Device-independent quantum key distribution based on Hardy’s paradox

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Art of Cryptography

In a cryptosystem, if Alice wishes to send messages to Bob then:

- Alice must have an encoding key, which allows her to encrypt her message.
- Bob also must have the matching decoding key, which allows Bob to decrypt the encrypted message.
Private key cryptography

A simple, yet highly effective private key cryptosystem is the Vernam cipher, sometimes called a one time pad cryptosystem.

Figure: One time pad cryptosystem.
Difficulties in Private Key

- The key bits cannot be reused for any future protocol.
- Key bits must be delivered in advance, guarded assiduously until used.

Possible Solutions

- Public key cryptosystems
  - Security based on computational complexity
  - Can be broken by quantum computers!
- Quantum cryptography
  1. Security based on Laws of Physics
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Quantum key distribution Protocols (QKD)

- **BB84 protocol** [C. H. Bennett, G. Brassard, 1984].
  \[ \{|0_z\rangle, 1_z\rangle, 0_x\rangle, 1_x\rangle\} \]

- **Ekert’s QKD protocol** [A. K. Ekert, PRL 91].
  \[
  |\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} \left[ |0_z\rangle_A |0_z\rangle_B + |1_z\rangle_A |1_z\rangle_B \right]
  
  = \frac{1}{\sqrt{2}} \left[ |0_x\rangle_A |0_x\rangle_B + |1_x\rangle_A |1_x\rangle_B \right].
  
  Key: \{ |0_z\rangle, |0_x\rangle \} \mapsto 0 \text{ and } \{ |1_z\rangle, |1_x\rangle \} \mapsto 1.
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Is QKD unconditional secure?
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Possible attacks on BB84-QKD protocols

An adversary could easily manipulate the apparatus such that the QKD scheme becomes completely insecure.

For example, instead of encoding and measuring in two different bases, Alice always use the same basis.

The eavesdropper can measure the photon in this basis without disturbing it.
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Possible attacks on Ekert’s protocols

Here, Alice and Bob assumed that $|\Phi^+\rangle_{AB} \in C^2 \otimes C^2$ is a two qubit state and producing the following correlations:

\[
P(ab|\sigma_x\sigma_x) = P(ab|\sigma_z\sigma_z) = \frac{1}{2} \quad \text{if } a = b
\]

\[
P(ab|\sigma_x\sigma_z) = P(ab|\sigma_z\sigma_x) = \frac{1}{4} \quad \text{for all, } a, b
\]

A separable (hence insecure) state $\rho_{AB} \in C^4 \otimes C^4$ also gives the same correlations.

\[
\rho_{AB} = \frac{1}{4} \sum_{u,v=0}^{1} |u_z^0 v_z^1\rangle_A \langle u_z^0 v_z^1| \otimes |u_z^0 v_z^1\rangle_B \langle u_z^0 v_z^1|
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where the measurements are $\sigma_z \otimes I$ for the setting ‘0’ and $I \otimes \sigma_z$ for the setting ‘1’.
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Eve can now have a perfect copy of the local states of Alice and Bob, for instance if they share the tripartite state

$$\rho_{ABE} = \sum_{u,v=0}^{1} |u_z v_z^1_A \langle u_z^0 v_z^1 | \otimes |u_z^0 v_z^1_B \langle u_z^0 v_z^1 | \otimes |u_z^0 v_z^1_E \langle u_z^0 v_z^1 |$$

Thus, the security proofs only hold when the dimension of the system is known, which cannot be assumed if the adversary supplies the devices.
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$$\rho_{ABE} = \sum_{u,v=0}^1 \left| u_z^0 v_z^1 \right>_A \left( u_z^0 v_z^1 \right>_B \otimes \left| u_z^0 v_z^1 \right>_E \right| \left( u_z^0 v_z^1 \right>_A \otimes \left| u_z^0 v_z^1 \right>_B \otimes \left| u_z^0 v_z^1 \right>_E$$

Thus, the security proofs only hold when the dimension of the system is known, which cannot be assumed if the adversary supplies the devices.

For the security proof of quantum key distribution, it is, therefore, assumed that the devices are trustworthy and work exactly as specified.
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Assumptions (Hidden) for secure QKD

QKD is often claimed to be unconditionally secure but, it actually does make certain assumptions.

