Distribution of Quantum Correlations In Heisenberg Spin Systems

V. SUBRAHMANYAM
IIT Kanpur

Collaboration: Aritra Kundu (ICTS)
A. Lakshminarayan (IIT, M)

IPQI-14, Bhubaneshwar

Plan

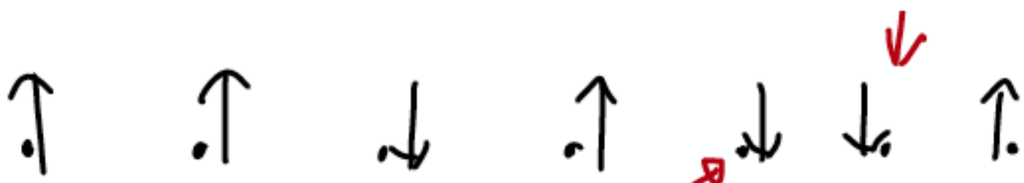
* **Distributions:** over various parts in a pure state
  pair correlations over a space of states

* **Quantum Correlations:** Pair-wise entanglement,
  pair Discord, Conditional Entropy

* **Heisenberg Spin Systems:** SU(2) spin-$\frac{1}{2}$
  Isotropic / Anisotropic Spatially Uniform
  Ground State $S = 0, \cdots \frac{N}{2}$ $|S^z| = 0, \cdots \frac{N}{2}$

* Results

Many-Spin States: Qubit at every site 2^N-Dimensional



Fixed Number of sites, up spins, down spins

Distinct labeled spatial part

Calculate: $\langle S_i^z \rangle$ $\langle S_i^z S_j^z \rangle$ $\langle S_i^+ S_j^- \rangle$

Magentization Diagonal Off-Diagonal

$$\Gamma_{ij} \approx m^2 + \frac{A}{r_{ij}^{p}} + B\, e^{-\frac{r_{ij}}{\xi}}$$

$m \neq 0$ Diagonal LRO

$A \neq 0$ Long-ranged Correlations

Questions: Which states exhibit more Entanglement?

How Quantum Correlations are Distributed?

Behavior Near Critical point?

Spin States : $|\psi\rangle = \sum_{s_1 s_2 \cdots s_N} \phi(s_1 s_2 \cdots s_N)\, |s_1 s_2 \cdots s_N\rangle$ 2^N-Dimension

As many Numbers

A general 30-qubit State : $4^{30} \approx 2^{18}$

30-bit Q-chip $\equiv$ A Billion 1-GB Pen drives!

ρ_i : Entanglement between i & Rest :

ρ_{ij} : Ent. between (ij) & Rest

Ent. between i & j

2-party Entanglement in a Mixed state

Similarly ρ_{ijk} contains 3-party Entanglement

$N\,(\text{1-party}) + {}^N C_2\,(\text{2-party}) + \cdots = 2^N$ Numbers

How Quantum Corretions Distributed Over various parts ?

States with definite # of ↑ spins:

Conservation

$$[\rho, S^z] = 0 \Rightarrow [\rho_{Block}, S^z_{Block}] = 0$$

Selection Rules

Structure of Reduced Density Matrix

$$\rho_i = \begin{array}{cc} |\uparrow\rangle & |\downarrow\rangle \end{array}$$
$$\rho_i = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$\langle \frac{1}{2} - S_i^z \rangle$

Magnetization

$$\begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array}$$

← Basis States

$$\rho_{ij} = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & w_1 & x & 0 \\ 0 & x^* & w_2 & 0 \\ 0 & 0 & 0 & v \end{pmatrix}$$

$\langle (\frac{1}{2} - S_i^z)(\frac{1}{2} - S_j^z) \rangle$

$\langle S_i^+ S_j^- \rangle$

Off-Diagonal Correlation fn

Time Reversal Invariance: $x = x^*$

How Entangled is a Spin State?

Ave. mixedness
Global Entanglement

$$\boxed{\mathcal{E}(\psi) = \frac{2}{N} \sum_{\ell} 1 - \mathrm{Tr}\,\rho_\ell^2}$$

Larger $\Rightarrow$ Better Ent. sharing

Two-party Entanglement: Eigenvalues of $\rho_{ij}\rho_{ij}^T$

$$C_{ij} = \mathrm{Max}\left(0, \lambda_1^{1/2} - \lambda_2^{1/2} - \lambda_3^{1/2} - \lambda_4^{1/2}\right) = 2\,\mathrm{Max}\left(0, |x| - \sqrt{uv}\right)$$

$$\mathcal{E} = 1 - 4m^2$$

$$\boxed{\frac{1}{2} C_{ij} = |\Gamma_{off}| - \sqrt{\left(\tfrac{1}{4} + \Gamma_D\right)^2 - m^2}}$$

