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Physica A 245 (1997) 534–542

PHYSICA A

Deadlocks and waiting times in traffic jam

Sutapa Mukherji¹, Somendra M. Bhattacharjee*

Institute of Physics, Bhubaneswar 751 005, India

Received 26 February 1997

Abstract

In a city of right moving and upmoving cars with hardcore constraint, traffic jam occurs in the form of bands. We show how the bands are destroyed by a small number of strictly left moving cars yielding a deadlock phase with a rough edge of left cars. We also show that the probability of waiting time at a signal for a particular tagged car has a power-law dependence on time, indicating the absence of any characteristic time scale for an emergent traffic jam. The exponent is the same for both the band and the deadlock cases. The significances of these results are discussed.

It is a common knowledge that vehicular traffic in a city can form a traffic jam for sufficiently high density of cars, even when the traffic rules are obeyed. The jammed-phase involves cars segregating and blocking one another over a large length-scale and time, reminiscent of cooperative phenomena. Such jams are rather ubiquitous in various situations from traffic flow to data transmission on a network, phase separation in granular materials, etc. In order to identify and characterize the basic features of traffic jams, simple cellular automata (CA)-inspired models [1–11] have gained importance over the last few years. Indeed, such CA models indicate that a traffic jam can be viewed as a dynamic transition. Apart from the practical relevance, traffic jam models constitute a special class in the general framework of driven systems. The obvious interest in such studies is in the nonequilibrium steady states, effects of broken ergodicity, nonequilibrium phase transitions, etc. [12].

The CA approach has so far been of two types: (1) one dimensional for highway traffic [7–10] and (2) two dimensional for city traffic [1–5,11], with distinct inputs for the two types. As yet, the role of dimensionality in traffic jam is not clear and, from the models, one might suspect that the traffic jams in one-dimension are rather

* Corresponding author. Fax: 91 674 581 142; e-mail: sb@iop.ren.nic.in.

¹ Present address: Institut für Theoretische Physik, Universität zu Köln, D-50937 Köln, Germany; e-mail: sutapa@thp.uni-koeln.de.

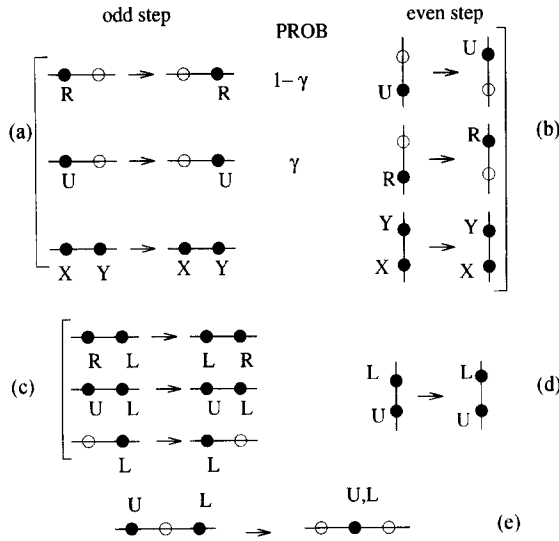


Fig. 1. The possible moves for odd and even steps. (a) and (b) correspond to the case with no left cars. The new moves with left cars are (c)–(e). (e) is special because it should not have occurred if strict hard core constraint is used. Since the cars do not look beyond nearest neighbors, this has to be accepted. X, Y stand for any type of car.

different from that in two dimensions. It is also tempting to propose that the three- or higher-dimensional models might be relevant to communication network, computer architecture, etc.

In this paper, we consider only the two-dimensional city traffic case. In the CA approach [1–5], a city is taken as a square lattice with, say, two types of cars, one set moving right and the other going up. A hard core constraint restricts occupation of two cars at the same site, preventing accidents. There is also a signal at each site (crossing) regulating traffic synchronously so that cars can move only right in the odd steps and only up in the even steps, see Fig. 1. A stochasticity is introduced by allowing a right (up) car, signal permitting, to go up (right) with probability γ . One starts with a random homogeneous initial condition and let the system evolve to its steady state according to the stochastic rules [13]. Note also that the rules ensure that the jams are not formed due to stray incidents like accidents or peculiar signals.

