discord and *right discord of ρ .*
- *If $D_A(\rho) = D_B(\rho) = 0$, then the state ρ is said to be completely classically correlated.*

Classical Quantum States

- The states of a quantum system with zero value of quantum discord, are known as Classical-Quantum states.
- A state ρ has zero-discord if and only if there exist a von Neumann measurement $\{\Pi_k = |\psi_k\rangle\langle\psi_k| \; ; \; k\}$

such that,
$$\sum_k (\Pi_k \otimes I_B)\rho(\Pi_k \otimes I_B) = \rho \qquad (4)$$

- The zero-discord state is of the form

$$\rho_{AB} = \sum_k p_k |\psi_k\rangle\langle\psi_k| \otimes \rho_k$$

- Where $\{|\psi_k\rangle\}$ is some orthonormal basis set, ρ_k are the quantum states of subsystem *B* and p_k are non-negative numbers such that $\sum_k p_k = 1$.
- The set of zero-discord states is not convex.

- We consider the singular value decomposition of ρ as diag[c_1, c_2, . . .]. Singular value decomposition defines new basis in local Hilbert-Schmidt spaces

$$S_n = \sum_{n'} U_{nn'} A_{n'} \qquad F_n = \sum_{m'} W_{mm'} B_{m'}$$

- *The state ρ in the new basis is of the* form

$$\rho = \sum_{n=1}^{L} c_n S_n \otimes F_n$$

where L is the rank of correlation matrix *R* (i.e., the number of non-zero eigenvalues c_n*).*

- The necessary and sufficient condition (4) becomes

$$\sum_k \Pi_k S_n \Pi_k = S_n \; ; \; \mathrm{n} = 1,2,\cdots \mathrm{L}$$

- This is equivalent to : $\left[S_n, \Pi_k\right] = 0 \; ; \forall \; \mathrm{k}, \mathrm{n}$

- This means that the set of operators $\{S_n\}$ *have common eigenbasis defined by the* set of projectors $\{\Pi_k\}$. Therefore, the set $\{\Pi_k\}$ exists if and only if:

$$[S_m, S_n] = 0 \;\; \forall \;\; \mathrm{m, n} = 1, 2, \cdots, \mathrm{L}$$

- By checking a maximum of $^{L(L-1)}/_2$ number of these commutators, one may identify the zero discord states, where $L = rank(R) \leq min\{d_A^2, d_B^2\}$.
- Now zero-discord state ρ is a sum of d_A *product* operators. This bounds the rank of the correlation tensor to $\boldsymbol{L \leq d_A}$.
- *Thus, the rank of the correlation tensor is itself the* discord witness: If $L > d_A$, *the state has a non-zero discord.*

Geometric Discord

- Geometric discord of a quantum state is defined as the minimum distance from the set(Ω_0) of states with zero quantum discord

$$D_{G_1}(\rho) := \min_{\chi \in \Omega_0} \|\rho - \chi\|_2^2$$

where $||.||_2$ is the Hilbert Schmidt distance.

- Geometric discord can also be defined as

$$D_G^X(\rho) := \min_{\{\Pi^X\}} \left\|\rho - \Pi^X(\rho)\right\|_2^2$$

minimization is taken over all local von Neumann measurement $\{\Pi^X\}$ on party X.

Geometric Discord of Two-Qubit system

- Any state of two-qubit system can be expressed as

$$\rho_{AB} = \frac{1}{4}\left[I_2 \otimes I_2 + x^t \lambda \otimes I_2 + I_2 \otimes y^t \lambda + \sum_{i,j} t_{ij} \lambda_i \otimes \lambda_j \right]$$

- where $\lambda=(\lambda_1,\lambda_2,\lambda_3)^t$; λ_i are generators of SU(2).
- Let $\mathrm{T}=\{t_{ij}\}$ and $k_{\max}$ is maximum eigenvalue of the matrix $xx^t + \mathrm{T}\mathrm{T}^t$
- Geometric Discord of two-qubit system can be described as

$$D_G^A(\rho_{AB}) = \tfrac{1}{4}\left[\|x\|^2 + \|T\|^2 - k_{\max} \right]$$

Coherence Vector Representation of Bipartite System

- In coherent vector representation a bipartite state ρ_{AB} of composite system $H_A \otimes H_B$ with $\dim(H_A) = d_A$ and $\dim(H_B) = d_B$, can be expressed as

$$\rho_{AB} = \frac{1}{d_A d_B} \mathrm{I}_A \otimes \mathrm{I}_B + \frac{1}{2d_B} \sum_{i=1}^{d_A^2-1} x_i \left(\lambda_{A_i} \otimes \mathrm{I}_B\right) + \frac{1}{2d_A} \sum_{i=1}^{d_B^2-1} y_i \left(\mathrm{I}_A \otimes \lambda_{B_i}\right)$$

$$+ \frac{1}{4} \sum_{i=1}^{d_A^2-1} \sum_{j=1}^{d_B^2-1} K_{ij} \left(\lambda_{A_i} \otimes \lambda_{B_j}\right)$$

where coherent vectors for reduced density matrix ρ_A and ρ_B are

$$(x_1, x_2, \cdots)^t = tr\left(\rho_{AB} \lambda_A \otimes \mathrm{I}_B\right), (y_1, y_2, \cdots)^t = tr\left(\rho_{AB} \mathrm{I}_A \otimes \lambda_B\right)$$

- The generators of SU(d_A) and SU(d_B) are denoted as λ_{A_i} and λ_{B_j} respectively. I_A and I_B are identity vectors of subsystem A and B. Also, a matrix with elements

$$K_{ij} = tr\left(\rho_{AB}\lambda_{A_i} \otimes \lambda_{B_j}\right)$$

are required for this representation.