Off-Diagonal Correlations Dominate for $C_{ij} \neq 0$

Ferro Order (poor Superposition) $\Rightarrow$ DLRO

Does not support Entanglement

GS Ferromagnet: $S = \frac{N}{2}$ $S^z = \frac{N}{2} - M$ $\left|\frac{N}{2}, \frac{N}{2} - M\right\rangle = \frac{1}{{}^N C_M} \sum_l |l\rangle$

$$\mathcal{E} = 4\,\frac{M(N-M)}{N^2} \qquad C_{ij} = C_{ave} = \frac{2m(N-m)}{N(N-1)}\left(1 - \sqrt{\frac{(m-1)(N-1-m)}{m(N-m)}}\right)$$

$$\boxed{M=1 \quad \mathcal{E} \simeq \frac{4}{N} \quad C_{ave} = \frac{2}{N}} \qquad \boxed{M = \frac{N}{2} \quad \mathcal{E} = 1 \quad C_{ave} = \frac{1}{N-1}}$$

One-Magnon State: $S^z = \frac{N}{2} - 1$ $|\psi\rangle = \sum_l \phi_l |l\rangle$

Location of $\downarrow$ spin

Plane wave State
$C_{ave} = \frac{2}{N}$

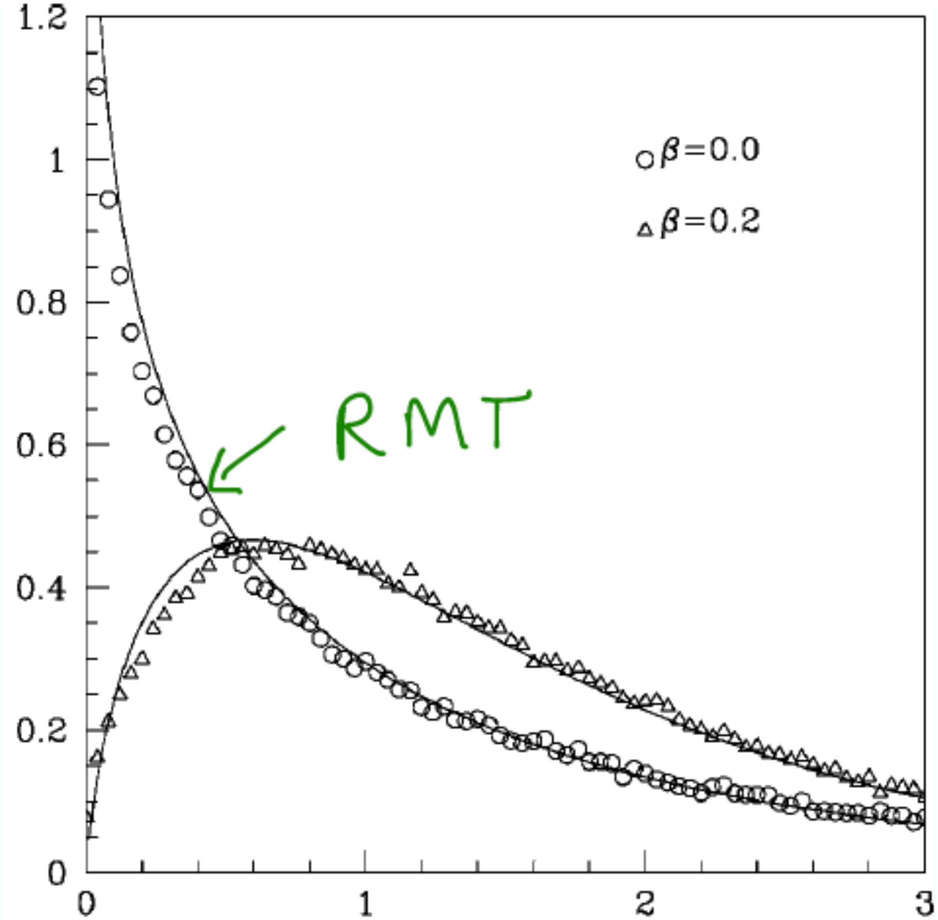
Random GOE/GUE Amplitudes ϕ_l:

$$\boxed{\begin{aligned} C_{ave} &= \frac{2}{N}\,\frac{2}{\pi} \quad \text{TR} \\ &= \frac{2}{N}\,\frac{4}{\pi} \quad \text{No TR} \end{aligned}}$$

Breaking Time Reversal $\to$ Better Ent. Sharing

AL, VS (2003)

Concurrence Distribution



$p(c)$

c

$\phi_{l+n} = e^{i 2\pi\beta} \phi_l$

One-Magnon

$\tau = 0.8 \quad N = 101$

$\beta = 0$ TR GOE

$p(c) = \frac{1}{\pi} K_0\left(\frac{c}{2}\right)$

$p(c) = c\, K_0(c)$

GUE

$$H = \sum_{j=1}^{N} \left[\frac{1}{2}(\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}) \;\; + g\cos(2\pi j/N)\hat{c}_j^\dagger \hat{c}_j \sum_{n=-\infty}^{\infty} \delta(2\pi t/\tau - n) \right]$$

Kicked-Harper Model

GS Antiferromagnet: S=0 Spatially uniform

$\langle S_i^x S_j^x \rangle = \langle S_i^y S_j^y \rangle = \langle S_i^z S_j^z \rangle$ $\quad \langle S_i^z \rangle = 0$ Time Reversal

$\Gamma_{ij}^{off} = \langle S_i^+ S_j^- \rangle = 2\Gamma_{ij}$ $\quad u = v = \frac{1}{4}\Gamma_{ij}$ $\quad |\Gamma_{ij}| \leq \frac{1}{4}$

