General Relativity

Institute of Physics Bhubaneshwar

Final Exam

Textbook: Sean Carroll's *Spacetime and Geometry*

1. {10 marks} Stereographic coordinates once again: Consider the stereographic projection of the two sphere on the plane of the equator. Schematic diagram is shown in the figure 1.

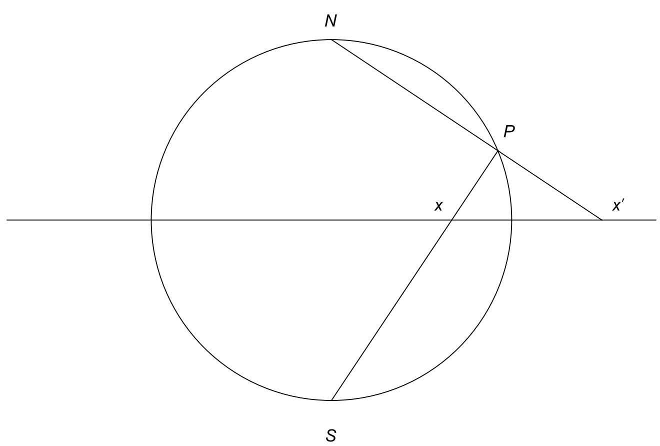

Figure 1: Stereographic projection of the equator.

There are two patches: (i) M_1: sphere with the south pole deleted, (ii) M_2: sphere with the north pole deleted. These projections map a point on the sphere (θ, ϕ) to points (X, Y) and (X', Y') respectively on the plane. Show that these maps are given by

$$\phi_1 : X + iY = e^{i\phi} \tan\frac{\theta}{2}, \tag{0.1}$$

$$\phi_2 : X' + iY' = e^{i\phi} \cot\frac{\theta}{2}. \tag{0.2}$$

Show that on the overlap

$$\phi_2 \circ \phi_1^{-1}(X, Y) = X' + iY' = \frac{1}{X - iY}. \tag{0.3}$$

2. {10 marks} Show that for an arbitrary vector field V^μ, the Jacobi identity holds

$$\left([[\nabla_\lambda, \nabla_\rho], \nabla_\sigma] + [[\nabla_\rho, \nabla_\sigma], \nabla_\lambda] + [[\nabla_\sigma, \nabla_\lambda], \nabla_\rho]\right) V^\mu = 0. \tag{0.4}$$

3. {10 marks} Show that the directional derivative of the Ricci scalar along a Killing vector field vanishes

$$K^\lambda \nabla_\lambda R = 0. \tag{0.5}$$

[HINT: Prove equation (3.176) first. Then (3.177). Then (3.178). This was one of the homework problems.]

4. {10 marks} If a vector field is of the form

$$\xi^{\mu} = g\nabla^{\mu} f, \tag{0.6}$$

where f and g and scalar functions, then show that

$$\xi_{[\mu}\nabla_{\nu}\xi_{\sigma]} = 0. \tag{0.7}$$

Also, show that in the differential form notation equation (0.7) can be written as

$$\xi \wedge d\xi = 0. \tag{0.8}$$

5. {15 marks} Imagine we have two vectors V^{μ} and W^{ν}, both of which are annihilated by a one form ξ_{μ}, i.e., $\xi_{\mu}V^{\mu} = 0$ and $\xi_{\mu}W^{\mu} = 0$, and the one form ξ_{μ} obeys

$$\xi_{[\mu}\nabla_{\nu}\xi_{\sigma]} = 0. \tag{0.9}$$

Show that this implies

$$\nabla_{[\mu}\xi_{\nu]}V^{\mu}W^{\nu} = 0. \tag{0.10}$$

[HINT: Consider $\xi_{[\mu}\nabla_{\nu}\xi_{\sigma]}V^{\mu}W^{\nu}$.]

If moreover the the vector ξ^{μ} is unit normalised, ie, $\xi_{\mu}\xi^{\mu} = 1$, show that it also follows that the tensor

$$K_{\mu\nu} = \nabla_{\mu}\xi_{\nu} - \xi_{\mu}\xi^{\sigma}\nabla_{\sigma}\xi_{\nu} \tag{0.11}$$

is symmetric.

6. {15 marks} <u>Lie derivatives and infinitesimal diffeomorphisms:</u> If you still haven't understood the relation between Lie derivatives and infinitesimal diffeomorphisms, this exam problem is your last chance. Consider a scalar function $f(x)$ and an infinitesimal diffeomorphisms

$$x^{\mu'} = x^{\mu} - \epsilon\,\xi^{\mu} + \mathcal{O}(\epsilon^2). \tag{0.12}$$

Since it is a scalar function it means that $f'(x') = f(x)$.

   (a) Show that to the leading order in ϵ

$$\delta f(x) := f'(x) - f(x) = \epsilon\,\pounds_{\xi} f. \tag{0.13}$$

(b) Next consider a vector field $A^\mu(x)$. Under the change of coordinates recall that the vector field transforms as

$$A^{\mu'}(x') = \frac{\partial x^{\mu'}}{\partial x^\nu} A^\nu(x). \tag{0.14}$$

Use this to show that

$$\delta A^\mu(x) := A^{\mu'}(x) - A^\mu(x) = \epsilon\, \pounds_\xi A^\mu(x). \tag{0.15}$$

(c) Show a similar result by designing appropriate notation for a one-form field.

7. {30 marks} Problem 5, Chapter 5. [HINT: For problem 5(b) use the correct version of equation (2.52).]