The deterministic model ($\gamma=0$) was shown to have a traffic jam phase [1–3] with no particular structure, while in the stochastic case ($\gamma \neq 0$), the jammed state involves well-defined bands with cars of two types phase segregated, blocking one another [6,14] provided the density, n , of cars is greater than a critical value $n_c(\gamma)$. The occurrence of bands with phase segregation is an indication of long-range order developing in the system as, e.g., in thermal phase separations, though the process here involves no thermal randomness. In Ref. [6], a simple Boltzmann-type approximation was developed, probably the only analytical approach available at this stage, and a linear stability analysis could reproduce the characteristic length of the bands

if $n > 0.5$. The phase diagram (called the fundamental diagram) is still out of reach of this approach [6].

Our purpose in this paper is twofold. One is to use this CA model for the city traffic to explore the complexity of the jammed phases, especially how a small change in the model can lead to drastic changes in the steady state. Next, we identify a procedure by which an *emergent* traffic jam can be predicted. This is done by analyzing the waiting times of a tagged car while the system is evolving towards a steady state. Both of these features are of practical importance and their significances are discussed at the end.

Our model involves an additional type of cars strictly going left. The left cars can coexist with the right cars but can block, and can be blocked by, an up car, see Fig. 1. The reason for this additional set is to introduce enough complexity in the system for the jam to have a wider variety than a simple band. There are, of course, more complicated situations, but we find this to be the minimal change that produces drastic effects. Note that in the absence of the left cars there is a symmetry for $\gamma < 0.5$ and $\gamma > 0.5$, and left cars destroy this symmetry.

The evolution of a random configuration can be monitored by the velocity that measures the average flow through a site. The velocity can be defined [6] as

$$v(t) = \frac{1}{2N} \sum_{G=R,U,L} \sum_{\mathbf{r}} (G_{\mathbf{r}}^{t+1} - G_{\mathbf{r}}^t)^2, \quad (1)$$

where N is the total number of cars, and $G_{\mathbf{r}}^t = 1$ or 0 according as site \mathbf{r} is occupied or not at time t by a car of type $G = R, U, L$ (for right, up or left moving car). A time-averaged velocity is defined as $\bar{v} = \sum_{T_1 \leq t \leq T_2} v(t) / (T_2 - T_1)$. This velocity gives a measure of the movement in the system and attains a small value if there is any traffic jam.

We also tag one randomly chosen up car and follow its trajectory as the system evolves. Because of the hard core constraint, a finite turning probability, and the signal, a car quite often stays at the same site for a certain time interval. Let us call this the waiting time, and let $P(t)$ be the probability density for waiting time t . We evaluate $P(t)$ from the trajectory of the tagged car by studying the histogram of waiting time t , i.e., by computing the number of times, Q , the car stays at a site for $t_1 < t < t_2$, so that the probability density for a particular realization is $Q/(t_2 - t_1)$. $P(t)$ is then obtained by averaging over various realizations. We show that the t dependence of $P(t)$ is different if the system evolves towards a jammed phase than in a moving phase. For the deterministic model, the power spectrum for the waiting time (i.e., its Fourier transform) at a particular site was shown to have “ $1/f$ ” behavior only at the threshold of the jamming transition [3]. We show that $P(t)$ is a more powerful way of predicting an emergent jam.