$\Gamma_{ij} \approx \frac{\sqrt{\log r}}{r}$

$$\boxed{\begin{aligned} C_{ij} &= 0 \qquad \Gamma_{ij} > 0 \quad |\Gamma_{ij}| < \tfrac{1}{12} \\ &= 6\left(|\Gamma_{ij}| - \tfrac{1}{12}\right) \quad |\Gamma_{ij}| > \tfrac{1}{12} \end{aligned}}$$

Nearest-Neighbor $C_{12} = 6\left(\frac{|E_g|}{3N_n} - \frac{1}{12}\right)$

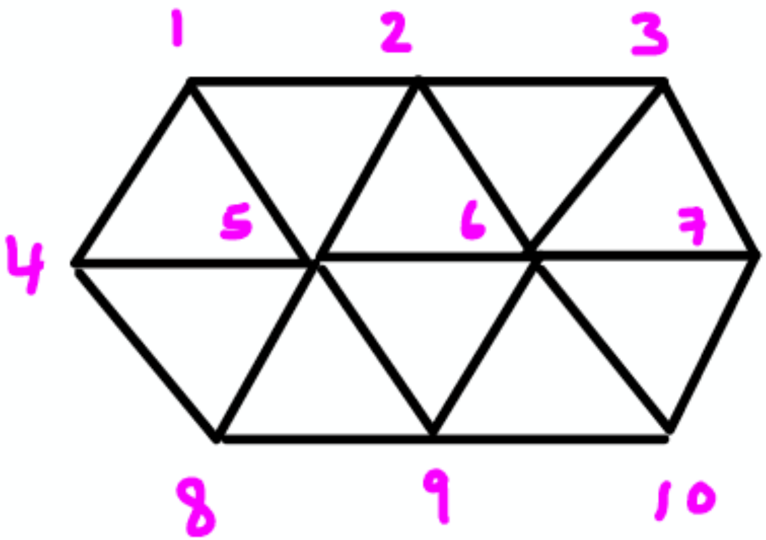
1-D $\quad C_{12} = 0.398 \quad C_{1l} = 0 \quad l > 2$

$\boxed{\xi = 1}$

2-D $\quad C_{12} = 0.16 \quad C_{ij} = 0$

Square $\quad$ Triangular/Kagome

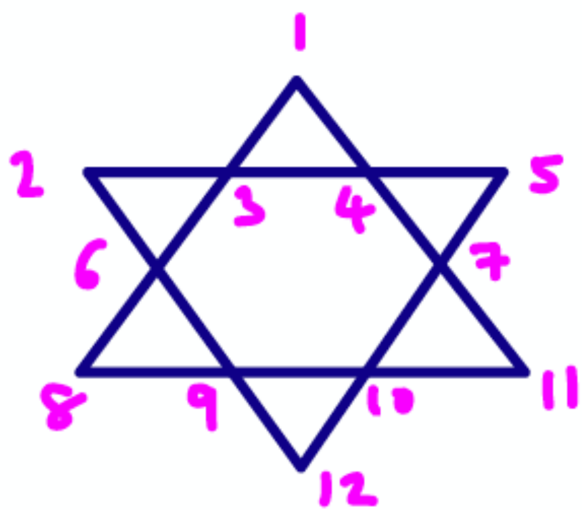
VS (2004)



$C_{12} = 0.26$
$C_{14} = 0.37$
$C_{56} = 0.33$

and equivalent bonds

C_{ij} zero for other Bonds



GS Twofold Deg.

A. $C_{14} = 0.55$

B. $C_{13} = 0.55$

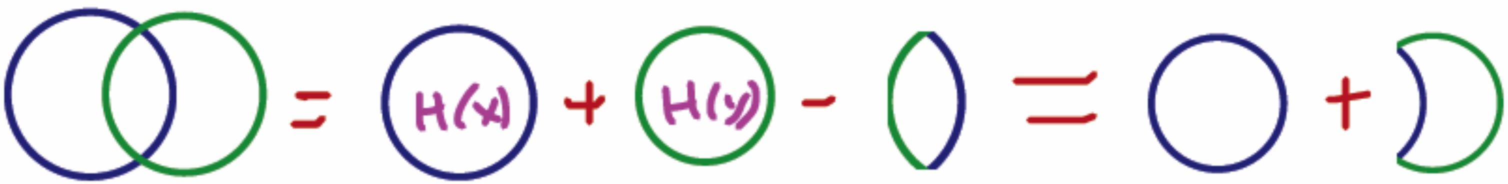
Alternate Bonds on Boundary

In the infinite lattice all pair Concurrences zero

Mutual Information: Quantum Correlations

Joint Distribution: $p(x,y),\ p(x),\ p(y),\ q_x(y)$

$H(y|x) = \sum p(x)\, H(q_x)$



Conditional Prob. $p(x,y) = p(x)\, q_x(y)$

$$\boxed{I = H(x) + H(Y) - H(x,Y)} = \boxed{H(Y) - H(Y|x) = \tilde{I}}$$

Quantum Joint State ρ_{AB}

Need to Do Measurement

$\rho_{AB} \longrightarrow \rho_{A|B} \to C(\rho_{A|B})$

Cond. Entropy Depends on Basis

Joint State: $\rho_{AB}, \rho_A, \rho_B, \rho_{A|B}$

Choose Basis: $|0\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2} e^{i\phi}|\downarrow\rangle$

