

Directed Polymers with Random Interaction: Marginal Relevance and Novel Criticality

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We show by an exact renormalization-group approach that a random two-chain interaction for $(d+1)$ -dimensional directed polymers is marginally relevant at $d=1$. There is a critical point for $d > 1$ separating the weak and strong disorder phases, and the length scale exponent is $\nu = [2(d-1)]^{-1}$ for $d > 1$. For the m th-order multicritical case involving random m -chain interactions, the disorder is marginally relevant at $d_m = 1/(m-1)$. Here also the disorder induces a critical point for $d > d_m$, with an exponent $\nu_m = [2d(m-1) - 2]^{-1}$. An essential singularity occurs for the length scale right at $d = d_m$.

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The problem of a $(d+1)$ -dimensional directed polymer (DP) in a random medium represents one of the simplest examples that captures most of the complexities of disordered systems [1-3]. In recent years, various attempts have been made to study this system, and thereby, gain an understanding of disordered systems in general [1-5]. Apart from the interest on its own as a simple disordered system, this particular model appears directly or in disguise in various situations, like commensurate-incommensurate transitions [1], surface growth [2,6], vertex models [7-9], flux lattice melting in high- T_c superconductors [10,11], wetting transition [12], etc. Several exact results are known for a single DP in random medium in $1+1$ dimension, but detailed understanding and settled results in higher dimensions are rather few.

The problem of pure interacting DPs, on the contrary, enjoys the status of almost complete solvability through renormalization group (RG) and appropriate ϵ expansion [7,13]. Even the multicritical case involving many-body interactions can also be handled exactly [9,14]. Prompted by this success, we study a model of interacting DPs with disorder that has the simplicity of the pure problem and yet has enough richness of a disordered system. We emphasize the paradigmatic nature of this particular model for disordered systems in view of its exact solvability through an ϵ expansion. We are not aware of any such example.

The model we suggest is a simple random version of the pure problem of two DPs coupled through a short-range interaction [13]. This present problem of two DPs, each directed along, say, the z axis but fluctuating in the transverse d -dimensional space, is described by the following Hamiltonian [10,13]:

$$\mathcal{H} = \frac{1}{2} \int_0^N dz \{ \dot{\mathbf{r}}_1^2(z) + \dot{\mathbf{r}}_2^2(z) \} + \int_0^N dz v_0 [1 + b(z)] V(\mathbf{r}_{12}(z)), \quad (1)$$

where $\mathbf{r}_1, \mathbf{r}_2$ are the transverse d -dimensional spatial coordinates of the two chains, each of length N , $\dot{\mathbf{r}}_i(z) = \partial \mathbf{r}_i(z) / \partial z$, and $\mathbf{r}_{12}(z) = \mathbf{r}_1(z) - \mathbf{r}_2(z)$. Here the in-

teraction is at the same z coordinate for the two chains, and $V(r)$ is a given interaction potential. We keep V for generality but soon will consider only δ -function-type short-range interaction. The quenched randomness is introduced through the coupling constant (CC) $v_0[1 + b(z)]$ with $v_0 > 0$. The interaction contains a pure repulsive part with CC v_0 as well as a random part with CC $v_0 b(z)$. Note that the disorder is only in the z direction, independent of the transverse coordinates. We consider only uncorrelated, normally distributed randomness with

$$\langle b(z) \rangle = 0, \quad \langle b(z_1) b(z_2) \rangle = \Delta \delta(z_1 - z_2). \quad (2)$$

Our aim is to see the effect of disorder on the thermodynamics of DPs and whether disorder induces any new phenomenon not seen in the pure case, such as any new critical behavior as seen for a single DP in a random medium [6,12].

A lattice version of the above model in Eq. (1), in $1+1$ dimension, in the relative coordinate, is similar to the wetting transition model investigated by Forgacs *et al.* [15], and Derrida, Hakim, and Vannimenus [12]. The latter authors, using a first-order perturbation in $b(z)$ and a type of real-space RG, have shown that the disorder is marginally relevant in one dimension. In this paper, using the dimensional regularization scheme, we show *exactly* the marginal relevance of disorder in $1+1$ dimension, and discuss the novel critical behavior that emerges for $d > 1$. We further generalize this random problem to multicritical cases and obtain the relevance of disorder and the necessary exponents exactly.