In terms of the boolean variables R, U, L , the exact stochastic evolution equations can be written as

$$R_{\mathbf{r}}^{t+1} = R_{\mathbf{r}}^t \left[\sigma^t \bar{\zeta}_{\mathbf{r}}^t + \sigma^t \zeta_{\mathbf{r}}^t (R_{\mathbf{r}+\mathbf{x}}^t + U_{\mathbf{r}+\mathbf{x}}^t) + \bar{\sigma}^t \zeta_{\mathbf{r}}^t + \bar{\sigma}^t \bar{\zeta}_{\mathbf{r}}^t (R_{\mathbf{r}+\mathbf{y}}^t + U_{\mathbf{r}+\mathbf{y}}^t) \right] \\ + \bar{R}_{\mathbf{r}}^t \bar{U}_{\mathbf{r}}^t [\sigma^t \zeta_{\mathbf{r}-\mathbf{x}}^t R_{\mathbf{r}-\mathbf{x}}^t + \bar{\sigma}^t \bar{\zeta}_{\mathbf{r}-\mathbf{y}}^t R_{\mathbf{r}-\mathbf{y}}^t],$$

$$\begin{aligned}
 U_r^{t+1} &= U_r^t [\bar{\sigma}^t \bar{\eta}_r^t + \bar{\sigma}^t \eta_r^t W_{r+y}^t + \sigma^t \eta_r^t + \sigma^t \bar{\eta}_r^t W_{r+x}^t] \\
 &\quad + \bar{R}_r^t \bar{U}_r^t \bar{L}_r^t [\bar{\sigma}^t \eta_{r-y}^t U_{r-y}^t + \sigma^t \bar{\eta}_{r-x}^t U_{r-x}^t], \\
 L_r^{t+1} &= L_r^t \sigma^t (L_{r-x}^t + U_{r-x}^t - L_{r-x}^t U_{r-x}^t) + \sigma^t \bar{U}_r^t \bar{L}_r^t L_{r+x}^t + \bar{\sigma}^t L_r^t,
 \end{aligned}
 \tag{2}$$

where $\sigma^t = t \bmod 2$ is the signal, $\zeta_r^t = 1$ or 0 with probability $1 - \gamma$ or γ denotes turning of a right-type car at site \mathbf{r} and time t , η as the corresponding variable for the up cars, \mathbf{x}, \mathbf{y} are the unit vectors for nearest neighbors, $\bar{a} = 1 - a$ for any a , and $W_r^t = U_r^t + L_r^t + R_r^t - R_r^t L_r^t - U_r^t L_r^t$ is a combination boolean variable for site \mathbf{r} . These equations can be obtained by considering the various possibilities of a car staying at the same site or coming from a neighboring site. The configuration at time $t + 1$ is determined by the configuration at time t , i.e., each car moves according to the position of the cars at time t .

A hard core constraint requires that $R_r^t U_r^t = 0$. We also have $U_r^t L_r^t = 0$ except for a U -hole- L -type configuration during a horizontal move (Fig. 1). The boolean variable W takes care of this eventuality. W is 1 if the site is occupied by any type of car and 0 if it is unoccupied. We have numerically evolved the system from arbitrary initial configurations for lattices upto 64×64 with periodic boundary conditions. We first present the results and then discuss them.

Deadlock phase: With no left cars, it is known that there is a critical density above which traffic jam occurs in the form of bands [6]. We chose a density ($n = 0.67$) higher than the critical value. As the left car density is increased we observe a roughening of the edges of the bands. In fact, if the bandwidth is small so that there are two bands in the lattice [17], then beyond a certain left-car density, the band changes to a single-one indicating that the bandwidth increases as the density increases. A more drastic change takes place at a still higher density (though much smaller compared to the overall density) when the structure loses the band pattern. This is a new phase which we call a *deadlock phase*. The structure is bounded by vertical lines on the left side and left cars on the right, as shown in Fig. 2. The deadlock phase has a strictly zero velocity compared to the band phase where there is a residual velocity ($\sim N^{-1/2}$) coming from the edges of the bands.