$|0\rangle, |1\rangle$ for B   Measure Operator $\vec{S}_B \cdot \hat{n}$

$$\boxed{\rho_{A|B} = p_0 \rho_{A|B_0} + p_1 \rho_{A|B_1}}$$

$p_0 = \mathrm{Tr}\, |0\rangle_B\langle 0| \rho_{AB}$

$$C_{\theta,\phi}(\rho_{A|B}) = p_0 S(\rho_{A|B_0}) + p_1 S(\rho_{A|B_1})$$

$$D(A,B) = \min_{\theta,\phi} C_{\theta,\phi}(\rho_{A|B}) + S(\rho_B) - S(\rho_{AB})$$

Pure ρ_{AB}: $\rho_{A|B_0} = |\phi_0\rangle\langle\phi_0| \Rightarrow$ $\boxed{D(A,B) = S(\rho_B) = D(B,A)}$

Anisotropic Heisenberg Model

$$\boxed{\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta\, S_i^z S_{i+1}^z}$$

Total S^z
Good Q Number

TR

$\Delta < 0$ Ferro $\Delta > 0$ Antiferromagnetic

$\Delta = 1$ Kosterlitz-Thouless Q Transition

$\Delta = 1$ Total S
Good Quantum Number

Bethe Ansatz Solvable $\Gamma_r \approx \frac{\sqrt{\log r}}{r}$

$$\rho_{ij} = \begin{pmatrix} u & & & \\ & w_1 & x & \\ & x^* & w_2 & \\ & & & v \end{pmatrix}$$

$\Gamma^{\text{off}} = k\, \Gamma^{\text{Diag}}$

$k = 2,\ \Delta = 1$

$$\frac{1}{4(1-k)} \leq \Gamma_r \leq \frac{1}{4(1+k)}$$

Positivity of ρ_{ij}

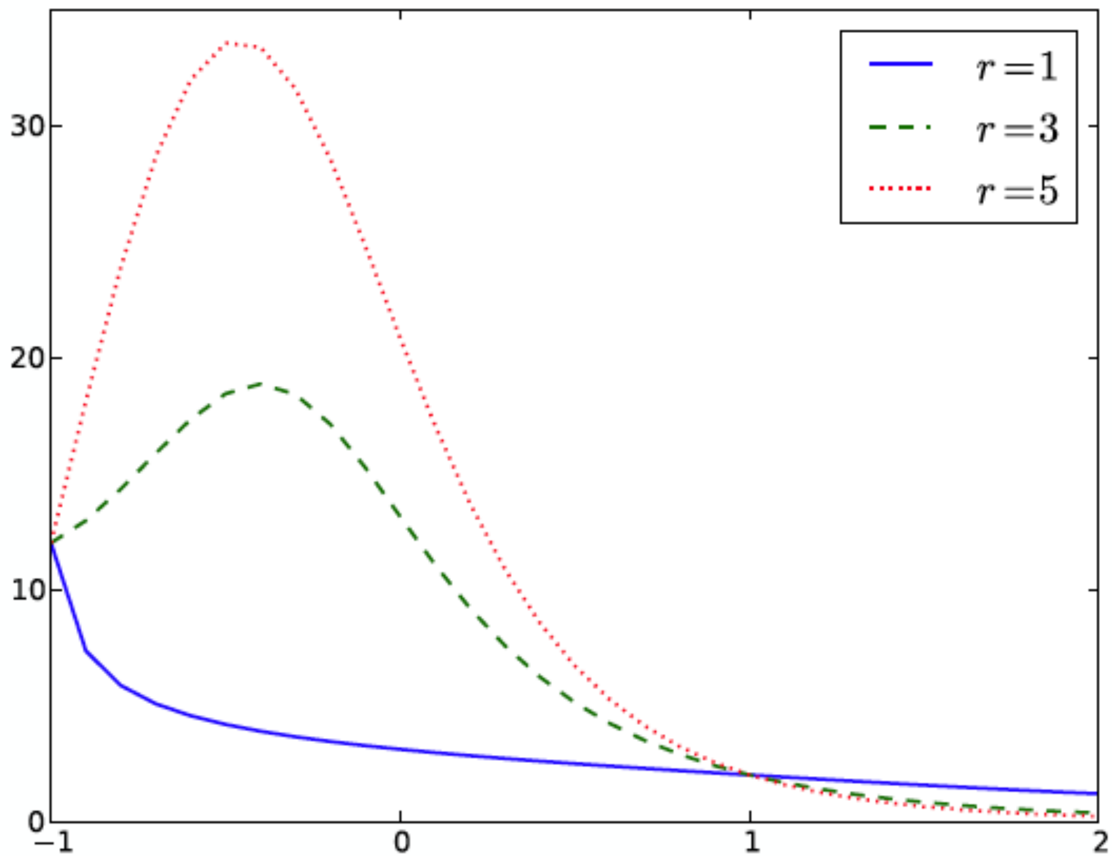
Optimization of $C_{\theta,\phi}$: Independent of ϕ U(1) Sym

$$\boxed{\begin{aligned} C_{\theta=0}(\rho_{S_i|S_j}) &= H\left(\tfrac{1}{2} + 2\Gamma_r\right) \\ C_{\theta=\frac{\pi}{2}}(\rho_{S_i|S_j}) &= H\left(\tfrac{1}{2} + k\Gamma_r\right) \end{aligned}}$$

$$\boxed{D(S_i, S_j) = \operatorname{Min}\{C_0, C_{\frac{\pi}{2}}\} + S(\rho_j) - S(\rho_{ij})}$$