The usual approach for studying disordered systems is to calculate different moments of the partition function Z , such as $\langle Z \rangle$, $\langle Z^2 \rangle - \langle Z \rangle^2$. These moments distinguish the quenched and annealed averages, defined by $\langle \ln Z \rangle$ and $\ln \langle Z \rangle$, respectively. We avoid the replica analysis and restrict ourselves to $\langle Z \rangle$ and $\langle Z^2 \rangle$ only. These are sufficient to give us the necessary results.

Let us start with $\langle Z \rangle$. A straightforward averaging over $b(z)$ shows that $\langle Z \rangle$ can be equivalently written as the partition function of a pure two-chain problem with a Hamiltonian

$$H = \frac{1}{2} \int_0^N dz \{ \dot{\mathbf{r}}_1^2(z) + \dot{\mathbf{r}}_2^2(z) \} + \bar{v}_0 \int_0^N dz V(\mathbf{r}_{12}(z)) - \frac{v_0^2 \Delta}{2} \int_0^N dz V^2(\mathbf{r}_{12}(z)), \quad (3)$$

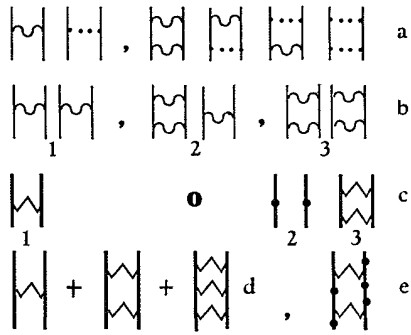


FIG. 1. (a) Diagrams up to one loop for $\langle Z \rangle$ for v_0 (dotted lines) and $v_0 b(z)$ (wiggly lines) type interactions. (b) Diagrams involving only $v_0 b(z)$ type interaction for $\langle Z^2 \rangle$. (c) Same as (b) but after disorder averaging. Thick lines indicate pairs; dots, two-chain interactions (\bar{v}_0); and wavy lines, the four-chain interaction (\bar{r}_0). The second diagram with dots only does not contribute in $\langle Z^2 \rangle_c$. (d) Diagrams for $\langle Z^2 \rangle_c$ [Eq. (7)]. (e) An example of a dressed diagram. (See text.)

so that $\langle Z \rangle = \int \mathcal{D}r e^{-H}$, $\mathcal{D}r$ representing the functional integral measure for the two chains. The effect of disorder is to induce an attraction between the two chains.

The role of disorder in changing the interaction can also be seen from the perturbation expansion of the connected part of $\langle Z \rangle_c$ as shown in Fig. 1(a). It is easy to see that only the diagrams with even number of consecutive wiggly lines survive after disorder averaging (DA). The nonzero contribution of the first-order diagram comes only from the one with a single dotted line, and it is

$$-v_0 \mathcal{V} \int_0^N dz \int dr V(r(z)),$$

where \mathcal{V} is the transverse d -dimensional volume. Among the one loop diagrams, the only surviving disorder diagram is the one with two wiggly lines. After DA, its contribution becomes

$$\mathcal{H}_{2,2} = \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \dot{r}_i^2(z) + \bar{v}_0 \int_0^N dz \{ \delta(r_{12}(z)) + \delta(r_{34}(z)) \} - \bar{r}_0 \int_0^N dz \delta(r_{12}(z)) \delta(r_{23}(z)), \quad (5)$$

where $\bar{r}_0 = v_0^2 \Delta$.

The special feature of Eq. (5) is the four-chain interaction introduced by DA. The system lowers "energy" when the two chains in each pair meet each other at the same chain length z . This is not the standard attraction between the four chains as, say, in multicritical cases (see below) but is of a distinct nature. The effect of this term generated through DA, as we will see, is to create a new divergence at $d=1$ in the perturbation expansion of $\langle Z^2 \rangle$. That $d=1$ is special follows from dimensional analysis which shows that the upper critical dimension (UCD) for $\bar{r}_0 = v_0^2 \Delta$ is $d=1$, though it is $d=2$ for \bar{v}_0 .

$$v_0^4 \int_0^N dz_1 \int_0^{z_1} dz_2 b(z_1) b(z_2) \int d\mathbf{r}_1 \int d\mathbf{r}_2 G_\delta^2(\mathbf{r}_1 - \mathbf{r}_2 | z_1 - z_2) \times \int_0^N dz'_1 \int_0^{z'_1} dz'_2 b(z'_1) b(z'_2) \int d\mathbf{R}_1 \int d\mathbf{R}_2 G_\delta^2(\mathbf{R}_1 - \mathbf{R}_2 | z'_1 - z'_2),$$