The evolution has been repeated several times and we find a broad region of left densities where the steady state can be of either type. Such coexistence is a reflection of a first-order transition. In fact for $\gamma = 0.8$, there is a range of left-car densities where we have observed sequentially the coexistence of “two-band”- “one-band”-phases, two-band – one-band – deadlock phases, and one-band–deadlock phase before going over to the deadlock phase.

For $\rho_l = 0$, there is a symmetry between $\gamma < 0.5$ and $\gamma > 0.5$, but not for $\rho_l \neq 0$ because of the hardcore repulsion between U and L . For $\gamma > 0.5$, the segregation of U and R is opposite to that for $\gamma < 0.5$. For a small number of left cars, with $\gamma > 0.5$, the L 's settle down at the RU interface, and as the number increases they expose some of the U 's. This leads to the roughening of the edge leading to the deadlock phase. See Fig. 2. The density of left cars for the transition from band to deadlock is a

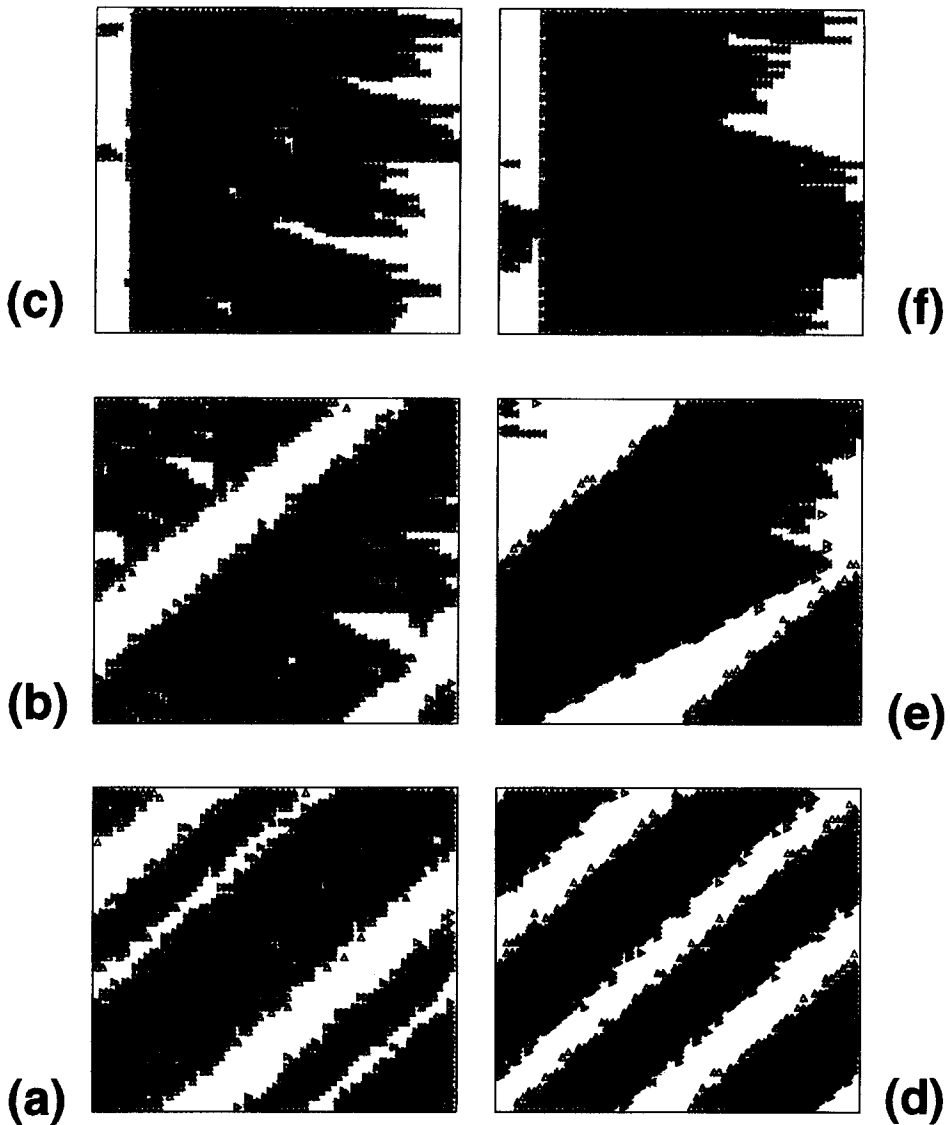


Fig. 2. The steady-state phase changes from two band (a,d) to one band (b,e) and to deadlock (c,f) as the number (#) of left cars (red) increases. All are for 64×64 lattices with periodic boundary conditions, and 2730 right (blue) and up (green) cars. (a, b, c) $\gamma = 0.2$, # = 100, 220, 420. (d, e, f) $\gamma = 0.8$, # = 270, 400, 500. Note the reversal in the car pattern for $\gamma > 0.5$. Also note the vertical wall on the left in (c) and (f).

monotonically increasing function of γ . We do not go into the details of these results because of the large coexistence region.

Waiting time: We now tag one up car at random from the beginning of the evolution. The trajectory of the tagged particle under various conditions are shown in Fig. 3. We observe that in the moving phase the car traverses a major part of the city while