- The assumption always present in any key agreement is that Alice and Bob have secure laboratories. This assumption is crucial and cannot be removed.

- A further assumption is that Alice and Bob have complete control over their physical devices (i.e., only the quantum channel is corrupted) and know their exact and complete specification.
Device-independent quantum key distribution based on Hardy’s paradox

Introduction

Quantum key distribution Protocol

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Minimum assumptions

**Goal:** To reduce the above assumptions to a minimum, in particular, to remove all assumptions about the exact working of the physical devices.

- Therefore, the devices could be manufactured by the adversary.
- The security should only rely on testable features of the devices, for example, the statistics of their behaviour.
- The honest parties would then only need to trust their ability to do classical calculations (to compute the statistics) and the shielding of their laboratories.
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Device-independent QKD

In a device-independent QKD analysis, Alice and Bob would not only distrust the source of particles, but they would also distrust their measuring apparatuses.

- They assume that the measurement directions may for instance drift with time due to imperfections in the apparatuses, or the entire apparatuses may be untrusted because they have been fabricated by a malicious party.

- Also they cannot even make assumptions about the dimension of the Hilbert space in which they are defined.
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Introduction

What is Device-independent QKD Protocol?

Device-independent quantum key distribution

\[ \text{Figure: Alice and Bob see their quantum devices as black boxes producing classical outputs, } a \text{ and } b \text{, as a function of classical inputs } X \text{ and } Y. \]

\textbf{Goal of Alice & Bob:} From the observed statistics, and without making any assumption on the internal working of the devices, they should be able to conclude whether they can establish a secret key secure against a quantum eavesdropper.
Introduction

What is Device-independent QKD Protocol?

Previous works on Device-independent QKD

- In 2007, Acín et al., introduced a device-independent QKD protocol based on Bell-CHSH inequality secure against collective attacks.
- In 2011, Masanes et al. provided a more general security scheme based on causally independent measurement processes.

But the security of all these protocols is undermined as the measurement at step $k$ may depend on the classical or quantum memory of all previous inputs and outputs.

Recently secure protocols where device re-use is allowed were introduced.
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Recently secure protocols where device re-use is allowed were introduced.
Hardy’s Paradox [L. Hardy, PRL 1992].

Consider a physical system consisting of two subsystems shared between Alice and Bob.

\[
\begin{align*}
A_0/A_1 & \downarrow \\
& \downarrow a = 0/1 \\
& \downarrow \\
B_0/B_1 & \downarrow \\
& \downarrow b = 0/1
\end{align*}
\]

Hardy’s conditions:  
\[
\begin{align*}
P(A_0 = 0, B_0 = 0) &= q > 0 \\
P(A_1 = 0, B_0 = 0) &= 0 \\
P(A_0 = 0, B_1 = 0) &= 0 \\
P(A_1 = 1, B_1 = 1) &= 0
\end{align*}
\]

This set of conditions cannot be satisfied by any LHVT.
Hardy's Paradox [L. Hardy, PRL 1992].

Consider a physical system consisting of two subsystems shared between Alice and Bob.

\[ A_0 / A_1 \]
\[ \downarrow \]
\[
\begin{array}{c}
\text{Alice} \\
\end{array}
\]
\[
\downarrow \\
\begin{array}{c}
a = 0 / 1
\end{array}
\]

\[ B_0 / B_1 \]
\[ \downarrow \]
\[
\begin{array}{c}
\text{Bob} \\
\end{array}
\]
\[
\downarrow \\
\begin{array}{c}
b = 0 / 1
\end{array}
\]

**Hardy's conditions:**

\[ P(A_0 = 0, B_0 = 0) = q > 0 \]
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Hardy's Paradox

Proof:

![Diagram]

Hardy's Conditions

- $P(A_0 = 0, B_0 = 0) = q > 0$
- $P(A_1 = 0, B_0 = 0) = 0$
- $P(A_0 = 0, B_1 = 0) = 0$
- $P(A_1 = 1, B_1 = 1) = 0$

First

$P(A_0 = 0, B_0 = 0) = q > 0$

Second

$P(A_1 = 0, B_0 = 0) = 0 \implies A_1 = 1$

Third

$P(A_0 = 0, B_1 = 0) = 0 \implies B_1 = 1$
Device-independent quantum key distribution based on Hardy's paradox