The triplet $\{x, y, K\}$ define a tensor as,

$$\Lambda = KK^t - y^2 xx^t$$

- This tensor is used to characterize the correlations of the bipartite state ρ_{AB}.

Condition for ρ being product state

- A bipartite state ρ_{AB} is a product state, if and only if the criterion tensor Λ=0 or rank(Λ)=0.

CONDITION FOR ρ BEING ZERO-DISCORD STATE

- A bipartite state ρ_{AB} is a state of zero discord, if and only if the criterion tensor Λ satisfies $rank(\Lambda) \leq d_A - 1$.

Local Quantum Uncertainty

- For a bipartite quantum state ρ_{AB}, Girolami et.al. give us the concept of local quantum uncertainty(LQU). It is defined as

$$U_A^{\Lambda} = \min_{K^{\Lambda}} I\left(\rho_{AB}, K^{A}\right)$$

- The minimization is performed over all non-degenerate spectrum Λ(characterized as local maximally informative observable)

$$K^{\Lambda} = K_A^{\Lambda} \otimes I$$

LQU as Measure of Bipartite Quantumness

- The LQU quantifies the minimum amount of uncertainty in a quantum state. Non-zero value of this quantity for a bipartite state ρ_{AB} indicates the non-existence of any quantum certain observable for ρ_{AB}.
- This quantity vanishes for all zero discord state w.r.t. measurement on party A.
- LQU is invariant under local unitary operation.
- It reduces to entanglement monotone for pure state. For pure bipartite states LQU reduces to linear entropy of reduced subsystems.

From the analytical formula for local quantum uncertainty it has been shown that the whole class of $\mathcal{U}\otimes\mathcal{U}$ invariant states (including Werner and Isotropic Class) in $n \otimes n$ systems, possess quantum correlation.

Effect of Universal Isotropic Disentanglement Process on Geometric Discord of Two-Qubit State

- For a two qubit state

$$\rho_{AB} = \frac{1}{4}\left[I_A \otimes I_B + x^t \lambda_A \otimes I_B + I_A \otimes y^t \lambda_B + \sum_{i,j} K_{ij} \lambda_{A_i} \otimes \lambda_{B_j} \right]$$

- Geometric Discord is

$$D_G^A(\rho_{AB}) = \frac{1}{4}\left[\|x\|^2 + \|K\|^2 - k_{\max} \right]$$

where K is the matrix, present in coherent vector representation of the state and $k_{\max}$ is the largest eigenvalue of the matrix $xx^t + KK^t$

- Under an universal isotropic disentanglement process initial state ρ_{AB} changes to the final state

$$\rho'_{AB} = \frac{1}{4}\left[\begin{array}{c} \mathrm{I}_A \otimes \mathrm{I}_B + \eta_A x^t \lambda_A \otimes \mathrm{I}_B + \mathrm{I}_A \otimes \eta_B y^t \lambda_B \\ + \sum_{i,j} \eta_A \eta_B K_{ij} \lambda_{A_i} \otimes \lambda_{B_j} \end{array} \right]$$

where η_A , η_B are the reduction factors.

- The Geometric Discord of the final state is

$$D_G^A(\rho'_{AB}) = \frac{1}{4}\left[\eta_A^2 \|x\|^2 + \eta_A^2 \eta_B^2 \|K\|^2 - k'_{\max} \right]$$

- where $k'_{\max}$ is maximum eigenvalue of the matrix

$$\eta_A^2 x x^t + \eta_A^2 \eta_B^2 K K^t$$

Optimal Disentanglement

- It is known that for the case of optimal asymmetric disentanglement process

$$\eta_A \eta_B \leq \tfrac{1}{3}$$

- The criterion tensor Λ' for ρ'_{AB} is given by

$$\Lambda' = \eta_A^2 \eta_B^2 \left(KK^t - y^2 xx^t \right) = \eta_A^2 \eta_B^2 \Lambda$$

- Final state ρ'_{AB} is a product state if and only if intial state ρ_{AB} is a product state.
- ρ'_{AB} is a zero discord state if and only if ρ_{AB} is a zero discord state.
- The above two findings lead us to conclude that

Optimal Disentanglement Process is a local process that preserve exactly zero discord state.

- All the non-classical correlations that have zero value for only zero discord states in two qubit systems, quantum disentanglement is the only local process that preserves their structures. For example:
- Quantum Discord,
- Geometric Discord,
- Relative Entropy of discord,
- Local Quantum Uncertainty

Co-authors of the paper
are
Ajay Sen & Debasis Sarkar

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