Ratio: Off-Diag / Diag. Corr. fn.

$$\frac{\Gamma^{\mathrm{Off}}_{1,1+r}}{\Gamma^{\mathrm{Diag}}_{1,1+r}}$$



Δ

N = 22 Spins

Periodic Chain

For all r, $\Delta < 1$

Ratio > 2

S^x Basis Preferred

for Min $S(\rho_{S_1 S_{1+r}})$

For $\Delta > 1$, Ratio < 2

S^z Basis Preferred

$$\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

Schematic Behavior Of $D(S_1, S_{1+r})$

Compltely Determined By Diagonal Corr. fn.

$$\Gamma_r \approx \frac{e^{-r/\xi}}{r}$$

Near Critical point

$\xi \to \infty$ Slow Decay

Long-ranged $\Gamma_r \Rightarrow$ Long-ranged Discord

At $\Delta = 1$ $D(S_1, S_{1+r})$ exhibits Critical Behavior

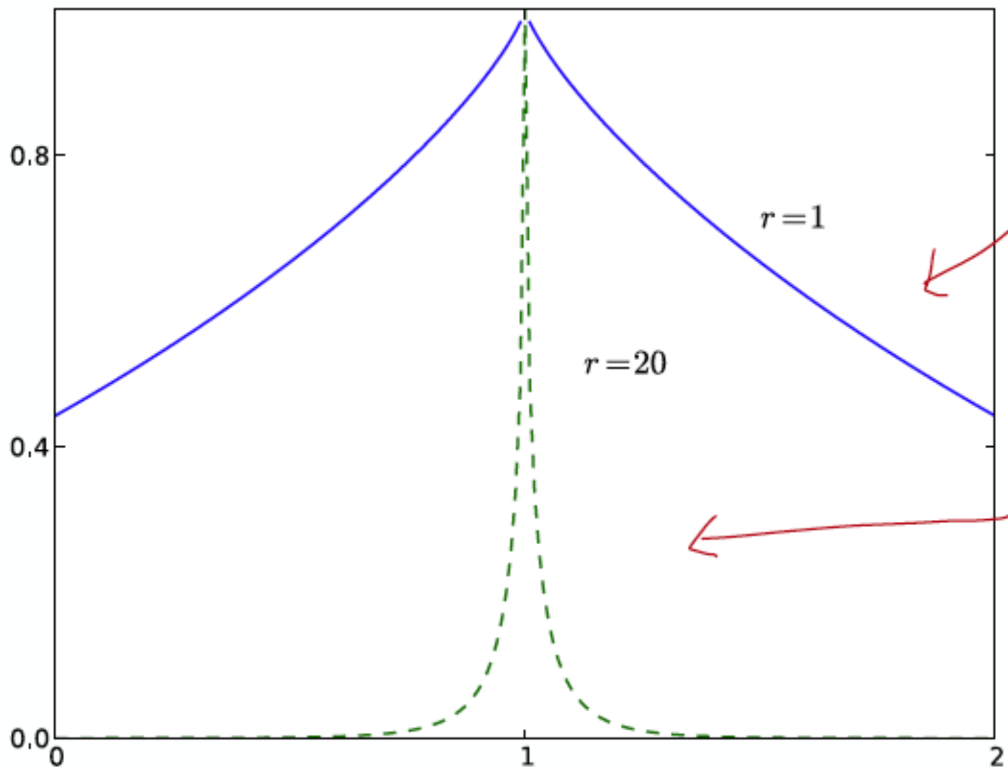
$r = 1$ Discord Depends on Specific Heat Singularity

$$\boxed{D(S_1,S_{1+r}) = 1 + 3\left(\tfrac{1}{4}+\Gamma_r\right)\ln\left(\tfrac{1}{4}+\Gamma_r\right) + \left(\tfrac{1}{4}-3\Gamma_r\right)\ln\left(\tfrac{1}{4}-3\Gamma_r\right) + H\left(\tfrac{1}{2}+2\Gamma_r\right)}$$