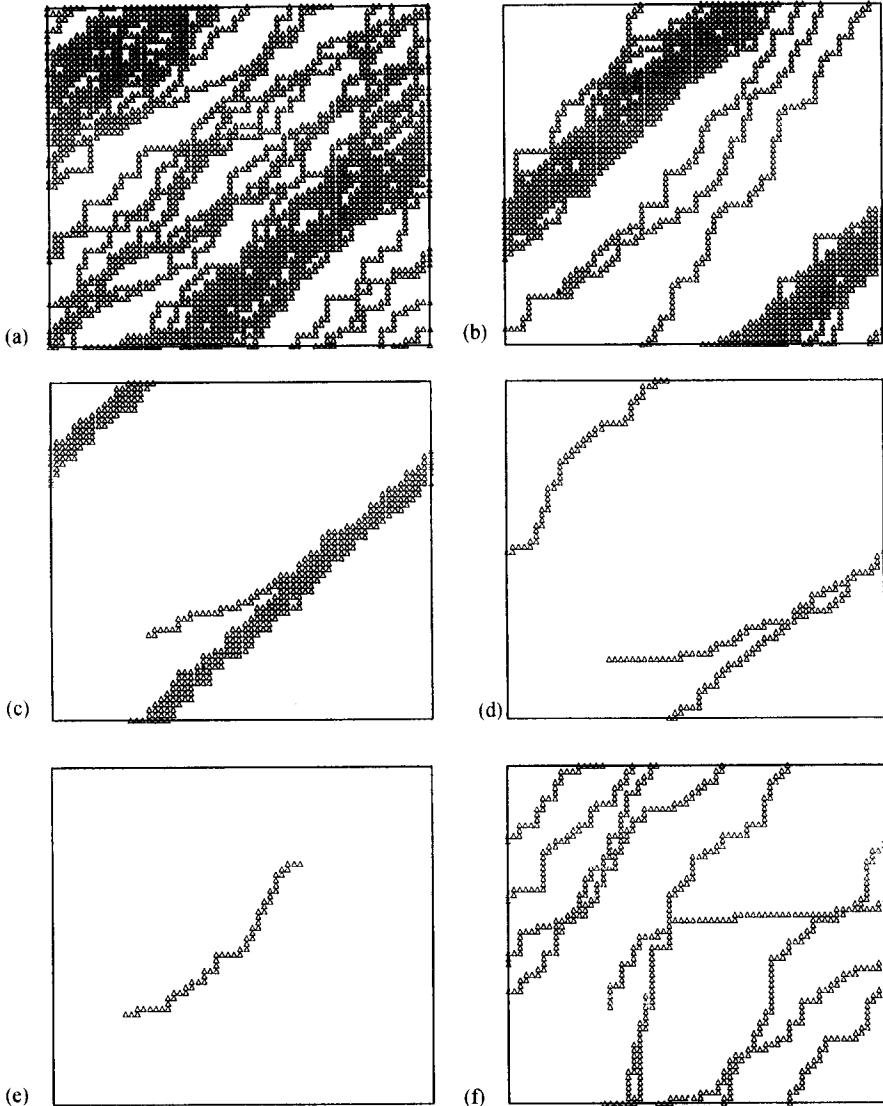


Fig. 3. The path of a tagged “up” car, for several values of γ and the number ($\#$) of left cars. (a) When there is no jam. This actually corresponds to $\gamma = 0.5$ with no left cars. (b) When the jam consists of one single band. $\gamma = 0.6$, $\# = 160$. (c,d) When the jam consists of two bands for $\gamma = 0.8$, $\# = 0$. (c) shows a car getting into the middle band, while (d) shows, from a different realization, a car getting into the outer band. (e,f) Now, the jammed state is a deadlock phase for $\gamma = 0.4$, $\# = 270$. Two possible paths are shown from two different samples. In case (e) the total path is rather small compared to (f). The starting point is always chosen, for convenience, near the lower left part of the lattice.

