

LETTER TO THE EDITOR

Loop mediated crossover in the three-dimensional Ising model

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Abstract. We establish the existence of crossover from a three (or any) dimensional Ising model to the anisotropic five vertex model by suppressing closed loops in the isomorphic dimer problem. The crossover exponent is determined exactly using a renormalization group approach.

The two-dimensional Ising model has a crossover phenomenon lurking in the wings. This crossover in the critical behaviour changes, e.g. the weak logarithmic divergence of the specific heat to a square root divergence ($\alpha = \frac{1}{2}$) of the vertex or K -model type. This is known from exact solutions. It is most easily seen by taking a certain limit of the coupling constant of the triangular lattice model [1], or, more directly, for the hexagonal lattice, by suppressing loop type excitations in the isomorphic dimer model [2]. The domain wall mapping of [2] makes it clear that the crossover is mediated by closed loops. In their absence, one would get only lines in one particular direction, spanning the whole lattice. It is these fluctuating domain walls that drive the critical behaviour of the K -model. (See, e.g. [3, 4] for connection of this with commensurate-incommensurate transitions.) A natural question then arises: is this crossover via the loops special to two dimensions or generic to the Ising model in any dimension? This question primarily motivated this study. And, yes, we find it to be generic—even the crossover can be studied quantitatively if not exactly.

We employ a combination of lattice and continuum approaches to tackle the question posed above. The lattice approach, via the high temperature expansion, establishes the loop mediated crossover. The details of the crossover, the crossover exponent, etc, are then obtained by the continuum path integral technique using renormalization group (RG).

Lattice model. The key point is to find out a generalization of the two-dimensional hexagonal lattice. But, then, universality tells us that every minute detail of the lattice is not important for the universal exponents.

We start with the Ising model on an expanded diamond lattice in the 110 orientation (figure 1). The reason for choosing the diamond lattice in this particular orientation is the speciality of sequential arrangements of the bonds in the transverse directions (x_1, x_2, \dots, x_{d-1}) as we move along the special $x_d (=z)$ direction. The lattice is further expanded by splitting each vertex and inserting a new bond, parallel to the special z direction (figure 1). Needless to say that such lattices can be constructed in any arbitrary

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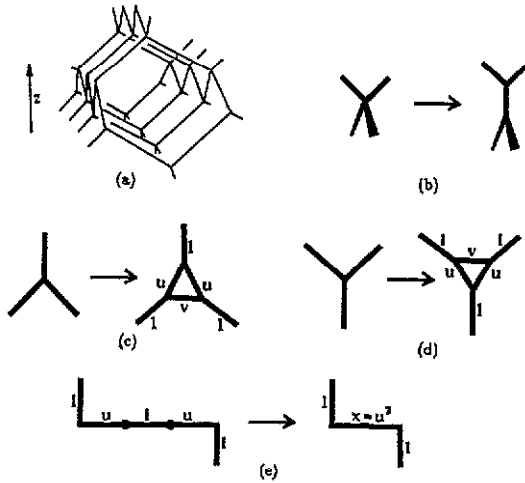


Figure 1. (a) The diamond lattice in the 110 orientation. (b) Expanding the lattice by inserting a bond at each vertex. (c) and (d) \mathcal{L}_D , the dimer lattice, with dimer activities (Fisher transformation). Sites of type (c) [(d)] are the birth (death) sites for the polymers. See [2]. (e) Reduction of \mathcal{L}_D to \mathcal{L} for $v=0$.

dimension d . At each vertex there will be a bond in the x_d ($\equiv z$) direction and two bonds in one of the remaining $d-1$ transverse directions with the transverse bonds arranged sequentially. We denote any such lattice by \mathcal{L} .

The Ising model is defined on this three coordinated lattice \mathcal{L} by the Hamiltonian $H/kT = -K \sum s_i s_j$ where k is the Boltzmann constant, T the temperature, $s_i = \pm 1$ are the spins situated on the vertices of \mathcal{L} , and the sum is over the nearest neighbours only, and $K > 0$.