A. Kundu, VS J Phys A (2013)

Schematic Behavior of Discord

$$\frac{D_t(S_1, S_{1+r})}{D_{t=1}(S_1, S_{1+r})}$$



t

Singularity Related to Sp. Heat Exponent

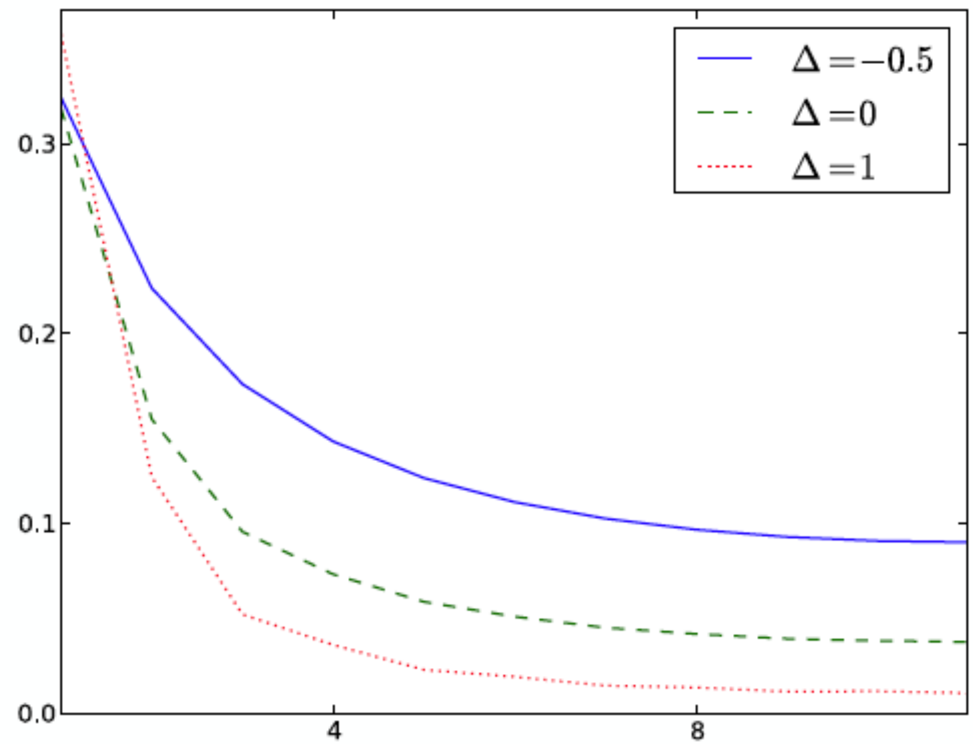
Singularity Related to Correlation Length Exponent