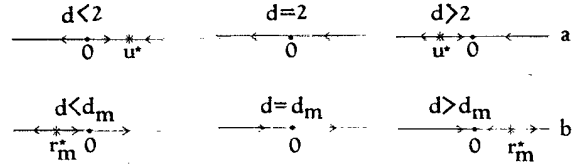


FIG. 2. (a) Flow diagrams for u . The nontrivial fixed point is at $u^* = 2 - d$. (b) Flow diagrams for r_m ($= r$ for $m=2$). * represents the nontrivial fixed point.

$$+ \frac{v_0^2 \Delta}{2} \mathcal{V} \int_0^N dz \int dr V^2(r(z)).$$

DA forces the two interaction points to merge together so that this diagram ultimately looks like a pure type diagram of first order but with an opposite sign. Such a reduction of order after DA takes place for every loop in higher order and can be checked explicitly—the combinatorics is always on our side. Since only even order disorder terms survive, they simulate an attractive interaction in the effective Hamiltonian, as seen in Eq. (3).

If we restrict V to a short-range interaction, then under renormalization, it is expected to map onto a δ function. With this in mind, we set both the repulsive and attractive potentials in Eq. (3) by a δ function so that the effective Hamiltonian, to be used for evaluating $\langle Z \rangle$, is

$$\mathcal{H}_2 = \frac{1}{2} \int_0^N dz \{ \dot{r}_1^2(z) + \dot{r}_2^2(z) \} + \bar{v}_0 \int_0^N dz \delta(r_{12}(z)). \quad (4)$$

For simplicity, we take $\bar{v}_0 \geq 0$, but note that strong disorder can make $\bar{v}_0 < 0$ as well.

Since the evaluation of $\langle Z \rangle$ with \mathcal{H}_2 is identical to a pure problem results can be borrowed *in toto* from Refs. [7,13]. The flow diagram for the dimensionless CC u (bare CC being $u_0 = \bar{v}_0 L^{(2-d)}$) is shown in Fig. 2(a).

We now consider $\langle Z^2 \rangle$. A direct averaging, as in Eq. (3) would give an effective Hamiltonian involving four chains. Restricting ourselves, as before, to the short-range potential, we get the effective Hamiltonian

In order to see the origin of the special four-chain interaction and the divergence at $d=1$, we go back to the original Hamiltonian equation (1) with a δ function potential. This two-chain δ function interaction generates the \bar{r}_0 term in Eq. (5). A perturbation expansion for $\langle Z^2 \rangle_c = \langle Z^2 \rangle - \langle Z \rangle^2$ involves only the connected diagrams. A few such diagrams formed purely out of random part of the short-range interaction are shown in Fig. 1(b). The contribution from Fig. 1(b)1 is $\mathcal{V}^2 v_0^2 \int_0^N dz_1 \times b(z_1) \int_0^N dz_2 b(z_2)$ which after DA using Eq. (2) becomes $\mathcal{V}^2 v_0^2 \Delta$. Similarly, Fig. 1(b)3 would correspond to the following expression:

where $G_0(r|z) = (2\pi z)^{-d/2} \exp(-r^2/2z)$, is the distribution function for a single polymer. DA leads to two different pairings of z 's for this diagram, one of which produces a disconnected diagram [Fig. 1(c)2] that does not contribute to $\langle Z^2 \rangle_c$. The other one [Fig. 1(c)3] that pairs up (z_1, z'_1) and (z_2, z'_2) gives, after integration over the spatial coordinates,

$$\frac{\bar{v}_0^2 \gamma^2}{(4\pi)^d} \frac{N^{1+(1-d)} \Gamma(1-d)}{1+(1-d) \Gamma[1+(1-d)]}. \quad (6)$$

This vividly exhibits the divergence at $d=1$. The diagram Fig. 1(b)2 vanishes after DA.

It is important to note that in the above averaging process the two pairs of chains become connected purely by the random part of the interaction. The loops formed thereby give the divergence at $d=1$, and these are the loops one gets from the last term of the effective Hamiltonian equation (5). The two-chain interaction for each pair does not produce any divergence at $d=1$.