The standard Fisher mapping [5] transforms the tangent hyperbolic expansion for the Ising partition function on a three coordinated lattice \mathcal{L} to a dimer covering problem† on \mathcal{L}_D . The dimer lattice \mathcal{L}_D is obtained by replacing each vertex of \mathcal{L} by a triangle (see figure 1). The dimer activities on the edges of the triangle are $u = \tanh K$, and one on other edges. The dimer partition function is $Z_D = \sum g(n) u^n$ where $g(n)$ is the number of configurations with n dimers on the edges of the triangles. This mapping is local and is valid for any graph. Still, it has, so far, been mainly used for planar lattices for which the dimer problem can be solved exactly by the Pfaffian technique [4]. Nevertheless, this transformation is essential for the present formulation.

Following [2], we now change one dimer activity (figure 1) to v . The Ising model, then, corresponds to the $v = u$ case. In the ground state, the dimers occupy the edges with activity one. The excited states are obtained by transferring dimers on the high energy bonds, thereby vacating the ground state bonds. The vacant ground state bonds can be connected uniquely to get lines or polymers or 'domain walls'. These polymers in one particular direction (commonly called directed polymers) are mutually avoiding (to respect the hard core constraint of the dimers), and either span the whole lattice in the preferred direction or terminate pairwise at a triangle if a v type edge is occupied. In other words, identical to the two-dimensional case [2], there are line type excitations that require an infinite energy (with macroscopic entropy), and there are finite energy

† In a dimer model, each site is occupied by one, and only one, dimer that sits on a bond connecting the nearest neighbours. See, e.g. [4].

excitations involving two polymers that start from a v type dimer (figure 1(c)) and end in a v type dimer (figure 1(d)) with mutual avoidance inbetween. These tragic reunion points will be termed dislocations. Of course, the two types of excitations can be combined to get more exotic ones. So far as the loops are concerned, $v = u$ is nothing special. Therefore, the model for any v (except zero) belongs to the same universality class as the Ising model.

If $v = 0$, then these edges of \mathcal{L}_D are absent. By removing the two coordinated vertices, we get a dimer problem on the original lattice \mathcal{L} with dimer activity $x = u^2$ for the transverse and one for the vertical (i.e. z) bonds [2]. This particular dimer problem has no loops and involves only the directed polymer type excitations. It has been studied in the past in connection with biomembrane phase transitions [6], and is known to belong to the same universality class as the ferroelectric five vertex model on diamond type lattices [7].

The critical behaviour of the vertex model is known exactly [7]. The system is frozen in the ground state in the low temperature $x < x_c = \frac{1}{2}$. The model is anisotropic with two different length scales, ξ_{\perp} and ξ_{\parallel} , in the transverse and vertical directions ($\xi_i \sim t^{-\nu_i}$, $i = \perp, \parallel$, $t \sim (x - x_c)$). The exponents for $d < 3$ are

$$\alpha = (3 - d)/2 \quad \nu_{\perp} = \frac{1}{2} \quad \nu_{\parallel} = 1 \quad (1)$$

with $d = 3$ as the upper critical dimension (UCD). The Ising model, in contrast, is isotropic, has $UCD = 4$, and has d dependent exponents for $d < 4$. It, therefore, belongs to a different universality class. Hence the existence of the loop mediated crossover in any dimension. \square

Crossover. Our focus is on the crossover behaviour around the critical point of the $v = 0$ vertex model. The free energy can be written in a scaling form

$$f \sim t^{2-\alpha} W_{\pm}(vt^{-\theta}) \quad (2)$$

as $v \rightarrow 0$ and $t \rightarrow 0$. We like to determine the crossover exponent θ . A positive θ indicates that the loops are relevant, and is expected to change the critical behaviour.

Adopting the procedure of [9], we expand the partition function for non-zero v in the low temperature phase as

$$Z \sim 1 + \sum_{n \geq 1} t^{(2-\alpha)-n\theta} W^{(n)}(0) \frac{v^n}{n!} \quad (3)$$

where $W^{(n)}$ is the n th derivative of the scaling function. (Note that the partition function of the vertex model in the low temperature phase is one.) In the low temperature phase, configurations with single dislocations are not important because they cost infinite energy. The dislocations always occur in pairs, as indeed seen explicitly in the exact two-dimensional solutions [2]. These finite energy excitations destroy the frozen behaviour of the vertex model. Consequently, the first non-trivial term is the v^2 term.