Critical Behavior as $t \to 1$

$$\xi \sim |t|^{-\nu} \qquad C_h \approx |t|^{-\alpha}$$

Pair Discord Vs Distance

$D(S_1, S_{1+r})$



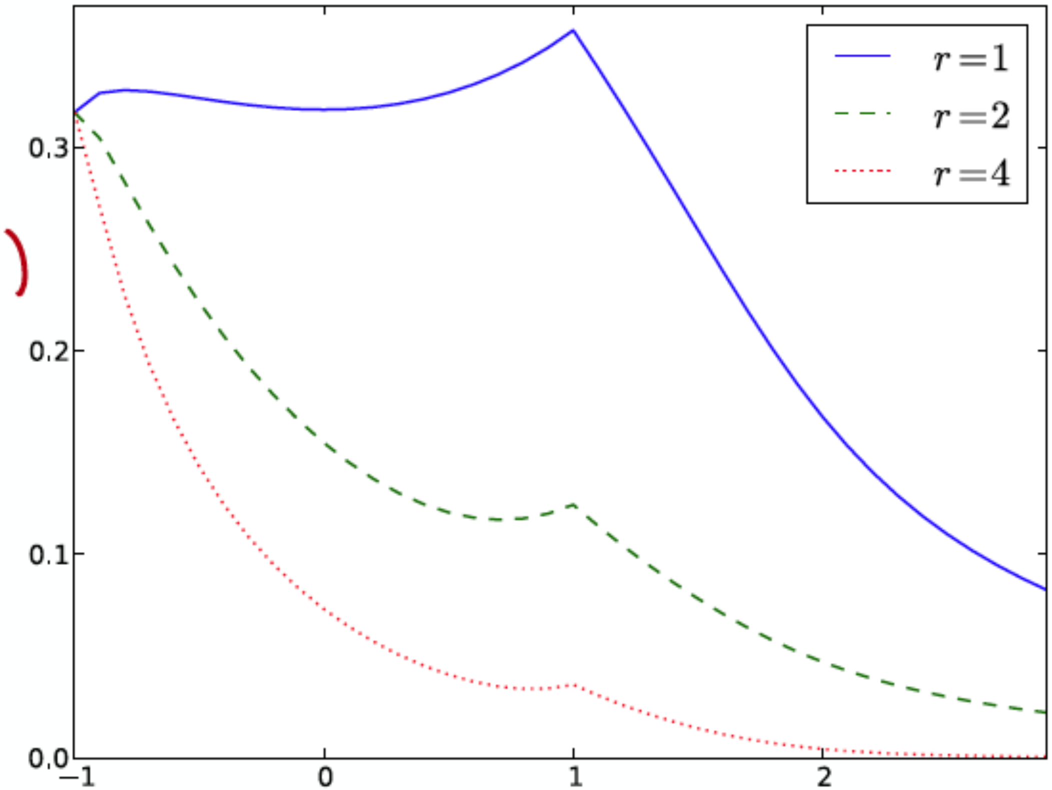
r

N = 22 spins
Periodic Chain

$\Delta < 0$ Ferromagnetic
Slow Decay

$$\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta\, S_i^z S_{i+1}^z$$

Discord Vs Anisotropy Δ



$D(S_1, S_{1+r})$

Δ

$N = 22$ Spins

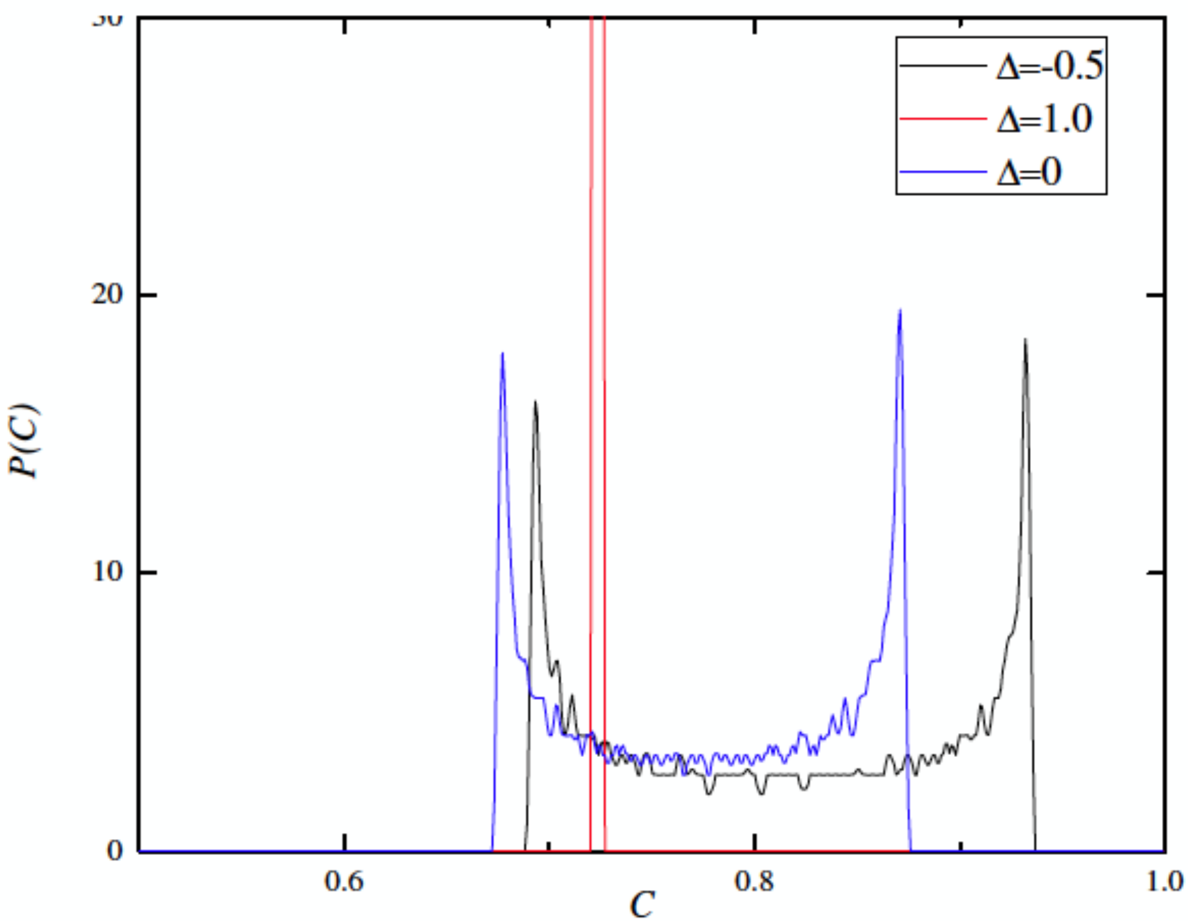
$\Delta < -1$, $D = 0$ for all pairs

Kink at $\Delta = 1$
Kosterlitz-Thouless Quantum Phase Transition

$$\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

$$P(C) = \frac{1}{4\pi}\int d\Omega \;\; \delta\left(C - C_{\theta,\phi}[\rho_{S_i S_{i+r}}]\right)$$