A systematic way of evaluating $\langle Z^2 \rangle_c$ is to do a double expansion in \bar{v}_0 and \bar{r}_0 with a few diagrams shown in Figs. 1(d) and 1(e). The whole series for $\langle Z^2 \rangle_c$ to all orders in \bar{r}_0 corresponding to the diagrams of Fig. 1(d) is given by

$$\langle Z^2 \rangle_c = \frac{N(4\pi)^d \gamma^2}{L^{2\epsilon}} \left[r_0 + \sum_{n=1}^{\infty} r_0^{n+1} \left(\frac{N}{L^2} \right)^{n\epsilon} \frac{\Gamma^n(\epsilon)}{\Gamma[2+n\epsilon]} \right], \quad (7)$$

where $r_0 = \bar{r}_0 L^{2\epsilon} (4\pi)^{-d}$, $\epsilon = 1-d$, and L is an arbitrary length scale introduced to make \bar{r}_0 and N dimensionless. This series in Eq. (7) actually corresponds to the $\bar{v}_0=0$ case. In fact, it is also possible to write down the whole series taking into account all orders in \bar{v}_0 . Since the leading-order divergence after dressing the solid lines with dots [Fig. 1(e)] remains the same as that obtained from Eq. (7), we do not go into any further details which will be discussed elsewhere.

The renormalization of the above Mittag-Leffler type series is well known [7,13]. Defining the renormalized interaction r through

$$r_0 = r(1 + a_1 r + a_2 r^2 + a_3 r^3 + \dots),$$

the coefficients can be determined order by order as $a_p = (-\epsilon)^{-p}$ [13]. The β function is, therefore, known exactly and is given by

$$\beta(r) \equiv L \frac{\partial r}{\partial L} = 2(\epsilon r + r^2). \quad (8)$$

The nontrivial fixed point is unphysical for $d < 1$ because r is strictly positive. It, however, moves to the physical domain for $d > 1$. See Fig. 2(b). Exactly at $d=1$, $\epsilon=0$, r grows with length L as

$$\mathcal{H} = \frac{1}{2} \int_0^N dz \sum_{i=1}^m \dot{x}_i^2(z) + \int_0^N dz v_m [1 + b(z)] \prod_{p=1}^{m-1} \delta(r_{pp+1}(z)), \quad (13)$$

with $v_m > 0$, and $b(z)$ obeying the same distribution as in Eq. (2).

So far as the averaged m -chain partition function $\langle Z_m \rangle$ is concerned, we follow the same procedure as for two chains. The sole effect of disorder at this level is to produce an m -chain attraction so that the system can be described as well by

$$r = r(0) \left[1 + 2r(0) \ln \frac{L_0}{L} \right]^{-1}, \quad (9)$$

$r(0)$ being the coupling at length L_0 . Hence, the disorder is marginally relevant, in agreement with Ref. [12].

For $d > 1$, there exists an unstable nontrivial fixed point at $r = |\epsilon|$ which separates two distinct regimes of disorders. If we start with a strong enough disorder, on the right side of the fixed point, it increases with length scale, going beyond the perturbative regime. This is the strong-disorder phase. On the other hand, the left side of the fixed point is the weak-disorder regime, since r flows to zero (the stable fixed point). The unstable fixed point, therefore, represents a critical point—a novel phase transition induced by the disorder.

One way of achieving the above-mentioned critical behavior is to change the strength of the disorder by controlling the temperature. The “strong-disorder” phase ($\langle \ln Z \rangle \neq \ln \langle Z \rangle$) would correspond to the low-temperature phase while the “weak-disorder” phase ($\langle \ln Z \rangle = \ln \langle Z \rangle$) is the high-temperature one. The details of the critical behavior can be obtained by integrating the β function,

$$r = |\epsilon| \left[1 - \frac{r(0) - |\epsilon|}{r(0)} \left(\frac{L}{L_0} \right)^{2|\epsilon|} \right]^{-1}. \quad (10)$$

For a small starting deviation $\Delta T \equiv T - T_c = r(0) - |\epsilon|$, there is a length scale $L \sim (\Delta T)^{-1/2|\epsilon|}$ at which r in Eq. (10) diverges. This we can identify as a length scale ξ associated with the critical point with the length scale exponent

$$\nu = (2|\epsilon|)^{-1}. \quad (11)$$

The divergence at $\epsilon=0$ is consistent with the essential singularity that follows from Eq. (8),

$$\xi \sim \exp[1/(2\Delta T)]. \quad (12)$$

It is interesting to note that higher moments such as $\langle Z^3 \rangle, \langle Z^4 \rangle, \dots$ do not require any new interaction and therefore lead to the same critical dimension $d=1$. The beta function is identical to the case for $\langle Z^2 \rangle$. Hence all the exponents, etc., remain unchanged. We might add that, in the replica approach, one needs $\langle Z^n \rangle$ with $n \rightarrow 0$. Since no new interaction is generated, $d=1$ will remain the critical dimension even in the replica analysis.