Proof:

\[ \begin{array}{ccc}
A_0 / A_1 & B_0 / B_1 \\
\downarrow & \downarrow \\
\text{Alice} & \text{Bob} \\
\downarrow & \downarrow \\
a = 0/1 & b = 0/1
\end{array} \]

Hardy's Conditions
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\end{align*}
\]

First \[ P(A_0 = 0, B_0 = 0) = q > 0 \implies A_1 = 1 \]

Third \[ P(A_0 = 0, B_1 = 0) = 0 \implies B_1 = 1 \]

Last \[ P(A_1 = 1, B_1 = 1) = 0 \text{ contradiction...} \]
Hardy's Paradox

Proof:

\[ A_0 / A_1 \quad B_0 / B_1 \]

Alice \quad Bob

\[ a = 0/1 \quad b = 0/1 \]

Hardy's Conditions

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Last \[ P(A_1 = 1, B_1 = 1) = 0 \text{ contradiction...} \]
Hardy’s argument and two qubit entangled state

The product states associated with all the conditions

\[ |\phi_3\rangle = |A_0 = 0\rangle |B_0 = 0\rangle; \quad P(A_0 = 0, B_0 = 0) = q > 0 \]
\[ |\phi_2\rangle = |A_1 = 0\rangle |B_0 = 0\rangle; \quad P(A_1 = 0, B_0 = 0) = 0 \]
\[ |\phi_1\rangle = |A_0 = 0\rangle |B_1 = 0\rangle; \quad P(A_0 = 0, B_1 = 0) = 0 \]
\[ |\phi_0\rangle = |A_1 = 1\rangle |B_1 = 1\rangle; \quad P(A_1 = 1, B_1 = 1) = 0, \]

where \( |X_i = j\rangle \) denotes the eigenstate of the observable \( X_i \) for the outcome \( j \).

Let \( |X_1 = 0\rangle = \alpha_X |X_0 = 0\rangle + \beta_X |X_0 = 1\rangle \) and
\( |X_1 = 1\rangle = \beta_X^* |X_0 = 0\rangle - \alpha_X^* |X_0 = 1\rangle \), where \( |\alpha_X|^2 + |\beta_X|^2 = 1 \) and
\( 0 < |\alpha_X| < 1 \) for \( X = A, B \).

Let \( S = \{ |\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle \} \), then \( \text{dim}(S) = 3 \).

If \( |\psi\rangle \) satisfies the Hardy’s conditions, then \( |\psi\rangle \perp S \). i.e., \( |\psi\rangle \in S^\perp \).

\[ \therefore |\psi\rangle \text{ is unique pure entangled state} \ [G. Kar, PLA 1997]. \]
Hardy's argument and two qubit entangled state

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Let $|X_1 = 0\rangle = \alpha_X|X_0 = 0\rangle + \beta_X|X_0 = 1\rangle$ and $|X_1 = 1\rangle = \beta_X^*|X_0 = 0\rangle - \alpha_X^*|X_0 = 1\rangle$, where $|\alpha_X|^2 + |\beta_X|^2 = 1$ and $0 < |\alpha_X| < 1$ for $X = A, B$.

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If $|\psi\rangle$ satisfies the Hardy's conditions, then $|\psi\rangle \perp S$. i.e., $|\psi\rangle \in S^\perp$.

$\therefore$ $|\psi\rangle$ is unique pure entangled state [G. Kar, PLA 1997].
Gram-Schmidt orthonormalization to find the Hardy-type state $|\psi\rangle$:

$|\phi'_0\rangle = |\phi_0\rangle$

$|\phi'_i\rangle = \frac{|\phi_i\rangle - \sum_{j=0}^{i-1} \langle\phi'_j|\phi_i\rangle |\phi'_j\rangle}{\sqrt{1 - \sum_{j=1}^{i-1} |\langle\phi'_j|\phi_i\rangle|^2}} \quad i = 1, 2.$

$|\phi_0\rangle = |A_1 = 1\rangle |B_1 = 1\rangle$

$|\phi_1\rangle = |A_1 = 0\rangle |B_0 = 0\rangle$

$|\phi_2\rangle = |A_0 = 0\rangle |B_1 = 0\rangle$

$|\phi_3\rangle = |A_0 = 0\rangle |B_0 = 0\rangle$

$\therefore$ The Hardy state $|\psi\rangle$:

$|\psi\rangle = \frac{|\phi_3\rangle - \sum_{i=0}^{2} \langle\phi'_i|\phi_3\rangle |\phi'_i\rangle}{\sqrt{1 - \sum_{i=0}^{2} |\langle\phi'_i|\phi_3\rangle|^2}}.$

The probability $q$ for this state:

$q = |\langle\psi|\phi_3\rangle|^2 = 1 - \sum_{i=1}^{7} |\langle\phi'_i|\phi_3\rangle|^2 = \frac{|\alpha_A|\alpha_B|^2 |\beta_A|\beta_B|^2}{1 - |\alpha_A\alpha_B|^2}.$

The maximum value of $q$ is $\frac{5\sqrt{5} - 11}{2} = 0.0901699$ for $|\alpha_A| = |\beta_B| = \sqrt{\frac{\sqrt{5} - 1}{2}}$ [T. F. Jordan, PRA 1994].
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∴ The Hardy state $|\psi\rangle$:

\[
|\psi\rangle = \frac{|\phi_3\rangle - \sum_{i=0}^{2} \langle \phi'_i | \phi \rangle \phi'_i}{\sqrt{1 - \sum_{i=0}^{2} |\langle \phi'_i | \phi_3 \rangle|^2}}.
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Hardy's Paradox

Device-independence of Hardy-test [Rabelo et.al., PRL 2012]

If the maximum success probability $q_{\text{max}} = \frac{5\sqrt{5} - 11}{2}$ is observed in a Hardy test, the state of the system is equivalent to

$$|\psi^H_{12} \otimes |\eta\rangle_{1'2'},$$

where $|\psi^H_{12}$ is the unique two-qubit Hardy state for $q_{\text{max}}$. 
Device-independent QKD based on Hardy’s argument

S1. Alice and Bob share many copies of Hardy states $|\psi\rangle$

S2. For each $|\psi\rangle$, Alice randomly chooses whether to measure $A_0$, or $A_1$ on her qubit. Bob does the same, measures ($B_0$ or $B_1$) on his qubit.

S3. Both announce their results but not measurement settings for all runs.
Device-independent QKD based on Hardy’s argument

**S1.** Alice and Bob share many copies of Hardy states $|\psi\rangle$

![Diagram showing the process of Alice and Bob sharing many copies of Hardy states $|\psi\rangle$.]

**S2.** For each $|\psi\rangle$, Alice randomly chooses whether to measure $A_0$, or $A_1$ on her qubit. Bob does the same, measures $(B_0$ or $B_1)$ on his qubit.

**S3.** Both announce their results but not measurement settings for all runs.
Cont....Protocol

S4. Check for eavesdropping:

For some randomly selected runs, Alice and Bob both announce their measurement choices \((A_i, B_j)\) and check the corresponding outcomes \((a, b)\) are satisfied the Hardy’s argument.

S5. From rest of the runs they generate the desire key:

For \(|\psi\rangle\),

\[
P(A_1 = 0, B_1 = 0) > P(A_0 = 0, B_0 = 0) > 0 \quad \text{and} \quad P(A_0 = 0, B_1 = 0) = P(A_1 = 0, B_0 = 0) = 0.
\]

Hence, they have correlated settings for \((0, 0)\) outcomes.
### Table for generating key