Nearest-Neighbor Conditional Entropy



N=22 Spins

Coarse-grained

Bin Size $\delta C = 0.005$

$\Delta \neq 1$

Peak at lower value

Slightly Less Probable

Which Determines

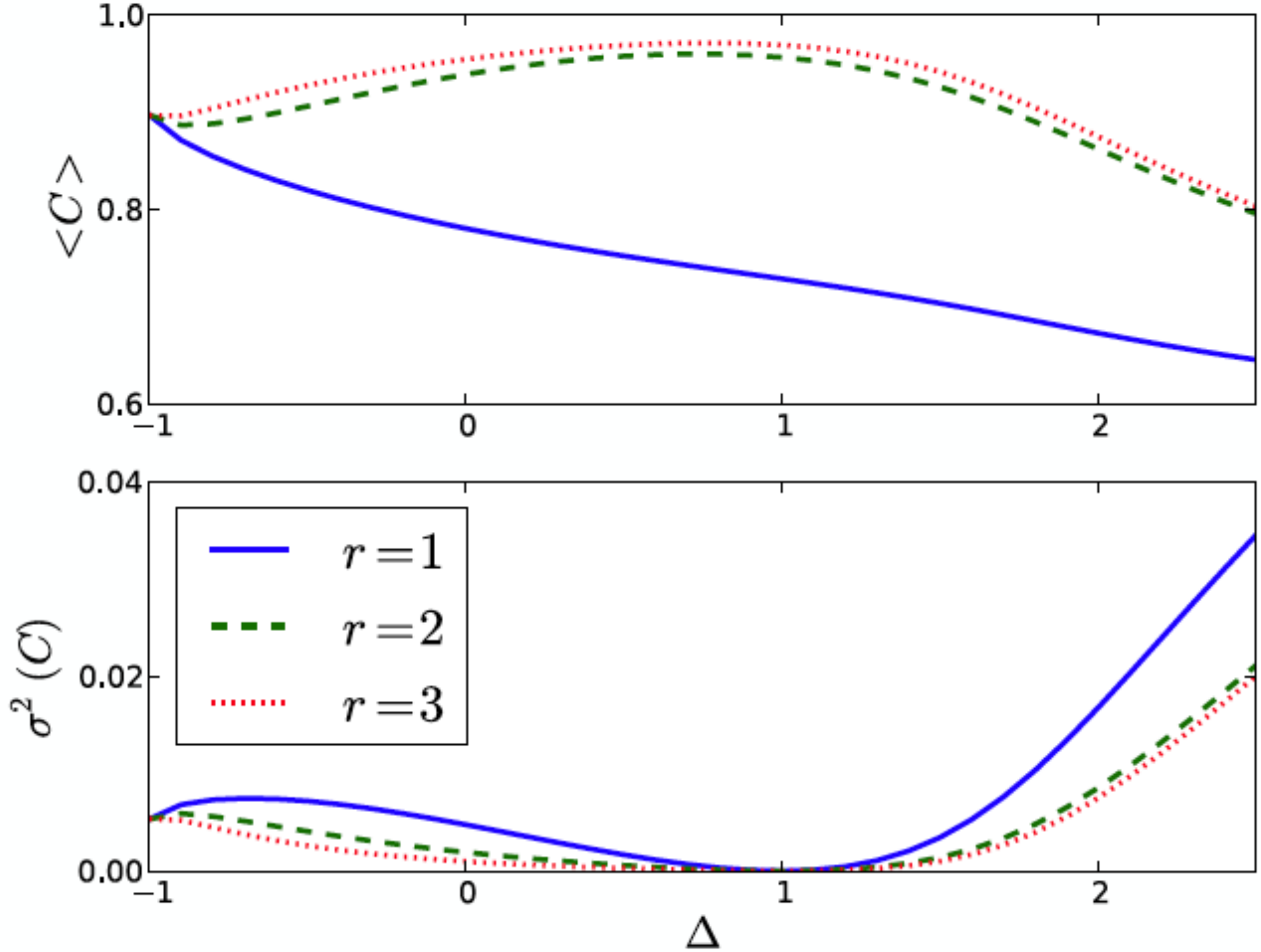
Quantum Discord

Larger-r Distribution

Shifts to right slightly

$$\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta\, S_i^z S_{i+1}^z$$

Moments of Cond. Entropy Distribution



$N = 22$ spins

$\langle c \rangle = \langle C(\rho_{s_1 s_{1+r}}) \rangle$

over all possible Measurement Bases

$\Delta = 1 \quad \sigma^2 = 0$

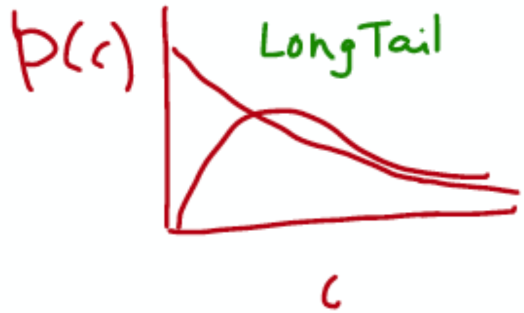
$P(c)$: δ-function

$$\mathcal{H} = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

Conclusions

Distributions of Quantum Correlations

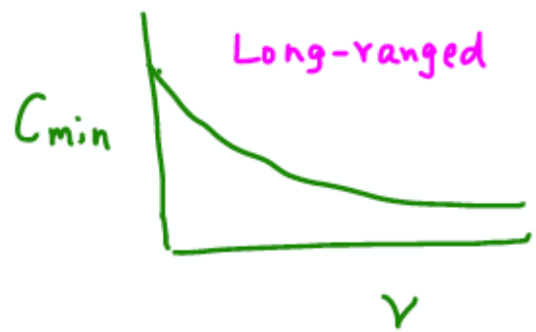
Concurrence

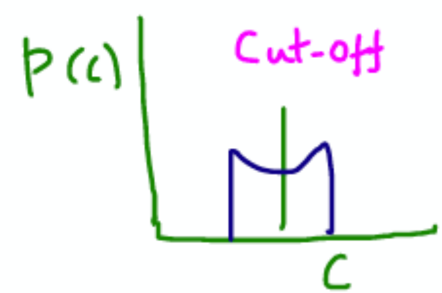


Pair Concurrences from all possible Multi-qubit States

In a Multi-Qubit Pure State

Discord / Conditional Entropy





Pair Cond. Entropies from ρ_{ij} as a distribution

Thank You

for

Your Attention