The v^2 term, Z_{v^2} , in the above expansion involves configurations where two directed polymers are created at some point and they diffuse or execute random walks avoiding each other until they meet and vanish after N steps. All such configurations for all possible values of N contribute to Z_{v^2} . Accordingly, we require the restricted partition function for two directed polymers, $Z_{R,2}(\mathbf{r})$ tied at origin at one end and at transverse coordinate \mathbf{r} at the other end. We assume (proved below) a power law decay $Z_{R,2}(\mathbf{r}) \sim N^{-\psi_{R,2}}$. In a continuum limit, Z_{v^2} comes from an integration of $Z_{R,2}(\mathbf{r})$ over the

$(d - 1)$ -dimensional r coordinates and integration over N . Rescaling r by ξ_{\perp} and N by ξ_{\parallel} , we obtain [9]

$$Z_{v,2} \sim \xi_{\perp}^{d-1} \xi_{\parallel}^{1-\psi_{R,2}} \sim t^{-1+\psi_{R,2}-(d-1)/2} \tag{4}$$

using the exponents from equation (1). Comparing with the t dependence of the v^2 term of equation (3) with $\alpha = (3 - d)/2$, we obtain

$$\theta = (d + 1 - \psi_{R,2})/2. \tag{5}$$

We obtain $\psi_{R,2}$ in a continuum limit by a path integral approach.

Continuum limit. Taking a continuum limit for the polymers, two interacting directed polymers can be studied in the path integral approach by the dimensionless Hamiltonian [8, 10]

$$H_2 = \frac{1}{2} \sum_{i=1}^2 \int_0^N dz \left(\frac{\partial r_i}{\partial z} \right)^2 + v_0 \int_0^N dz \delta^d(r_1(z) - r_2(z)) \tag{6}$$

where $r_i(z)$ is the $(d - 1)$ -dimensional coordinate of a point at z on the contour of the i th chain. The first term is the elastic energy, taking care of the connectivity of the chains†. The viciousness of the walkers is simulated in the second term by a δ -function interaction at the same z coordinate with $v_0 > 0$. For the non-interacting case, $\psi_{R,2} = d - 1$, but interaction changes it to $\psi_{R,2} = d - 1 + \eta_{R,2}$. The anomalous part $\eta_{R,2}$ comes from the renormalization group approach [11].

The polymer partition function of interest is

$$Z_{R,2}(r) = \int \mathcal{D}r \prod_{i=1,2} [\delta^d(r_i(0)) \delta^d(r_i(N) - r)] e^{-H_2} \tag{7}$$

$\mathcal{D}r$ standing for the sum over all paths. We evaluate $Z_{R,2}(r)$ by a perturbation expansion in the coupling constant v_0 . Some of the diagrams are shown in figure 2. It is possible to resum the series and get the results. We, however, find it more illuminating to use a renormalization group approach, which links the origin of the anomalous part of the exponent $\psi_{R,2}$ to the multiplicative renormalization constant for the partition function. We will skip the details of the evaluation of the diagrams, and quote the series from [11]:

$$\frac{Z_{R,2}}{(2\pi N)^{-d}} = 1 + \sum_{n=1}^{\infty} \left(\frac{-u_0}{4\pi} \right)^n (4\pi N L^{-2})^{n\epsilon/2} \frac{\Gamma^{n+1}(\epsilon/2)}{\Gamma((n+1)\epsilon/2)} \tag{8}$$

where $u_0 = v_0 L^{\epsilon}$ ($\epsilon = 3 - d$ and L an arbitrary length scale) is the dimensionless coupling constant and, for simplicity, r is taken at origin.

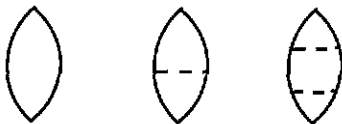


Figure 2. Diagrams for $Z_{R,2}$. Solid lines are the polymers and dotted lines are the interactions. See [7, 10, 11] for rules to evaluate these.

† Note that d of this paper is $d + 1$ of [10, 11].