the motion is restricted to bands in Fig. 3b–Fig. 3e. There is no such definite pattern for the deadlock phase as shown in Fig. 3e and Fig. 3f. From these trajectories we computed the waiting probability, $P(t)$. When there is no jam, the tagged car does not spend much time at the crossings and $P(t)$ decays fast, almost exponentially. This is

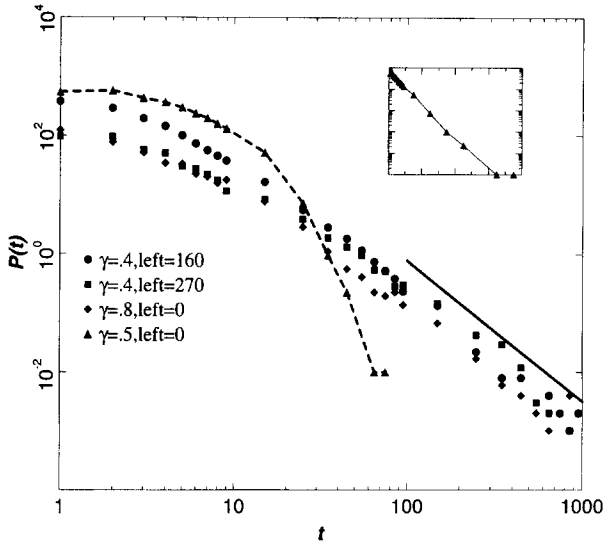


Fig. 4. $P(t)$ vs. t Plot. The values of γ and the number of left cars are given in the legend. The solid line has a slope of 2.4, indicating $P(t) \sim t^{-2.4}$. The plot for the moving phase represented by the dashed line is shown in the inset on a semilog scale.

shown in Fig. 4a for $\gamma = 0.5$ without any left cars. For values of γ , when there is a band phase or a deadlock phase, the waiting time seems to have a power-law decay $P(t) \sim t^{-w}$ with $w = 2.4$ for both the jammed states, see Fig. 4. It therefore appears that, for an emerging jammed situation, the waiting time for a car has no definite time scale. Haven't we all have that feeling when stuck in a jam?