<table>
<thead>
<tr>
<th>Run</th>
<th>Alice Measurement basis ($A_s$)</th>
<th>Outcome ($a$)</th>
<th>Bob Measurement basis ($B_t$)</th>
<th>Outcome ($b$)</th>
<th>Useful for Key (Bit value)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i_1$</td>
<td>$A_0$</td>
<td>0</td>
<td>$B_1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$i_2$</td>
<td>$A_0$</td>
<td>0</td>
<td>$B_0$</td>
<td>0</td>
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</tr>
<tr>
<td>$i_3$</td>
<td>$A_1$</td>
<td>1</td>
<td>$B_1$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$i_4$</td>
<td>$A_0$</td>
<td>1</td>
<td>$B_1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$i_5$</td>
<td>$A_1$</td>
<td>0</td>
<td>$B_1$</td>
<td>0</td>
<td>✓ (1)</td>
</tr>
<tr>
<td>$i_6$</td>
<td>$A_0$</td>
<td>0</td>
<td>$B_0$</td>
<td>0</td>
<td>✓ (0)</td>
</tr>
<tr>
<td>$i_7$</td>
<td>$A_0$</td>
<td>1</td>
<td>$B_0$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$i_8$</td>
<td>$A_1$</td>
<td>0</td>
<td>$B_1$</td>
<td>0</td>
<td>✓ (1)</td>
</tr>
<tr>
<td></td>
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<tr>
<td>$i_k$</td>
<td>$A_1$</td>
<td>1</td>
<td>$B_0$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$i_{k+1}$</td>
<td>$A_0$</td>
<td>0</td>
<td>$B_1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$i_{k+2}$</td>
<td>$A_1$</td>
<td>0</td>
<td>$B_1$</td>
<td>0</td>
<td>✓ (1)</td>
</tr>
<tr>
<td>$i_{k+3}$</td>
<td>$A_1$</td>
<td>1</td>
<td>$B_0$</td>
<td>1</td>
<td></td>
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</table>
Alice and Bob choose their settings and the Hardy state $|\psi\rangle$ corresponding to maximum probability of success $q_{\text{max}} = 0.0901699$ i.e.,

$$|\alpha_A| = |\beta_B| = \sqrt{\frac{\sqrt{5} - 1}{2}} \quad \text{and} \quad |\psi\rangle = |\psi^H\rangle.$$ 

For $|\psi^H\rangle$, 

$P(A_0 = 0, B_0 = 0) = 0.0901699 < P(A_1 = 0, B_1 = 0) = 0.236068.$

So, Alice and Bob further drop some runs of settings 1 to make equal number of 0s and 1s in the key.

Key rate: $\frac{2 \times 0.0901699}{4} = 0.04509.$

But for non-uniform settings e.g., 

$(A_0/B_0 : A_1/B_1) = (0.618 : 0.382)$, the key rate is 0.06888.
Device-independent QKD

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Hardy paradox in Noise Case

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\begin{align*}
P(A_0 = 0, B_0 = 0) & \geq q - \epsilon, \\
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\]

Goal: For \( \epsilon > 0 \) small enough the protocol remains secure against general attacks.

In other words,

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P_{\text{guess}}(\mathcal{A} | a = b = 0) = \max_{A_s} P(A_s | a = b = 0) \leq G(\epsilon).
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where \( G \) is a concave function.
Device-independent quantum key distribution based on Hardy’s paradox

Device-independent QKD based on Hardy’s test

Noisy Case

Hardy paradox in Noise Case

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where \( G \) is a concave function.
How to calculate $G$?

To find $G$ we first use Bayes rule to express the conditional probability $P(A_0|a = 0, b = 0)$ as

$$P(A_0|a = 0, b = 0) = \frac{x}{x + y}$$

where

$$x \equiv P(a = 0, b = 0|A_0, B_0)P(A_0, B_0) + P(a = 0, b = 0|A_0, B_1)P(A_0, B_1)$$

$$y \equiv P(a = 0, b = 0|A_1, B_0)P(A_1, B_0) + P(a = 0, b = 0|A_1, B_1)P(A_1, B_1)$$

Similarly we have $P(A_1|a = 0, b = 0) = \frac{y}{x+y}$. 
Device-independent quantum key distribution based on Hardy’s paradox

Device-independent QKD based on Hardy’s test

Noisy Case

Guessing probability

Let \( g_0 = P(A_0|a = 0, b = 0) \), \( g_1 = P(A_1|a = 0, b = 0) \) and \( G \geq \max\{g_0, g_1\} \).

**Figure:** Guessing probability with uniform settings.

**Figure:** Non-uniform settings: 0 is chosen with the probability 0.61803405.
Device-independent quantum key distribution based on Hardy’s paradox

= Device-independent QKD based on Hardy’s test

= Noisy Case

Key rate

\[
\rho(\eta) = \frac{1 - \eta}{2} \mathbb{I} + \eta |\psi^H\rangle\langle \psi^H|
\]
Conclusion

- In our QKD protocol, the bits used for secret key do not come from the results of the measurements on an entangled state but from the choices of settings which are harder for an eavesdropper to influence.

- Instead of a single security parameter (a violation of some Bell inequality) a set of them is used to estimate the level of trust in the secrecy of the key.

Device-independent quantum key distribution based on Hardy’s paradox

THANK YOU