Multicritical case.—Higher-order interactions in polymers lead to multicritical phenomena, and for directed polymers a whole hierarchy of multicritical points (MCP) can be studied exactly [9,14]. The corresponding random m th-order multicritical case would involve the following Hamiltonian for m chains interacting through a random m -chain interaction at the same z coordinate:

a pure m -chain Hamiltonian with an effective coupling constant \bar{v}_m . As for two chains, here also, the disorder can make \bar{v}_m negative. For simplicity, we consider only $\bar{v}_m \geq 0$. Deviations from the pure case become transparent when we study fluctuations through $\langle Z_m^2 \rangle_c \equiv \langle Z_m^2 \rangle - \langle Z_m \rangle^2$. An averaging over disorder would yield, as in Eq. (5), an effective Hamiltonian

$$\begin{aligned} \mathcal{H}_{m,m} = & \frac{1}{2} \int_0^N dz \sum_{i=1}^{2m} \dot{\mathbf{r}}_i^2(z) + \bar{v}_m \int_0^N dz \prod_{p=1}^{m-1} \delta(\mathbf{r}_{pp+1}(z)) + \bar{v}_m \int_0^N dz \prod_{q=m+1}^{2m-1} \delta(\mathbf{r}_{qq+1}(z)) \\ & - v_m^2 \Delta \int_0^N dz \prod_{p=1}^{m-1} \delta(\mathbf{r}_{pp+1}(z)) \prod_{q=m+1}^{2m-1} \delta(\mathbf{r}_{qq+1}(z)). \end{aligned} \quad (14)$$

Note that this Hamiltonian is different from the corresponding $2m$ -chain pure Hamiltonian of the type in Eq. (13)—it involves one less δ function. There is no δ function connecting the two sets.

Dimensional analysis shows that $\bar{r}_{m0} \equiv v_m^2 \Delta$ is dimensionless at $d = 1/(m-1)$ but \bar{v}_m is dimensionless at $d = 2/(m-1)$, the UCD for the pure MCP.

The usual perturbation expansion in \bar{r}_{m0} , taking for simplicity $\bar{v}_m = 0$, yields a series for $\langle Z_m^2 \rangle_c$ identical to Eq. (7) except for the replacements of the factor $(4\pi)^d$ by $[(2\pi)^{m-1} m]^d$, ϵ by $\epsilon_m \equiv 1 - d(m-1)$, and r_0 by $r_{m0} \equiv \bar{r}_{m0} L^{2\epsilon_m} [(2\pi)^{m-1} m]^{-d}$. The divergence now occurs at $d = 1/(m-1)$. The β function for r_m is still identical to Eq. (8) with (ϵ_m, r_m) replacing (ϵ, r) . This proves the marginal relevance of the disorder at dimension $d_m = 1/(m-1)$. It immediately follows that all the results obtained for the two-chain case would, indeed, be valid with the replacement of UCD by d_m , and ϵ by ϵ_m .

It is instructive to compare the present problem with that of a single DP in a random medium. The recursion relation has remarkable similarity with the one derived in Ref. [6] for the growth of disorder at the one-loop level. The phase transition expected for a single DP in a random medium is, therefore, the analog of the phase transition discussed here [3]. The notable difference is in the single-chain behavior. In the former problem, even a single chain needs anomalous exponents but nothing special happens at the single-chain level in our case.

To summarize, we have introduced a simple disordered system involving directed polymers with random interaction. The disorder is shown to be marginal at $d = 1$ for the two-chain problem. A novel critical behavior is predicted for $d > 1$ with a transition between a fluctuation dominated strong disorder (low-temperature) phase to a high-temperature phase where disorder is irrelevant. The critical point has $\nu = (2|\epsilon|)^{-1}$, where $\epsilon = 1 - d$. We have also generalized this random problem to a hierarchy of multicritical points where the m th-order MCP involves an m -chain interaction. The upper critical dimension is reduced from the pure value $2/(m-1)$ to $d_m = 1/(m-1)$. As for the two-chain case, a similar weak-disorder to

strong-disorder phase transition occurs for the m th MCP for $d > d_m$. The corresponding exponent is $\nu_m = (2|\epsilon_m|)^{-1}$ with $\epsilon_m = 1 - d(m-1) < 0$. At $\epsilon_m = 0$, there is an essential singularity for the length scale. These results are obtained in the dimensional regularization scheme to all orders, and are, therefore, exact.

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