Each term of the perturbation series shows a divergence at $d=3$ at which v_0 is dimensionless. The removal of divergences requires (i) a renormalization of the coupling constant $u_0 = u[1 - u/(2\pi\epsilon)]^{-1}$ [10], and (ii) an overall multiplicative renormalization constant $R_{R,2}(u)$. It is shown in [11] that $R_{R,2}(u) = [1 - u/(2\pi\epsilon)]^{-2}$, so that $Z_{R,2}|_r = R_{R,2}(u)Z_{R,2}$ is well defined. $R_{R,2}(u)$ is known exactly. A standard renormalization group argument then gives $\eta_{R,2}$ as

$$\eta_{R,2} = \frac{1}{2} \left(\beta(u) \frac{\partial}{\partial u} \ln R_{R,2}(u) \right)_{u=u^*} = \epsilon \quad (9)$$

where $u^* = 2\pi\epsilon$ is the fixed point of $\beta(u) \equiv L(\partial u/\partial L) = u\epsilon[1 - u/(2\pi\epsilon)]$ [10]. Putting everything together, we get

$$\psi_{R,2} = 2 \quad (10)$$

for all $d < 3$.

The crossover exponent then follows from equation (5) as

$$\theta = (d-1)/2. \quad (11)$$

Since $\eta_{R,2}$ is known exactly, this formula for θ is also exact. It agrees with the exact result of [2] and the random walk argument of [9] for $d=2$.

In the context of two-dimensional commensurate-incommensurate transitions, where the lines are the fluctuating domain walls, it is possible to have more than two walls meeting at the dislocation points. For two dimensions, the random walk argument of Huse and Fisher [9] shows that dislocations for p walls are relevant if $p > \sqrt{6}$. A similar question can also be asked for $d > 2$. It would involve the decay exponent $\psi_{R,p}$ for p directed polymers. We have shown elsewhere [11] that $\psi_{R,p} = p(d-1)/2 + (\frac{p}{2})\epsilon + O(\epsilon^2)$. This exponent gives the critical number p to $O(\epsilon)$ as $4 - \epsilon$. In other words, even though $p=3$ dislocations are not relevant in two dimensions, they do induce a crossover for $d > 2$. To which class the model crosses over to is yet to be seen.

In summary, we proposed a model that shows a crossover from anisotropic d -dimensional five vertex models to the isotropic d -dimensional Ising universality class, and the crossover is mediated by loops involving reunion of two vicious walkers. Such a crossover in the three-dimensional Ising model has not, hitherto, been recognized. We also show that dislocations involving three walkers can induce a crossover in the vertex model for $d > 2$, though to which class remaining open. Such three walker reunions are irrelevant in two dimensions. The crossover exponent to the Ising class is determined to be $\theta = (d-1)/2$ (for $d \leq 3$). This is an exact result and agrees with the known result for two dimensions [2, 9].

References

- [1] Blote H W J and Hilhorst H J 1982 *J. Phys. A: Math. Gen.* **15** L631
- [2] Bhattacharjee S M 1984 *Phys. Rev. Lett.* **53** 1161
- [3] Fisher M E 1984 *J. Stat. Phys.* **34** 667
- [4] Nagle J F, Yokoi C S O and Bhattacharjee S M 1989 *Phase Transitions and Critical Phenomena* vol 13, ed C Domb and J Lebowitz (New York: Academic)
- [5] Fisher M E 1966 *J. Math. Phys.* **7** 1776
- [6] Bhattacharjee S M *et al* 1983 *J. Stat. Phys.* **32** 361

- [7] Bhattacharjee S M and Rajasekaran J J 1991 *Phys. Rev. A* **44** 6202
- [8] Nelson D R and Seung H S 1988 *Phys. Rev. B* **39** 9153
Nelson D R 1988 *Phys. Rev. Lett.* **60** 1973
- [9] Huse D A and Fisher M E 1984 *Phys. Rev. B* **29** 239
- [10] Rajasekaran J J and Bhattacharjee S M 1991 *J. Phys. A: Math. Gen.* **24** L371
- [11] Mukherji S and Bhattacharjee S M Reunion of vicious walkers: Results from ϵ -expansion *Preprint*