Let us now discuss these results. First, we have shown that the simple phase diagram can be modified drastically by the addition of a *small* density of left cars. Our simulations are at a density for which, except for $\gamma = 0.5$, bands are supposed to form along the diagonal in the absence of any left cars. In the deadlock phase, there is a vertical wall on the left side but the left cars bunch together on the right side. Since the left cars, by construction, are restricted to one dimension, we concentrate on the motion of the left cars of one row only. These cars can hop to the left, only if the site is not occupied by a U or L car. Note that the left cars are transparent to the right cars. If, in the spirit of the Boltzmann approximation, the effect of the up cars is taken into account by a random blocking of a vacant site with probability $n/2$ (the density of up cars), *uncorrelated* both in time and space, no bunching can occur (we have also checked it explicitly). In other words, the bunching of left cars we observe is mainly due to a correlated blocking both in space and time. More quantitatively, one might use the Boltzmann approximation of Ref. [6] to study the stability of a homogeneous phase. In this approximation, the exact stochastic equations are replaced by time evolution equations for the average densities, ignoring all correlations. From Eq. (2), one sees that a small density of left cars would be a small perturbation to the

equations derived in Ref. [6]. Such a small perturbation does not destroy the instability towards a band phase, and, hence, cannot yield a deadlock phase.

Similarly, one can invoke a Boltzmann approximation for $P(t)$ also. An up car at time $t+1$ will stay at its site if (i) the sites it can move to are blocked, (ii) signal stops it, or (iii) the turning probability does not allow it to move. Therefore, the probability of staying over at the same site at time t is proportional to

$$p(\mathbf{r}, t) \equiv [\bar{\sigma}^t \bar{\eta}_r^t + \bar{\sigma}^t \eta_r^t W_{\mathbf{r}+\mathbf{y}}^t + \sigma^t \eta_r^t + \sigma^t \bar{\eta}_r^t W_{\mathbf{r}+\mathbf{x}}^t]. \quad (3)$$

It then follows that the probability of waiting time t at a site \mathbf{r} , if the up car comes to the site at time T , is proportional to $\prod_{m=0}^t p(\mathbf{r}, T+m)$. $P(t)$ is now obtained by averaging over all sites, all T and all realizations. In the simplest situation if we replace all the boolean variables by the averages as in the Boltzmann approximation [6], we see that in a moving phase (in absence of any left cars) $P(t) \sim [(1+n)/2]^t \sim \exp(-t |\ln(1+n)/2|)$. This shows an exponential tail consistent with our observation for the moving phase. Inclusion of a small density of left cars do not change this result significantly. To get a power-law tail, again the correlations have to play a role.

For one-dimensional highway traffic, the life time of jammed regions has been found to have a power-law tail and this has been attributed to the avalanches or self-organized critical (SOC) behavior of the model [8]. This SOC behavior is characteristic of the steady state of the one-dimensional model. In the two-dimensional case, we are observing the power-law behavior in the *approach* to the steady state. It is tempting to associate an underlying SOC-type behaviour as the origin of this power-law for waiting times, but such an identification is not apparent.

The bunching of the left cars on the right edge in the deadlock phase has another implication. If these left cars are removed from the deadlock structure, the jammed phase goes into *one* single-band structure along the diagonal, not necessarily the steady-state solution of the original zero left car density case. The deadlock phase is a compact lattice spanning structure, and to create more than one band, it is necessary to produce cracks (i.e., vacant sites) in the middle of the structure, which due to hard core repulsion and tight packing (see Fig. 2) is impossible. An interesting way to look at this is to think of the minute quantity of left cars acting as an adhesive to bring together the bands. Since, the left cars get segregated in any case, their removal from the road is rather easy, and the system relaxes to a single structure. This might be a practical way of bringing together the jammed region into the central part of the city relieving all other regions. We wonder if this helps in real life in any way.

To summarize, we have shown that the band phase for two types of cars can be modified drastically by addition of a small number of left-moving cars. The bunching of the left cars in the deadlock phase is significant. We have also shown that the waiting time has a power-law distribution as the traffic jam is approached. In contrast, in the moving phase, the long waiting times are almost absent. This gives a useful way of *predicting beforehand* whether a traffic jam is emerging or not. An experimental verification of this prediction would be highly interesting.

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